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Singularly perturbed quasilinear Choquard equations with nonlinearity satisfying Berestycki–Lions assumptions

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Abstract

In the present paper, we consider the following singularly perturbed problem:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u - \varepsilon^2\Delta(u^2)u = \varepsilon^{-\alpha}(I_\alpha * G(u))g(u), & x\in\mathbb{R}^N; \\ u\in H^1(\mathbb{R}^N), \end{cases}$$

where $\varepsilon > 0$ is a parameter, $N\geq 3$, $\alpha\in(0,N)$, $G(t)=\int_0^t g(s)\,\mathrm{d}s$, $I_\alpha:\mathbb{R}^N\to\mathbb{R}$ is the Riesz potential, and $V\in\mathcal{C}(\mathbb{R}^N,\mathbb{R})$ with $0<\min_{x\in\mathbb{R}^N}V(x)<\lim_{|y|\to\infty}V(y)$. Under the general Berestycki–Lions assumptions on g, we prove that there exists a constant $\varepsilon_0>0$ determined by V and g such that for $\varepsilon\in(0,\varepsilon_0]$ the above problem admits a semiclassical ground state solution $\hat{u}_\varepsilon$ with exponential decay at infinity. We also study the asymptotic behavior of $\{\hat{u}_\varepsilon\}$ as $\varepsilon\to 0$.

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Keywords: Quasilinear Choquard equation; Singularly perturbed; Semiclassical state; Concentration; Berestycki–Lions assumptions

1 Introduction

In this paper, we consider the following singularly perturbed quasilinear Choquard equation:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u - \varepsilon^2\Delta(u^2)u = \varepsilon^{-\alpha}(I_\alpha * F(u))f(u), & x\in\mathbb{R}^N; \\ u\in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\varepsilon>0$ is a parameter, $N\geq 3$, $\alpha\in(0,N)$, and $I_\alpha:\mathbb{R}^N\to\mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x)=\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha\pi^{N/2}|x|^{N-\alpha}},\quad x\in\mathbb{R}^N\setminus\{0\},$$

$G(t)=\int_0^t g(s)\,\mathrm{d}s$, $V:\mathbb{R}^N\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ satisfy the following basic assumptions:

(V1) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $0 < V_0 := \min_{x\in\mathbb{R}^N} V(x) < V_\infty := \lim_{|y|\to\infty} V(y)$;

(G1) $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $G(t) = o(t^{(N+\alpha)/N})$ as $t \to 0$, and there exists a constant $\mathcal{C}_0 > 0$ such that

$$|g(t)t| \le \mathcal{C}_0\big(|t|^{(N+\alpha)/N} + |t|^{2(N+\alpha)/(N-2)}\big), \quad \forall t \in \mathbb{R};$$

(G2) $G(t) = o(t^{2(N+\alpha)/(N-2)})$ as $|t| \to \infty$;

(G3) There exists $s_0 > 0$ such that $G(s_0) \not= 0$.

Note that (V1) was introduced by Rabinowitz in [23], and (G1)–(G3) were almost necessary and sufficient conditions and regarded as the Berestycki–Lions type conditions to Choquard equations, which were introduced by Moroz and Van Schaftingen in [19] for the study of (1.1) with $\varepsilon = 1$. Without loss of generality, in this paper we assume that

$$0 \in \Omega_0 := \Big\{x \in \mathbb{R}^N : V(x) = V_0 = \min_{x\in\mathbb{R}^N} V(x)\Big\}. \tag{1.2}$$

If the nonlocal term $(I_\alpha * F(u))f(u)$ is replaced with a local nonlinear term $h(x,u)$, then (1.1) reduces to the well-known quasilinear Schrödinger equation introduced in [3, 4, 12] to study a model of self-trapped electrons in quadratic or hexagonal lattices. If the term $\Delta(u^2)u$ is absent, then (1.1) is usually called the nonlinear Choquard equation, which was introduced by Pekar [22], and it describes the quantum mechanics of a polaron at rest. We refer to [1, 2, 6, 7, 9, 10, 13–20, 25] and the references therein in either cases for $\varepsilon = 1$.

This paper was motivated by some recent works of Yang, Zhang, and Zhao [28] and Yang, Tang, and Gu [26, 27], and Zhang and Ji [29], in which quasilinear Choquard equations were considered. In particular, for $\varepsilon = 1$, Yang, Zhang, and Zhao [28] in 2018 first considered the existence of nontrivial solutions for the quasilinear Choquard equation (1.1), where $N \ge 3$, $\alpha \in (0,N)$, and $f(u) = |u|^{p-2}u$ with $2 < p < 2(N+\alpha)/(N-2)$, by using variational and perturbation method. Later, this result was extended partly by [26, 29].

As $\varepsilon \to 0$, the existence and asymptotic behavior of the solutions of the singularly perturbed equation (1.1) are known as the semi-classical problem, which was used to describe the transition between of quantum mechanics and Newtonian mechanics, see Floer and Weinstein [11] for the pioneering work. In [27], Yang, Tang, and Gu considered singularly perturbed equation (1.1) under (V1), (G1), (G2) and the following assumption:

(G3′) $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $g(t) = 0$ for $t \le 0$, and $\frac{g(t)}{t^3}$ is positive and increasing in $(0,\infty)$.

Based on the dual approach and the Nehari manifold method, the existence, multiplicity, and concentration behavior of positive solutions were obtained.

Different from those in the previous papers [26–29], in this paper, we consider the existence and concentration of ground state solutions for the more generalized quasilinear Choquard equation (1.1) under the general "Berestycki–Lions assumptions" on the nonlinearity g which are almost necessary. To state our result, we introduce the following conditions:

(V2) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ and $t \mapsto \frac{NV(tx)+\nabla V(tx)\cdot(tx)}{t^\alpha}$ is nonincreasing on $(0,\infty)$ for all $x \in \mathbb{R}^N \setminus \{0\}$;

(G4) $g(t) = o(t)$ as $t \to 0$.

Obviously, (G1)–(G4) are much weaker than those used in [26–29]. Our result is as follows.

Theorem 1.1 *Assume that V and g satisfy* (V1), (V2), *and* (G1)–(G3). *Then there exists a number $\varepsilon_0 > 0$ determined by V and g, see* (3.23), *such that for all $\varepsilon \in (0, \varepsilon_0]$*, (1.1) *has a ground state solution $\hat{u}_\varepsilon$. If* (G4) *holds also, then the following statements hold*:

(i) *For $\varepsilon \in (0, \varepsilon_0]$, the function $|\hat{u}_\varepsilon|$ achieves its maximum at a point x_ε, which satisfies*

$$\lim_{\varepsilon\to 0} V(x_\varepsilon) = V_0 = \min_{x\in\mathbb{R}^N} V(x);$$

(ii) *There exist $\Pi_0 > 0$ and $\kappa_0 > 0$ independent of $\varepsilon \in (0, \varepsilon_0]$ such that the maximum point x_ε of $|\hat{u}_\varepsilon|$ satisfies the inequality*

$$\bigl|\hat{u}_\varepsilon(x)\bigr| \le \Pi_0 \exp\left(-\frac{\kappa_0}{\varepsilon}|x - x_\varepsilon|\right), \quad \forall x \in \mathbb{R}^N, \varepsilon \in (0, \varepsilon_0];$$

(iii) *For any sequence $\varepsilon_n \to 0$, the sequence $\hat{u}_{\varepsilon_n}(\varepsilon_n x + x_{\varepsilon_n})$ converges in $H^1(\mathbb{R}^N)$ to a ground state solution u of the following autonomous equation*:

$$-\Delta u + V_0 u - \Delta\bigl(u^2\bigr)u = \bigl(I_\alpha * F(u)\bigr)f(u). \tag{1.3}$$

The rest of the paper is organized as follows. In Sect. 2, we give variational setting and preliminary lemmas. Section 3 is devoted to the proof of the existence of ground state solutions. In the last section, we establish the concentration of ground state solutions and prove Theorem 1.1.

Next, we give some notations. $H^1(\mathbb{R}^N)$ is the usual Sobolev space with the standard scalar product and the norm

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)\,\mathrm{d}x, \qquad \|u\| = (u, u)^{1/2}, \quad \forall u, v \in H^1\bigl(\mathbb{R}^N\bigr).$$

$L^s(\mathbb{R}^N)$ $(1 \le s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s\,\mathrm{d}x)^{1/s}$. We use $C_1, C_2, \ldots$ to indicate positive constants possibly different in different places.

2 Variational setting and preliminary lemmas

Observe that formally (1.1) is the Euler–Lagrange equation associated with the following functional:

$$\mathcal{I}_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^N} \bigl(1 + 2u^2\bigr)|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \bigl(I_\alpha * G(u)\bigr)G(u)\,\mathrm{d}x. \tag{2.1}$$

It is well known that $\mathcal{I}_\varepsilon$ is not well defined in general in $H^1(\mathbb{R}^N)$. To overcome this difficulty, we employ an argument developed by Colin and Jeanjean [9] (see also [14]). We make the change of variables by $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1 + 2|f(t)|^2}} \quad \text{on } [0, +\infty), \qquad f(-t) = -f(t) \quad \text{on } (-\infty, 0]. \tag{2.2}$$

After the change of variables from $\mathcal{I}_\varepsilon$, we obtain the following functional:

$$\begin{aligned} J_\varepsilon(v) &= \mathcal{I}_\varepsilon(u) = \mathcal{I}_\varepsilon\bigl(f(v)\bigr) \\ &= \frac{\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)f^2(v)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \bigl(I_\alpha * G\bigl(f(v)\bigr)\bigr)G\bigl(f(v)\bigr)\,\mathrm{d}x, \end{aligned} \tag{2.3}$$

which is well defined on the space $H^1(\mathbb{R}^N)$. Then the critical points of (2.3) are weak solutions of the equation

$$-\varepsilon^2\Delta v = \frac{1}{\sqrt{1+2|f(v)|^2}}\big[\big(I_\alpha * G(f(v))\big)g(f(v)) - V(x)f(v)\big], \quad x\in\mathbb{R}^N. \tag{2.4}$$

Note that if v is a critical point of (2.3), then $u=f(v)$ is a weak solution of (1.1) (see [9, pp. 217–218]). Replacing $v(\varepsilon x)$ with $v(x)$, one easily sees that (2.4) is equivalent to

$$-\Delta v = \frac{1}{\sqrt{1+2f^2(v)}}\big[\big(I_\alpha * G(f(v))\big)g(f(v)) - V(\varepsilon x)f(v)\big], \quad x\in\mathbb{R}^N. \tag{2.5}$$

The energy functional associated with problem (2.5) is given by

$$\Phi_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}V(\varepsilon x)f^2(v)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G(f(v))\big)G(f(v))\,\mathrm{d}x. \tag{2.6}$$

Lemma 2.1 *The function $f(t)$ and its derivative satisfy the following properties*:

(f1) *f is uniquely defined, $\mathcal{C}^\infty$ and invertible, and $0<f'(t)\le 1$ for all $t\in\mathbb{R}$*;

(f2) *$|f(t)|\le|t|$ and $|f(t)|\le 2^{1/4}|t|^{1/2}$ for all $t\in\mathbb{R}$*;

(f3) *$f(t)/t\to 1$ as $t\to 0$ and $f(t)/\sqrt{t}\to 2^{1/4}$ as $t\to+\infty$*;

(f4) *$f(t)/2\le tf'(t)\le f(t)$ for all $t>0$ and $f(t)\le tf'(t)\le f(t)/2$ for all $t\le 0$*;

(f5) *There exists a positive constant θ_0 such that*

$$|f(t)| \ge \begin{cases}\theta_0|t|, & |t|\le 1,\\ \theta_0|t|^{1/2}, & |t|>1.\end{cases}$$

From (G1), Hardy–Littlewood–Sobolev inequality, and (f2) of Lemma 2.1, we deduce that, for some $p\in(2,2^*)$ and any $\epsilon>0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\big(I_\alpha * G(f(v))\big)G(f(v))\,\mathrm{d}x\\
&= \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha\pi^{N/2}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(f(v))G(f(v))}{|x-y|^{N-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C_1\big\|G(f(v))\big\|^2_{2N/(N+\alpha)}\\
&\le \epsilon\Big(\big\|f(v)\big\|_2^{2(N+\alpha)/N} + \big\|f(v)\big\|_{2\cdot 2^*}^{2(N+\alpha)/(N-2)}\Big) + C_\epsilon\big\|f(v)\big\|_{2p}^{(N+\alpha)p/N}\\
&\le \epsilon\Big(\|v\|_2^{2(N+\alpha)/N} + 2^{(N+\alpha)/2(N-2)}\|v\|_{2^*}^{2(N+\alpha)/(N-2)}\Big) + C_\epsilon 2^{(N+\alpha)p/(4N)}\|v\|_p^{(N+\alpha)p/N},\\
&\quad\forall v\in H^1(\mathbb{R}^N).
\end{aligned} \tag{2.7}$$

As in [25, Lemma 2.1, Lemma 2.2], we have the following result.

Lemma 2.2 *Assume that* (V1) *and* (V2) *hold. Then, for all $t\ge 0$ and $y\in\mathbb{R}^N$*,

$$h(t,y) := \big(\alpha + Nt^{N+\alpha}\big)V(y) - (N+\alpha)t^N V(ty) + \big(t^{N+\alpha}-1\big)\nabla V(y)\cdot y \ge 0. \tag{2.8}$$

Moreover, $|\nabla V(y)\cdot y|\to 0$ as $|y|\to\infty$.

For all $\varepsilon > 0$, we define a functional on $H^1(\mathbb{R}^N)$ as follows:

$$\begin{aligned}\mathcal{P}_\varepsilon(v) :=& \frac{N-2}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}\big[NV(\varepsilon x) + \nabla V(\varepsilon x)\cdot \varepsilon x\big]f^2(v)\,\mathrm{d}x \\ &- \frac{N+\alpha}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v)\big)\big)G\big(f(v)\big)\,\mathrm{d}x, \end{aligned}\tag{2.9}$$

which is associated with the Pohožaev identity $\mathcal{P}_\varepsilon(v) = 0$ of (2.5), and set

$$\mathcal{M}_\varepsilon := \big\{v \in H^1\big(\mathbb{R}^N\big)\setminus\{0\} : \mathcal{P}_\varepsilon(v) = 0\big\}.\tag{2.10}$$

By elemental calculations, we can get the following inequality:

$$h_1(t) := 2 + \alpha - (N+\alpha)t^{N-2} + (N-2)t^{N+\alpha} > h_1(1) = 0, \quad \forall t \in [0,1)\cup(1,+\infty).\tag{2.11}$$

In what follows, we define $v_t(x) := v(tx)$ for all $x \in \mathbb{R}^N$ and $t > 0$ along any $v \in H^1(\mathbb{R}^N)\setminus\{0\}$.

Lemma 2.3 *Assume that* (V1), (V2), (G1), *and* (G2) *hold. Then*

$$\Phi_\varepsilon(v) \geq \Phi_\varepsilon(v_t) + \frac{1-t^{N+\alpha}}{N+\alpha}\mathcal{P}_\varepsilon(v) + \frac{h_1(t)}{2(N+\alpha)}\|\nabla v\|_2^2, \quad \forall v \in H^1\big(\mathbb{R}^N\big), t > 0.\tag{2.12}$$

Proof Observe that

$$\begin{aligned}\Phi_\varepsilon(v_t) =& \frac{t^{N-2}}{2}\|\nabla v\|_2^2 + \frac{t^N}{2}\int_{\mathbb{R}^N}V(t\varepsilon x)f^2(v)\,\mathrm{d}x \\ &- \frac{t^{N+\alpha}}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v)\big)\big)G\big(f(v)\big)\,\mathrm{d}x.\end{aligned}\tag{2.13}$$

By (2.6), (2.8), (2.9), and (2.13), we have

$$\begin{aligned}&\Phi_\varepsilon(v) - \Phi_\varepsilon(v_t) \\ &= \frac{1-t^{N-2}}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}\big[V(\varepsilon x) - t^N V(t\varepsilon x)\big]f^2(v)\,\mathrm{d}x \\ &\quad - \frac{1-t^{N+\alpha}}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v)\big)\big)G\big(f(v)\big)\,\mathrm{d}x \\ &= \frac{1-t^{N+\alpha}}{N+\alpha}\bigg\{\frac{N-2}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}\big[NV(\varepsilon x) + \nabla V(\varepsilon x)\cdot\varepsilon x\big]f^2(v)\,\mathrm{d}x \\ &\quad - \frac{N+\alpha}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v)\big)\big)G\big(f(v)\big)\,\mathrm{d}x\bigg\} \\ &\quad + \frac{2+\alpha-(N+\alpha)t^{N-2}+(N-2)t^{N+\alpha}}{2(N+\alpha)}\|\nabla v\|_2^2 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N}\bigg\{\bigg[\frac{\alpha+Nt^{N+\alpha}}{N+\alpha}V(\varepsilon x) - t^N V(t\varepsilon x)\bigg] - \frac{1-t^{N+\alpha}}{N+\alpha}\nabla V(\varepsilon x)\cdot\varepsilon x\bigg\}f^2(v)\,\mathrm{d}x \\ &\geq \frac{1-t^{N+\alpha}}{N+\alpha}\mathcal{P}_\varepsilon(u) + \frac{h_1(t)}{2(N+\alpha)}\|\nabla v\|_2^2.\end{aligned}$$

This shows that (2.12) holds. □

Lemma 2.3 and (2.11) give the following consequence.

Corollary 2.4 *Assume that* (V1), (V2), (G1), *and* (G2) *hold. Then, for* $v \in \mathcal{M}_\varepsilon$,

$$\Phi_\varepsilon(v) = \max_{t>0} \Phi_\varepsilon(v_t). \tag{2.14}$$

Lemma 2.5 *Assume that* V *satisfies* (V1) *and* (V2). *Then there exist constants* $\gamma_1, \gamma_2 > 0$ *independent of* $\varepsilon > 0$ *such that*

$$NV(y) + \nabla V(y) \cdot y \geq \gamma_1, \quad \forall y \in \mathbb{R}^N \tag{2.15}$$

and

$$\alpha V(y) - \nabla V(y) \cdot y \geq \gamma_2, \quad \forall y \in \mathbb{R}^N. \tag{2.16}$$

Proof By (2.8), we have $\lim_{t\to\infty} h(t,x)/t^{N+\alpha} \geq 0$ for any $y \in \mathbb{R}^N$, and so

$$NV(y) + \nabla V(y) \cdot y \geq 0, \quad \forall y \in \mathbb{R}^N. \tag{2.17}$$

Let $V_{\max} := \max_{x\in\mathbb{R}^N} V(x) \in (0,\infty)$. Using (V1), (2.8), and (2.17), we have

$$\begin{aligned} NV(y) + \nabla V(y) \cdot y &\geq \frac{(N+\alpha)t^N V(ty) + \nabla V(y) \cdot y - \alpha V(y)}{t^{N+\alpha}} \\ &\geq \frac{(N+\alpha)[t^N V(ty) - V(y)]}{t^{N+\alpha}} \\ &\geq \frac{(N+\alpha)[t^N V_0 - V_{\max}]}{t^{N+\alpha}}, \quad \forall y \in \mathbb{R}^N, t > 0. \end{aligned} \tag{2.18}$$

Choose $t = s_1 = (2V_{\max}/V_0)^{1/N}$, so that (2.18) gives

$$\begin{aligned} NV(y) + \nabla V(y) \cdot y &\geq \frac{(N+\alpha)[s_1^N V_0 - V_{\max}]}{s_1^{N+\alpha}} \\ &\geq (N+\alpha) V_{\max} \left(\frac{2V_{\max}}{V_0}\right)^{(N+\alpha)/N} := \gamma_1, \quad \forall y \in \mathbb{R}^N. \end{aligned}$$

This completes the proof of (2.15). We next prove that (2.16) holds also. Note that (2.8) yields that

$$\begin{aligned} \alpha V(y) - \nabla V(y) \cdot y &\geq \frac{(N+\alpha)t^N}{1-t^{N+\alpha}} \left[V(ty) - t^N V(y)\right] \\ &\geq \frac{(N+\alpha)t^N}{1-t^{N+\alpha}} \left[V_0 - t^N V_{\max}\right], \quad \forall y \in \mathbb{R}^N, 0 < t < 1. \end{aligned} \tag{2.19}$$

Choose $t = s_2 = (V_0/2V_{\max})^{1/\alpha} \in (0,1)$, then (2.19) implies that

$$\begin{aligned}\alpha V(y) - \nabla V(y)\cdot y &\geq \frac{(N+\alpha)s_2^N}{1-s_2^{N+\alpha}}\big[V_0 - s_2^N V_{\max}\big] \\ &= \frac{N+\alpha}{(\frac{2V_{\max}}{V_0})^{N/\alpha} - \frac{V_0}{2V_{\max}}}\frac{V_0}{2} := \gamma_2, \quad \forall y \in \mathbb{R}^N.\end{aligned}$$

This completes the proof of (2.16), and also of the lemma. □

To show $\mathcal{M}_\varepsilon \neq \emptyset$, we define a set Λ as follows:

$$\Lambda := \left\{v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \big(I_\alpha * G(f(v))\big)G(f(v))\,\mathrm{d}x > 0\right\}. \tag{2.20}$$

Lemma 2.6 *Assume that* (V1), (V2) *and* (G1)–(G3) *hold. Then* $\Lambda \neq \emptyset$ *and* $\mathcal{M}_\varepsilon \subset \Lambda$.

$$\big\{v \in H^1(\mathbb{R}^N) \setminus \{0\} : \mathcal{P}_\varepsilon(v) \leq 0\big\} \subset \Lambda. \tag{2.21}$$

Proof In view of the proof of [19, The proof of Claim 1 in Proposition 2.1], (G3) implies $\Lambda \neq \emptyset$. If $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $\mathcal{P}_\varepsilon(v) \leq 0$, then it follows from (2.9) and (2.15) that

$$\begin{aligned}&-\frac{N+\alpha}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G(f(v))\big)G(f(v))\,\mathrm{d}x \\ &= \mathcal{P}_\varepsilon(v) - \frac{N-2}{2}\|\nabla v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N}\big[NV(\varepsilon x) + \nabla V(\varepsilon x)\cdot \varepsilon x\big]f^2(v)\,\mathrm{d}x \\ &\leq -\frac{N-2}{2}\|\nabla v\|_2^2 < 0,\end{aligned}$$

which implies $v \in \Lambda$. □

Arguing as in the proof of [25, Lemma 2.8], we get the following result.

Lemma 2.7 *Assume that* (V1), (V2), *and* (G1)–(G3) *hold. Then, for any* $v \in \Lambda$, *there exists unique* $t_v > 0$ *such that* $v_{t_v} \in \mathcal{M}_\varepsilon$.

From Corollary 2.4, Lemmas 2.6 and 2.7, we have $\mathcal{M}_\varepsilon \neq \emptyset$ and the following lemma.

Lemma 2.8 *Assume that* (V1), (V2), *and* (G1)–(G3) *hold. Then*

$$\inf_{v\in\mathcal{M}_\varepsilon} \Phi_\varepsilon(v) := m_\varepsilon = \inf_{v\in\Lambda}\max_{t>0}\Phi_\varepsilon(v_t).$$

Lemma 2.9 *Assume that* (V1), (V2), *and* (G1)–(G3) *hold. Then there exists* $\varrho_0 > 0$ *independent of* ε *such that* $m_\varepsilon = \inf_{v\in\mathcal{M}_\varepsilon}\Phi_\varepsilon(v) \geq \varrho_0$.

Proof Let $A(v) := \int_{\mathbb{R}^N}[|\nabla v|^2 + f^2(v)]\,\mathrm{d}x$ for any $v \in H^1(\mathbb{R}^N)$. Since $\mathcal{P}_\varepsilon(v) = 0$ for all $v \in \mathcal{M}_\varepsilon$, by (2.7), (2.9), (2.15), and Sobolev embedding theorem, one has, for any $v \in \mathcal{M}_\varepsilon$,

$$\begin{aligned}\frac{\min\{N-2,\gamma_1\}}{2}A(v) &\leq \frac{1}{2}\int_{\mathbb{R}^N}\big\{(N-2)|\nabla v|^2 + \big[NV(\varepsilon x) + \nabla V(\varepsilon x)\cdot\varepsilon x\big]f^2(v)\big\}\,\mathrm{d}x \\ &= \frac{N+\alpha}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G(f(v))\big)G(f(v))\,\mathrm{d}x\end{aligned}$$

$$\leq \frac{\min\{N-2,\gamma_1\}}{4}\|f(v)\|_2^{2(N+\alpha)/N} + C_1\|v\|_{2^*}^{2(N+\alpha)/(N-2)}$$
$$\leq \frac{\min\{N-2,\gamma_1\}}{4}\big[A(v)\big]^{(N+\alpha)/N} + C_2\big[A(v)\big]^{(N+\alpha)/(N-2)}, \tag{2.22}$$

which implies that there exists $\rho_0 > 0$ independent of ε such that

$$\int_{\mathbb{R}^N}\big[|\nabla v|^2 + f^2(v)\big]\,\mathrm{d}x \geq \rho_0, \quad \forall v \in \mathcal{M}_\varepsilon. \tag{2.23}$$

Let $\{v_n\} \subset \mathcal{M}_\varepsilon$ be such that $\Phi_\varepsilon(v_n) \to m_\varepsilon$. From (2.23) and (2.12) with $t \to 0$, we have

$$\begin{aligned} m_\varepsilon + o(1) &= \Phi_\varepsilon(v_n) - \frac{1}{N+\alpha}\mathcal{P}_\varepsilon(v_n) \\ &= \frac{(2+\alpha)}{2(N+\alpha)}\|\nabla v_n\|_2^2 + \frac{1}{2(N+\alpha)}\int_{\mathbb{R}^N}\big[\alpha V(\varepsilon x) - \nabla V(\varepsilon x)\cdot \varepsilon x\big]f^2(v_n)\,\mathrm{d}x \\ &\geq \frac{\rho_0}{2(N+\alpha)}\min\{2+\alpha,\gamma_2\} := \varrho_0. \end{aligned} \tag{2.24}$$

This shows that $m_\varepsilon = \inf_{v\in\mathcal{M}_\varepsilon}\Phi_\varepsilon(v) \geq \varrho_0$. The proof is complete. □

Following the idea of [8, Lemma 2.14], we can prove the following lemma by using Lemma 2.3, the deformation lemma, and intermediary theorem for continuous functions.

Lemma 2.10 *Assume that* (V1), (V2), *and* (G1)–(G3) *hold. If* $\bar{v} \in \mathcal{M}_\varepsilon$ *and* $\Phi_\varepsilon(\bar{v}) = m_\varepsilon$, *then* $\bar{v}$ *is a critical point of* Φ_ε.

3 Existence of ground state solutions for (2.5)

From now on in the paper, we always assume that (V1), (V2), and (G1)–(G3) hold without further mentioning. In view of Lemma 2.10, to obtain the existence of a critical point of Φ_ε, it suffices to prove that m_ε can be attained. To this end, we have to overcome the difficulty caused by the lack of the compactness of Sobolev embeddings. For this, we shall compare m_ε with the minimax level of the following autonomous problems:

$$-\Delta v = \frac{1}{\sqrt{1+2|f(v)|^2}}\big[\big(I_\alpha * G(f(v))\big)g\big(f(v)\big) - bf(v)\big], \quad x \in \mathbb{R}^N, \tag{3.1}$$

where b is a positive constant. To do that, we first seek for a ground state solution of (3.1) which minimizes the value of the functional $\hat{\Phi}_b(v)$ on the Pohožaev manifold $\hat{\mathcal{M}}_b$, where

$$\hat{\Phi}_b(v) = \frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla v|^2 + bf^2(v)\big)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v)\big)\big)G\big(f(v)\big)\,\mathrm{d}x \tag{3.2}$$

and

$$\hat{\mathcal{M}}_b := \big\{u \in H^1\big(\mathbb{R}^N\big)\setminus\{0\} : \hat{\mathcal{P}}_b(v) = 0\big\} \tag{3.3}$$

with

$$\hat{\mathcal{P}}_b(v) = \frac{N-2}{2}\|\nabla v\|_2^2 + \frac{Nb}{2}\big\|f(v)\big\|_2^2 - \frac{N+\alpha}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v)\big)\big)G\big(f(v)\big)\,\mathrm{d}x. \tag{3.4}$$

Lemma 3.1 $\hat{m}_b := \inf_{v\in\hat{\mathcal{M}}_b} \hat{\Phi}_b(v)$ *is attained for any* $b > 0$.

Proof In view of Lemma 2.9, we have $\hat{m}_b > 0$. Let $\{v_n\} \subset \hat{\mathcal{M}}_b$ be such that $\hat{\Phi}_b(v_n) \to \hat{m}_b$. Note that

$$\begin{aligned}\hat{\Phi}_b(v) &= \hat{\Phi}_b(v_t) + \frac{1-t^{N+\alpha}}{N+\alpha}\hat{\mathcal{P}}_b(v) + \frac{2+\alpha-(N+\alpha)t^{N-2}+(N-2)t^{N+\alpha}}{2(N+\alpha)}\|\nabla v\|_2^2 \\ &\quad + \frac{\alpha-(N+\alpha)t^N+Nt^{N+\alpha}}{2(N+\alpha)}b\big\|f(v)\big\|_2^2, \quad \forall v\in H^1(\mathbb{R}^N), t>0. \end{aligned} \tag{3.5}$$

By elemental calculations, we can get the following inequality:

$$h_2(t) := \alpha-(N+\alpha)t^N+Nt^{N+\alpha} > h_2(1) = 0, \quad \forall t\in[0,1)\cup(1,+\infty). \tag{3.6}$$

Then (2.11), (3.5), and (3.6) yield

$$\hat{\Phi}_b(v) \ge \hat{\Phi}_b(v_t) + \frac{1-t^{N+\alpha}}{N+\alpha}\hat{\mathcal{P}}_b(v), \quad \forall v\in H^1(\mathbb{R}^N), t>0. \tag{3.7}$$

Since $\hat{\mathcal{P}}_b(v_n) = 0$, then (3.5) with $t \to 0$ gives

$$\hat{m}_b + o(1) = \hat{\Phi}_b(v_n) = \frac{(2+\alpha)}{2(N+\alpha)}\|\nabla v_n\|_2^2 + \frac{\alpha}{2(N+\alpha)}b\big\|f(v_n)\big\|_2^2, \tag{3.8}$$

which, together with (f2) of Lemma 2.1, implies that both $\{\|\nabla v_n\|_2\}$ and $\{\|f(v_n)\|\}$ are bounded. Moreover, using (f5) of Lemma 2.1 and Sobolev embedding inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} v_n^2\,\mathrm{d}x &= \int_{|v_n|\le 1} v_n^2\,\mathrm{d}x + \int_{|v_n|>1} v_n^2\,\mathrm{d}x \le \frac{1}{\theta_0^2}\int_{|v_n|\le 1}\big|f(v_n)\big|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}|v_n|^{2^*}\,\mathrm{d}x \\ &\le \frac{1}{\theta_0^2}\big\|f(v_n)\big\|_2^2 + S^{-2^*/2}\|\nabla v_n\|_2^{2^*}. \end{aligned} \tag{3.9}$$

Hence, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Since $\hat{\mathcal{P}}_b(v_n) = 0$, using (2.7), (2.23), and Lions' concentration compactness principle, one can easily prove that there exist $\delta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |v_n|^2\,\mathrm{d}x > \delta/2$. Let $\hat{v}_n(x) = v_n(x+y_n)$. Then

$$\hat{\Phi}_b(\hat{v}_n) \to \hat{m}_b, \qquad \hat{\mathcal{P}}_b(\hat{v}_n) = 0, \tag{3.10}$$

and there exists $\hat{v} \in H^1(\mathbb{R}^N)\setminus\{0\}$ such that $\hat{v}_n \rightharpoonup \hat{v}$ in $H^1(\mathbb{R}^N)$, $\hat{v}_n \to \hat{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [1, 2^*)$, $\hat{v}_n \to \hat{v}$ a.e. on $\mathbb{R}^N$. Let $w_n = \hat{v}_n - \hat{v}$. By a standard argument, we have

$$\hat{\Phi}_b(\hat{v}_n) = \hat{\Phi}_b(\hat{v}) + \hat{\Phi}_b(w_n) + o(1) \quad \text{and} \quad \hat{\mathcal{P}}_b(\hat{v}_n) = \hat{\mathcal{P}}_b(\hat{v}) + \hat{\mathcal{P}}_b(w_n) + o(1). \tag{3.11}$$

Set

$$\Psi_b(v) := \hat{\Phi}_b(v) - \frac{1}{N+\alpha}\hat{\mathcal{P}}_b(v) = \frac{(2+\alpha)}{2(N+\alpha)}\|\nabla v\|_2^2 + \frac{\alpha}{2(N+\alpha)}b\big\|f(v)\big\|_2^2. \tag{3.12}$$

Then (2.6), (3.4), (3.10), (3.11), and (3.12) give

$$\hat{\mathcal{P}}_b(w_n) = -\hat{\mathcal{P}}_b(\hat{v}) + o(1), \qquad \Psi_b(w_n) = \hat{m}_b - \Psi_b(\hat{v}) + o(1). \tag{3.13}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then we have

$$\hat{\Phi}_b(\hat{v}) = \hat{m}_b, \qquad \hat{\mathcal{P}}_b(\hat{v}) = 0. \tag{3.14}$$

Thus, we assume that $w_n \neq 0$ for all $n \in \mathbb{N}$.

We claim that $\hat{\mathcal{P}}_b(\hat{v}) \leq 0$. Otherwise, if $\hat{\mathcal{P}}_b(\hat{v}) > 0$, then (3.13) implies $\hat{\mathcal{P}}_b(w_n) < 0$ for large n. Arguing as in Lemmas 2.6 and 2.7, there exists $t_n > 0$ such that $(w_n)_{t_n} \in \hat{\mathcal{M}}_b$ for large n. From (3.2), (3.4), (3.7), (3.12), and (3.13), we obtain

$$\begin{aligned} \hat{m}_b - \Psi_b(\hat{v}) + o(1) = \Psi_b(w_n) &= \hat{\Phi}_b(w_n) - \frac{1}{N+\alpha}\hat{\mathcal{P}}_b(w_n) \\ &\geq \hat{\mathcal{P}}_b\big((w_n)_{t_n}\big) - \frac{t_n^N}{N+\alpha}\hat{\mathcal{P}}_b(w_n) \geq \hat{m}_b - \frac{t_n^N}{N+\alpha}\hat{\mathcal{P}}_b(w_n) \geq \hat{m}_b, \end{aligned}$$

which is absurd because of $\Psi_b(\hat{v}) > 0$. Hence, $\hat{\mathcal{P}}_b(\hat{v}) \leq 0$ and the claim holds. Since $\hat{v} \neq 0$ and $\hat{\mathcal{P}}_b(\hat{v}) \leq 0$, arguing as in Lemmas 2.6 and 2.7, there exists $\hat{t} > 0$ such that $\hat{v}_{\hat{t}} \in \hat{\mathcal{M}}_b$. From (3.2), (3.4), (3.7), (3.10), (3.12), and the weak semicontinuity of norm, we derive that

$$\begin{aligned} \hat{m}_b &= \lim_{n\to\infty}\left[\hat{\Phi}_b(\hat{v}_n) - \frac{1}{N+\alpha}\hat{\mathcal{P}}_b(\hat{v}_n)\right] \\ &= \lim_{n\to\infty}\left[\frac{(2+\alpha)}{2(N+\alpha)}\|\nabla\hat{v}_n\|_2^2 + \frac{\alpha}{2(N+\alpha)}b\big\|f(\hat{v}_n)\big\|_2^2\right] \\ &\geq \frac{(2+\alpha)}{2(N+\alpha)}\|\nabla\hat{v}\|_2^2 + \frac{\alpha}{2(N+\alpha)}b\big\|f(\hat{v})\big\|_2^2 \\ &= \hat{\Phi}_b(\hat{v}) - \frac{1}{N+\alpha}\hat{\mathcal{P}}_b(\hat{v}) \geq \hat{\Phi}_b(\hat{v}_{\hat{t}}) - \frac{\hat{t}^N}{N+\alpha}\hat{\mathcal{P}}_b(\hat{v}) \geq \hat{m}_b, \end{aligned}$$

which implies again the validity of (3.14) in this case. Clearly, (3.14) proves the lemma. □

In view of Lemmas 2.10 and 3.1, we have the following theorem.

Lemma 3.2 *For all $b > 0$, (3.1) has a ground state solution $\hat{v} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that*

$$\hat{\Phi}_b(\hat{v}) = \hat{m}_b = \inf_{v\in\hat{\mathcal{M}}_b} \hat{\Phi}_b(v) = \inf_{v\in\Lambda}\max_{t>0} \hat{\Phi}_b(v_t).$$

Using (1.2), (2.6), (2.9), (2.10), (3.2), (3.3), and (3.4), we know that $\Phi_0 = \hat{\Phi}_{V_0}$, $\mathcal{P}_0 = \hat{\mathcal{P}}_{V_0}$ and $\mathcal{M}_0 = \hat{\mathcal{M}}_{V_0}$. Let

$$\hat{V} := \frac{1}{2}(V_\infty + V_0) = \frac{1}{2}\big(V_\infty + V(0)\big). \tag{3.15}$$

Applying Lemma 3.2, there exist $\hat{v}_0 \in \mathcal{M}_0$ and $\hat{v} \in \hat{\mathcal{M}}_{\hat{V}}$ such that

$$\Phi_0'(\hat{v}_0) = 0, \qquad \Phi_0(\hat{v}_0) = m_0 = \inf_{v\in\mathcal{M}_0} \Phi_0(v) = \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} \Phi_0(v_t) > 0 \tag{3.16}$$

and

$$\hat{\Phi}'_{\hat{V}}(\hat{v}) = 0, \qquad \hat{\Phi}_{\hat{V}}(\hat{v}) = \hat{m}_{\hat{V}} = \inf_{v\in\hat{\mathcal{M}}_{\hat{V}}} \hat{\Phi}_{\hat{V}}(v) = \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} \hat{\Phi}_{\hat{V}}(v_t) > 0. \tag{3.17}$$

In view of Lemma 2.7, there exists $t_0 > 0$ such that

$$\hat{v}_{t_0} \in \mathcal{M}_0, \qquad \Phi_0(\hat{v}_{t_0}) \geq m_0. \tag{3.18}$$

For $v \in H^1(\mathbb{R}^N)$, we define the following functional:

$$\Phi_*(v) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla v|^2 + V_{\max} f^2(v)\big)\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \big(I_\alpha * G\big(f(v)\big)\big)G\big(f(v)\big)\,\mathrm{d}x. \tag{3.19}$$

Using (G1) and (G2), it is easy to check that there exists $T_0 > 1$ such that

$$\Phi_*\big((\hat{v}_0)_t\big) < 0, \quad \forall t \geq T_0. \tag{3.20}$$

In view of Lemma 2.7 and (3.20), for any $\varepsilon > 0$, there exists $t_\varepsilon \in (0, T_0)$ such that

$$(\hat{v}_0)_{t_\varepsilon} \in \mathcal{M}_\varepsilon, \qquad \Phi_\varepsilon\big((\hat{v}_0)_{t_\varepsilon}\big) \geq m_\varepsilon. \tag{3.21}$$

Lemma 3.3 *$\hat{m}_{\hat{V}} \geq m_0 + \delta_0$, where $\delta_0 = (V_\infty - V_0)t_0^N \|f(\hat{v})\|_2^2/4 > 0$ is independent of $\varepsilon > 0$.*

Proof By (3.17) and (3.18), one has

$$\begin{aligned} \hat{m}_{\hat{V}} = \hat{\Phi}_{\hat{V}}(\hat{v}) \geq \hat{\Phi}_{\hat{V}}(\hat{v}_{t_0}) &= \Phi_0(\hat{v}_{t_0}) + \frac{\hat{V} - V_0}{2} t_0^N \int_{\mathbb{R}^N} f^2(\hat{v})\,\mathrm{d}x \\ &\geq m_0 + \frac{V_\infty - V_0}{4} t_0^N \big\|f(\hat{v})\big\|_2^2 = m_0 + \delta_0, \end{aligned}$$

as desired. □

Now, we choose $R_0 > 0$ sufficiently large such that

$$V(x) \geq \hat{V}, \qquad \big[1 + T_0^{N+\alpha}\big]V_{\max} \int_{|x|>R_0} f^2(\hat{v}_0)\,\mathrm{d}x \leq \delta_0, \quad \forall |x| > R_0, \tag{3.22}$$

where $T_0 > 0$ is given by (3.20), and $\delta_0 > 0$ in Lemma 3.3. By (V1) and (V3), there exists $\varepsilon_0 > 0$ small enough such that, for all $\varepsilon \in [0, \varepsilon_0]$,

$$\big\|f(\hat{v}_0)\big\|_2^2 \sup_{|x|\leq R_0} \big[\big(NT_0^{N+\alpha} + \alpha\big)\big|V(\varepsilon x) - V(0)\big| + \big(1 + T_0^{N+\alpha}\big)\big|\nabla V(\varepsilon x)\cdot(\varepsilon x)\big|\big] \leq \delta_0. \tag{3.23}$$

Lemma 3.4 *$m_0 \geq m_\varepsilon - 3\delta_0/4$ for all $\varepsilon \in [0, \varepsilon_0]$.*

Proof Note that (2.15) and (2.16) lead to

$$-NV(y) < -NV(y) + \gamma_1 \leq \nabla V(y)\cdot y \leq \alpha V(y) - \gamma_2 < \alpha V(y), \quad \forall y \in \mathbb{R}^N. \tag{3.24}$$

Since $\mathcal{P}_0(\hat{v}_0) = 0$ by (3.16), then (2.1), (2.9), (2.12), and (3.21)–(3.24) yield that

$$\begin{aligned}
m_0 &= \Phi_0(\hat{v}_0) = \Phi_\varepsilon(\hat{v}_0) + \frac{1}{2}\int_{\mathbb{R}^N}\big[V(0) - V(\varepsilon x)\big]f^2(\hat{v}_0)\,\mathrm{d}x \\
&\geq \Phi_\varepsilon\big((\hat{v}_0)_{t_\varepsilon}\big) + \frac{1 - t_\varepsilon^{N+\alpha}}{N+\alpha}\mathcal{P}_\varepsilon(\hat{v}_0) + \frac{1}{2}\int_{\mathbb{R}^N}\big[V(0) - V(\varepsilon x)\big]f^2(\hat{v}_0)\,\mathrm{d}x \\
&= \Phi_\varepsilon\big((\hat{v}_0)_{t_\varepsilon}\big) + \frac{1 - t_\varepsilon^{N+\alpha}}{N+\alpha}\mathcal{P}_0(\hat{v}_0) + \frac{1}{2}\int_{\mathbb{R}^N}\big[V(0) - V(\varepsilon x)\big]f^2(\hat{v}_0)\,\mathrm{d}x \\
&\quad + \frac{1 - t_\varepsilon^{N+\alpha}}{2(N+\alpha)}\int_{\mathbb{R}^N}\big\{N\big[V(\varepsilon x) - V(0)\big] + \nabla V(\varepsilon x)\cdot(\varepsilon x)\big\}f^2(\hat{v}_0)\,\mathrm{d}x \\
&= \Phi_\varepsilon\big((\hat{v}_0)_{t_\varepsilon}\big) + \frac{1}{2(N+\alpha)}\int_{\mathbb{R}^N}\big\{\alpha\big[V(0) - V(\varepsilon x)\big] + \nabla V(\varepsilon x)\cdot(\varepsilon x)\big\}f^2(\hat{v}_0)\,\mathrm{d}x \\
&\quad - \frac{t_\varepsilon^{N+\alpha}}{2(N+\alpha)}\int_{\mathbb{R}^N}\big\{N\big[V(\varepsilon x) - V(0)\big] + \nabla V(\varepsilon x)\cdot(\varepsilon x)\big\}f^2(\hat{v}_0)\,\mathrm{d}x \\
&\geq m_\varepsilon - \frac{1 + T_0^{N+\alpha}}{2}V_{\max}\int_{|x|>R_0} f^2(\hat{v}_0)\,\mathrm{d}x \\
&\quad - \frac{\|f(\hat{v}_0)\|_2^2}{2(N+\alpha)}\sup_{|x|\leq R_0}\big[\big(NT_0^{N+\alpha} + \alpha\big)\big|V(\varepsilon x) - V(0)\big| + \big(1 + T_0^{N+\alpha}\big)\big|\nabla V(\varepsilon x)\cdot(\varepsilon x)\big|\big] \\
&\geq m_\varepsilon - \frac{3\delta_0}{4}, \quad \forall \varepsilon \in [0, \varepsilon_0],
\end{aligned}$$

as desired. □

Next, we extend Theorem 3.2 to the case $\varepsilon > 0$.

Lemma 3.5 *m_ε is achieved for all $\varepsilon \in (0, \varepsilon_0]$.*

Proof In view of Lemmas 2.7 and 2.9, we have $\mathcal{M}_\varepsilon \neq \emptyset$ and $m_\varepsilon > 0$ for all $\varepsilon \in (0, \varepsilon_0]$. For any fixed $\varepsilon \in (0, \varepsilon_0]$, let $\{v_n\} \subset \mathcal{M}_\varepsilon$ be such that $\Phi_\varepsilon(v_n) \to m_\varepsilon$. Since $\mathcal{P}_\varepsilon(v_n) = 0$, then it follows from (2.23) and (2.24) that

$$m_\varepsilon + o(1) = \Phi_\varepsilon(v_n) \geq \frac{(2+\alpha)}{2(N+\alpha)}\|\nabla v_n\|_2^2 + \frac{\gamma_2}{2(N+\alpha)}\big\|f(v_n)\big\|_2^2,$$

which, together with (3.9), implies that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, we have $v_n \rightharpoonup \hat{v}$ in $H^1(\mathbb{R}^N)$. Then $v_n \to \hat{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$ and $v_n \to \hat{v}$ a.e. in $\mathbb{R}^N$. Now, we prove that $\hat{v} \neq 0$.

Arguing by contradiction, suppose that $\hat{v} = 0$, then $v_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$ and $v_n \to 0$ a.e. in $\mathbb{R}^N$. Using Lemma 2.7, we know that there exists $t_n > 0$ such that $(v_n)_{t_n} \in \hat{\mathcal{M}}_{\hat{V}}$ for $n \in \mathbb{N}$. We claim that there exist two constants $0 < T_1 < T_2$ such that

$$T_1 \leq t_n \leq T_2, \quad \forall n \in \mathbb{N}. \tag{3.25}$$

Indeed, if $t_n \to 0$, from Lemma 3.2 and the boundedness of $\{\|v_n\|\}$ we then deduce that

$$
\begin{aligned}
0 < \hat{m}_{\hat{V}} &\le \hat{\Phi}_{\hat{V}}\big((v_n)_{t_n}\big) \\
&= \frac{t_n^{N-2}}{2}\|\nabla v_n\|_2^2 + \frac{\hat{V}t_n^N}{2}\big\|f(v_n)\big\|_2^2 - \frac{t_n^{N+\alpha}}{2}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v_n)\big)\big)G\big(f(v_n)\big)\,\mathrm{d}x = o(1),
\end{aligned}
$$

which is absurd. Hence, the first inequality holds in (3.25). Moreover, we can verify that

$$
\beta_0 := \liminf_{n\to\infty}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v_n)\big)\big)G\big(f(v_n)\big)\,\mathrm{d}x > 0. \tag{3.26}
$$

Otherwise, if (3.26) does not hold, then there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\big(I_\alpha * G\big(f(v_{n_k})\big)\big)G\big(f(v_{n_k})\big)\,\mathrm{d}x = 0. \tag{3.27}
$$

Then (2.9), (2.15), (2.23), and (3.27) yield

$$
0 = \mathcal{P}_\varepsilon(v_{n_k}) \ge \frac{N-2}{2}\|\nabla v_{n_k}\|_2^2 + \frac{\gamma_1}{2}\big\|f(v_{n_k})\big\|_2^2 + o(1) \ge \frac{1}{2}\min\{N-2,\gamma_1\}\rho_0 + o(1).
$$

This contradiction shows that (3.26) holds. By (3.2), (3.26), the boundedness of $\{v_n\}$, and Sobolev embedding theorem, we have

$$
\begin{aligned}
\hat{\Phi}_{\hat{V}}\big((v_n)_t\big) &\le \frac{t^{N-2}}{2}\|\nabla v_n\|_2^2 + \frac{\hat{V}t^N}{2}\big\|f(v_n)\big\|_2^2 - \frac{t^{N+\alpha}}{2}\beta_0 \\
&\le C_1\big(t^{N-2} + t^N\big) - \frac{t^{N+\alpha}}{2}\beta_0, \quad \forall t > 0, n \in \mathbb{N},
\end{aligned}
$$

which implies that there exists $T_2 > 0$ such that

$$
\hat{\Phi}_{\hat{V}}\big((v_n)_t\big) < 0, \quad \forall t > T_2, n \in \mathbb{N}. \tag{3.28}
$$

Since $\hat{\Phi}_{\hat{V}}((v_n)_{t_n}) \ge \hat{m}_{\hat{V}} > 0$ due to (3.17), then (3.28) yields that $t_n \le T_2$ for all $n \in \mathbb{N}$. This shows that (3.25) holds. Thus it follows from (2.1), (2.14), (3.2), (3.22), and (3.25) that

$$
\begin{aligned}
m_\varepsilon + o(1) &= \Phi_\varepsilon(v_n) \ge \Phi_\varepsilon\big((v_n)_{t_n}\big) \\
&\ge \hat{\Phi}_{\hat{V}}\big((v_n)_{t_n}\big) + \frac{t_n^N}{2}\int_{\mathbb{R}^N}\big[V(\varepsilon t_n x) - \hat{V}\big]f^2(v_n)\,\mathrm{d}x \\
&\ge \hat{m}_{\hat{V}} - \frac{\hat{V}T_2^N}{2}\int_{|x|\le R_0/(\varepsilon T_1)} f^2(v_n)\,\mathrm{d}x = \hat{m}_{\hat{V}} + o(1),
\end{aligned}
$$

which, together with Lemmas 3.3 and 3.4, implies

$$
m_\varepsilon \ge \hat{m}_{\hat{V}} \ge m_0 + \delta_0 \ge m_\varepsilon + \frac{\delta_0}{4}.
$$

This contradiction shows that the claim is true, that is, $\hat{v} \ne 0$. Let $w_n = v_n - \hat{v}$. As in the proof of (3.14), we can deduce that $\Phi_\varepsilon(\hat{v}) = m_\varepsilon$ and $\mathcal{P}_\varepsilon(\hat{v}) = 0$. This completes the proof. □

In view of Lemmas 2.8, 2.10, and 3.5, we easily obtain the following result.

Proposition 3.6 *For every $\varepsilon \in (0, \varepsilon_0]$, (2.5) has a ground state solution $\hat{v}_\varepsilon$ such that*

$$\Phi_\varepsilon(\hat{v}_\varepsilon) = m_\varepsilon = \inf_{v\in\Lambda} \max_{t>0} \Phi_\varepsilon\big(v(\cdot/t)\big) > 0. \tag{3.29}$$

4 Concentration of ground state solutions

In this section, we always assume that (V1), (V2), and (G1)–(G4) hold, and consider the concentration of ground state solutions for (2.5) and give the proof of Theorem 1.1. For this purpose, for every $\varepsilon \in (0, \varepsilon_0]$, let $\hat{v}_\varepsilon$ be a ground state solution of (2.5) obtained in Proposition 3.6, which satisfies (3.29). Moreover, when $\varepsilon = 0$, we denote by $\hat{v}_0$ the ground state solution of (1.3), that is, the ground state solution constructed in Theorem 3.2 when $b = V_0$. Therefore, as in Sect. 1, we put for all $\varepsilon \in [0, \varepsilon_0]$

$$\mathcal{L}_{m_\varepsilon} = \big\{v \in H^1\big(\mathbb{R}^N\big) \setminus \{0\} : \Phi_\varepsilon'(v) = 0, \Phi_\varepsilon(v) = m_\varepsilon\big\}$$

and set

$$\Upsilon = \big\{v \in \mathcal{L}_{m_\varepsilon} : \varepsilon \in [0, \varepsilon_0]\big\}.$$

Lemma 4.1 *There exists $K_0 > 0$ independent of ε such that $\varrho_0 \le m_\varepsilon \le K_0$ for all $\varepsilon \in [0, \varepsilon_0]$.*

Proof From (2.13), (3.20), Lemmas 2.7 and 2.9, we derive that

$$\begin{aligned}\varrho_0 \le m_\varepsilon &\le \max\big\{\Phi_\varepsilon\big((\hat{v}_0)_t\big) : t \in (0, T_0]\big\} \\ &\le \frac{T_0^{N-2}}{2}\|\nabla \hat{v}_0\|_2^2 + \frac{V_{\max} T_0^N}{2}\big\|f(\hat{v}_0)\big\|_2^2 + C_1 T_0^{N+\alpha}\big(\|\hat{v}_0\|_2^{2(N+\alpha)/N} + \|\hat{v}_0\|_{2^*}^{2(N+\alpha)/(N-2)}\big) \\ &:= K_0, \quad \forall \varepsilon \in [0, \varepsilon_0],\end{aligned}$$

where $\hat{v}_0 \in H^1(\mathbb{R}^N)$ and $T_0 > 0$ are given by (3.16) and (3.20), respectively. □

Lemma 4.2 *There exists $K_1 > 0$ independent of ε such that $\|v\| \le K_1$ for all $v \in \Upsilon$.*

Proof For any $v_\varepsilon \in \mathcal{L}_{m_\varepsilon}$ with $\varepsilon \in [0, \varepsilon_0]$, by (2.16) and Lemma 4.1, one has

$$K_0 \ge m_\varepsilon = \Phi_\varepsilon(v_\varepsilon) - \frac{1}{N+\alpha}\mathcal{P}_\varepsilon(v_\varepsilon) \ge \frac{(2+\alpha)}{2(N+\alpha)}\|\nabla v_\varepsilon\|_2^2 + \frac{\gamma_2}{2(N+\alpha)}\big\|f(v_\varepsilon)\big\|_2^2. \tag{4.1}$$

As in the proof of (3.9), we have

$$\int_{\mathbb{R}^N} v_\varepsilon^2 \,\mathrm{d}x \le \frac{1}{\theta_0^2}\big\|f(v_\varepsilon)\big\|_2^2 + S^{-2^*/2}\|\nabla v_\varepsilon\|_2^{2^*}. \tag{4.2}$$

The lemma thus follows from (4.1) and (4.2). □

Lemma 4.3 $\limsup_{\varepsilon\to\bar{\varepsilon}} m_\varepsilon \le m_{\bar{\varepsilon}}$ *for every $\bar{\varepsilon} \in [0, \varepsilon_0]$.*

Proof Fix $\bar{\varepsilon} \in [0, \varepsilon_0]$ and $\hat{v}_{\bar{\varepsilon}} \in \mathcal{L}_{m_{\bar{\varepsilon}}}$. Arguing by contradiction, suppose that $\limsup_{\varepsilon\to\bar{\varepsilon}} m_\varepsilon > m_{\bar{\varepsilon}}$. Let $\epsilon_0 = \limsup_{\varepsilon\to\bar{\varepsilon}} m_\varepsilon - m_{\bar{\varepsilon}}$. Clearly $\epsilon_0 > 0$. From Lemma 2.7, for any $\varepsilon > 0$, there exists

$\bar{t}_\varepsilon > 0$ such that $(\hat{v}_{\bar{\varepsilon}})_{\bar{t}_\varepsilon} \in \mathcal{M}_\varepsilon$, and so

$$\Phi_\varepsilon\big((\hat{v}_{\bar{\varepsilon}})_{\bar{t}_\varepsilon}\big) \geq m_\varepsilon, \qquad \Phi_\varepsilon\big((\hat{v}_{\bar{\varepsilon}})_{\bar{t}_\varepsilon}\big) \geq \Phi_\varepsilon\big((\hat{v}_{\bar{\varepsilon}})_t\big), \quad \forall t > 0. \tag{4.3}$$

It is easy to check that there exists a constant $\bar{T} > 0$ such that $0 < \bar{t}_\varepsilon \leq \bar{T}$ for some $\bar{T} = T_{\hat{u}_{\bar{\varepsilon}}} > 0$, moreover, for any bounded set $\Omega \subset \mathbb{R}^N$,

$$\lim_{\varepsilon \to \bar{\varepsilon}} \sup_{x \in \Omega} \big[\big|V(\varepsilon x) - V(\bar{\varepsilon} x)\big| + \big|\nabla V(\varepsilon x) \cdot (\varepsilon x) - \nabla V(\bar{\varepsilon} x) \cdot (\bar{\varepsilon} x)\big|\big] = 0. \tag{4.4}$$

Choose $R_1 > R_0$ such that

$$\big(1 + \bar{T}^{N+\alpha}\big) V_{\max} \int_{|x| \geq R_1} f^2(\hat{v}_{\bar{\varepsilon}})\,\mathrm{d}x \leq \frac{\epsilon_0}{2}. \tag{4.5}$$

Noting that $\mathcal{P}_{\bar{\varepsilon}}(\hat{v}_{\bar{\varepsilon}}) = 0$, then it follows from (2.12), (3.24), (4.3), (4.4), and (4.5) that

$$\begin{aligned}
m_{\bar{\varepsilon}} &= \Phi_{\bar{\varepsilon}}(\hat{v}_{\bar{\varepsilon}}) = \Phi_\varepsilon(\hat{v}_{\bar{\varepsilon}}) + \frac{1}{2} \int_{\mathbb{R}^N} \big[V(\bar{\varepsilon} x) - V(\varepsilon x)\big] f^2(\hat{v}_{\bar{\varepsilon}})\,\mathrm{d}x \\
&\geq \Phi_\varepsilon\big((\hat{v}_{\bar{\varepsilon}})_{\bar{t}_\varepsilon}\big) + \frac{1 - \bar{t}_\varepsilon^{N+\alpha}}{N+\alpha} \mathcal{P}_\varepsilon(\hat{v}_{\bar{\varepsilon}}) + \frac{1}{2} \int_{\mathbb{R}^N} \big[V(\bar{\varepsilon} x) - V(\varepsilon x)\big] f^2(\hat{v}_{\bar{\varepsilon}})\,\mathrm{d}x \\
&\geq m_\varepsilon + \frac{1 - \bar{t}_\varepsilon^{N+\alpha}}{N+\alpha} \mathcal{P}_{\bar{\varepsilon}}(\hat{v}_{\bar{\varepsilon}}) + \frac{1}{2} \int_{\mathbb{R}^N} \big[V(\bar{\varepsilon} x) - V(\varepsilon x)\big] f^2(\hat{v}_{\bar{\varepsilon}})\,\mathrm{d}x \\
&\quad + \frac{1 - \bar{t}_\varepsilon^{N+\alpha}}{N+\alpha} \int_{\mathbb{R}^N} \big[NV(\varepsilon x) + \nabla V(\varepsilon x) \cdot (\varepsilon x) - NV(\bar{\varepsilon} x) - \nabla V(\bar{\varepsilon} x) \cdot (\bar{\varepsilon} x)\big] f^2(\hat{v}_{\bar{\varepsilon}})\,\mathrm{d}x \\
&\geq m_\varepsilon - \frac{1 + \bar{T}^{N+\alpha}}{N+\alpha} \int_{|x| \leq R_1} \big[N\big|V(\varepsilon x) - V(\bar{\varepsilon} x)\big| \\
&\quad + \big|\nabla V(\varepsilon x) \cdot (\varepsilon x) - \nabla V(\bar{\varepsilon} x) \cdot (\bar{\varepsilon} x)\big|\big] f^2(\hat{v}_{\bar{\varepsilon}})\,\mathrm{d}x \\
&\quad - \frac{1}{2} \int_{|x| \leq R_1} \big|V(\bar{\varepsilon} x) - V(\varepsilon x)\big| f^2(\hat{v}_{\bar{\varepsilon}})\,\mathrm{d}x - \big(1 + \bar{T}^{N+\alpha}\big) V_{\max} \int_{|x| \geq R_1} f^2(\hat{v}_{\bar{\varepsilon}})\,\mathrm{d}x \\
&\geq m_\varepsilon - \frac{1 + \bar{T}^{N+\alpha}}{N+\alpha} \big\|f(\bar{v}_{\bar{\varepsilon}})\big\|_2^2 \sup_{|x| \leq R_1} \big[N\big|V(\varepsilon x) - V(\bar{\varepsilon} x)\big| \\
&\quad + \big|\nabla V(\varepsilon x) \cdot (\varepsilon x) - \nabla V(\bar{\varepsilon} x) \cdot (\bar{\varepsilon} x)\big|\big] \\
&\quad - \frac{1}{2} \big\|f(\bar{v}_{\bar{\varepsilon}})\big\|_2^2 \sup_{|x| \leq R_1} \big[\big|V(\varepsilon x) - V(\bar{\varepsilon} x)\big|\big] - \frac{\epsilon_0}{2},
\end{aligned}$$

and so $m_{\bar{\varepsilon}} + \epsilon_0 = \limsup_{\varepsilon \to \bar{\varepsilon}} m_\varepsilon \leq m_{\bar{\varepsilon}} + \frac{\epsilon_0}{2}$. This contradiction shows $\limsup_{\varepsilon \to \bar{\varepsilon}} m_\varepsilon \leq m_{\bar{\varepsilon}}$. □

Lemma 4.4 *If $v \in \Upsilon$, then $v \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $\lim_{|x| \to \infty} v(x) = 0$. Moreover, there is $\alpha_0 > 0$ independent of $x \in \mathbb{R}^N$ and $v \in \Upsilon$ such that*

$$\big|v(x)\big| \leq \alpha_0 \int_{B_1(x)} \big|v(y)\big|\,\mathrm{d}y, \quad \forall x \in \mathbb{R}^N, v \in \Upsilon. \tag{4.6}$$

Proof By a standard argument, we can prove that

$$I_\alpha * G\big(f(v)\big) \in L^\infty\big(\mathbb{R}^N\big), \quad \forall v \in H^1\big(\mathbb{R}^N\big). \tag{4.7}$$

Using (4.7), Lemma 4.2, and the standard bootstrap argument (see [21]), we can deduce that, for any $s \geq 2$, there exists $C_s > 0$ independent of $v \in \Upsilon$ such that

$$v \in W^{1,s}(\mathbb{R}^N), \qquad \|v\|_{W^{1,s}(\mathbb{R}^N)} \leq C_s, \quad \forall v \in \Upsilon,$$

which, together with Sobolev imbedding theorem, implies that there is $C_\infty > 0$ independent of $v \in \Upsilon$ such that

$$\|v\|_\infty \leq C_\infty, \quad \forall v \in \Upsilon. \tag{4.8}$$

By (4.7), Lemma 2.1, (G1), and (G4), there exists a constant $\Theta_1 > V_0$ such that

$$\left|\big(I_\alpha * G(f(v))\big)g(f(v))\right| \leq \Theta_1 |v|, \quad \forall |v| \leq C_\infty. \tag{4.9}$$

In view of (4.8), (4.9), Lemma 4.2, and [21, Lemma 1], we have $v \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $\lim_{|x|\to\infty} v(x) = 0$. Since $v \in \mathcal{L}_{m_\varepsilon}$ is a solution of (2.5) for some $\varepsilon > 0$, then (4.9) and Lemma 4.2 yield that

$$\Delta |v| = \frac{v \cdot \Delta v}{|v|} = \frac{V(\varepsilon x) f(v) v - (I_\alpha * G(f(v)))g(f(v))v}{|v|} \geq -\Theta_1 |v|, \quad \forall x \in \mathbb{R}^N, \tag{4.10}$$

which implies that $|v|$ is a sub-solution of the equation $(-\Delta - \Theta_1)w = 0$, and hence (4.6) follows from the sub-solution estimate (see [24, Theorem C.1.2]). □

Lemma 4.5 *For every $v_\varepsilon \in \mathcal{L}_{m_\varepsilon} \subset \Upsilon$, there exists $y_\varepsilon \in \mathbb{R}^N$ such that $|v_\varepsilon(y_\varepsilon)| = \max_{x\in\mathbb{R}^N} |v_\varepsilon(x)|$. Let $\tilde{v}_\varepsilon(x) := v_\varepsilon(x + y_\varepsilon)$, and let $\varepsilon_n \in (0, \varepsilon_0]$ such that $\limsup_{n\to\infty} \varepsilon_n = \bar{\varepsilon}$. Then we have*

(i) *If $\bar{\varepsilon} > 0$, then $\{v_{\varepsilon_n}\}$ has a convergence subsequence, whose limit belongs to Υ;*
(ii) *If $\bar{\varepsilon} = 0$, then $\{\tilde{v}_{\varepsilon_n}\}$ has a convergence subsequence, whose limit is not zero.*

Proof For $\{\varepsilon_n\} \subset [0, \varepsilon_0]$ and $v_{\varepsilon_n} \in \mathcal{L}_{m_{\varepsilon_n}}$, Lemma 4.2 implies that $\{v_{\varepsilon_n}\}$ is bounded in $H^1(\mathbb{R}^N)$. By a standard argument, we can get

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_{\varepsilon_n}|^2 \,\mathrm{d}x > 0. \tag{4.11}$$

By Lemma 4.4, there exists $y_\varepsilon \in \mathbb{R}^N$ such that $|v_\varepsilon(y_\varepsilon)| = \max_{x\in\mathbb{R}^N} |v_\varepsilon(x)|$. This, together with (4.11), gives

$$\limsup_{n\to\infty} \left|v_{\varepsilon_n}(y_{\varepsilon_n})\right|^2 \geq \frac{1}{\mathcal{C}_N} \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_{\varepsilon_n}|^2 \,\mathrm{d}x > 0, \tag{4.12}$$

where $\mathcal{C}_N$ is the volume of the unit N-ball.

(i) If $\bar{\varepsilon} \in (0, \varepsilon_0]$, then passing to a subsequence, we may assume that $\varepsilon_n \to \bar{\varepsilon} \in (0, \varepsilon_0]$ and $v_{\varepsilon_n} \rightharpoonup \hat{v}$ in $H^1(\mathbb{R}^N)$. Similar to the proof of Lemma 3.5, we can conclude that $v_{\varepsilon_n} \to \hat{v}$ in $H^1(\mathbb{R}^N)$, $\Phi'_{\bar{\varepsilon}}(\hat{v}) = 0$, and $\Phi_{\bar{\varepsilon}}(\hat{v}) = \lim_{n\to\infty} \Phi_{\varepsilon_n}(v_{\varepsilon_n}) = m_{\bar{\varepsilon}}$. This implies that $\hat{v} \in \mathcal{L}_{m_{\bar{\varepsilon}}} \subset \Upsilon$.

(ii) If $\bar{\varepsilon} = 0$, passing to a subsequence, we may assume that $\varepsilon_n \to 0$ and $\tilde{v}_{\varepsilon_n} \rightharpoonup \tilde{v}_0$ in $H^1(\mathbb{R}^N)$. Note that (4.12) implies that $\tilde{v}_0 \neq 0$. Since V is bounded, going to a subsequence if necessary, we may assume that $\lim_{n\to\infty} V(\varepsilon_n y_{\varepsilon_n}) = \beta$. Note that

$$\begin{aligned} m_{\varepsilon_n} &= \Phi_{\varepsilon_n}(v_{\varepsilon_n}) \\ &= \frac{1}{2}\|\nabla \tilde{n}_{\varepsilon_n}\|_2^2 + \int_{\mathbb{R}^N} V\big(\varepsilon_n(x + y_{\varepsilon_n})\big) f^2(\tilde{v}_{\varepsilon_n})\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \big(I_\alpha * G\big(f(\tilde{v}_{\varepsilon_n})\big)\big) G\big(f(\tilde{v}_{\varepsilon_n})\big)\,\mathrm{d}x. \end{aligned} \tag{4.13}$$

As in the proof of Lemma 3.5, and using (4.13) and Lemma 4.3, we obtain $\tilde{v}_{\varepsilon_n} \to \tilde{v}_0$ in $H^1(\mathbb{R}^N)$, $\hat{\Phi}'_\beta(\tilde{v}_0) = 0$, and $\hat{\Phi}_\beta(\tilde{v}_0) = \lim_{n\to\infty} \Phi_{\varepsilon_n}(v_{\varepsilon_n}) \le m_0$. This concludes the proof. □

As those of [5, Lemmas 6.5–6.7], we get the following three lemmas.

Lemma 4.6 $\inf\{\|u\|_\infty : u \in \Upsilon\} := \delta_0 > 0$.

Lemma 4.7 *There exist* $\Pi_1, \kappa_1 > 0$ *independent of* $x \in \mathbb{R}^N$ *and* $v \in \Upsilon$ *such that*

$$\big|v(x)\big| \le \Pi_1 \exp\big(-\kappa_1|x - y_v|\big), \quad \forall x \in \mathbb{R}^N, v \in \Upsilon, \tag{4.14}$$

where $|v(y_v)| = \max_{x\in\mathbb{R}^N} |v(x)|$.

Lemma 4.8 *Let* $v_\varepsilon \in \mathcal{L}_{m_\varepsilon}$ *for* $\varepsilon \in (0, \varepsilon_0]$ *and* $y_\varepsilon \in \mathbb{R}^N$ *be a global maximum point of* v_ε. *Then*

(i) $\sup_{\varepsilon\in[0,\varepsilon_0]}(\varepsilon|y_\varepsilon|) < \infty$;

(ii) *For* $\varepsilon_n \to 0^+$, *up to a subsequence*, $\tilde{v}_{\varepsilon_n} = v_{\varepsilon_n}(\cdot + y_{\varepsilon_n})$ *converges in* $H^1(\mathbb{R}^N)$ *to a ground state solution of* (1.3).

Proof of Theorem 1.1 Let $\hat{w}_\varepsilon(x) = \bar{v}_\varepsilon(x/\varepsilon)$ and $x_\varepsilon := \varepsilon y_\varepsilon$. In view of Proposition 3.6, for every $\varepsilon \in (0, \varepsilon_0]$, $\hat{w}_\varepsilon(x) = \bar{v}_\varepsilon(x/\varepsilon)$ is a ground state solution of (2.4). Hence, for all $\varepsilon \in (0, \varepsilon_0]$, (1.1) has a ground state solution $\hat{u}_\varepsilon(x) := f(\hat{w}_\varepsilon(x)) = f(\bar{v}_\varepsilon(x/\varepsilon))$. Letting $x_\varepsilon := x_0 + \varepsilon y_\varepsilon$, (i) follows from Lemma 4.8. Since f is strictly increasing and $|f(t)| \le |t|$ for all $t \in \mathbb{R}$, Lemmas 4.7 and 4.8 imply the validity of (ii) and (iii), respectively. □

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