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Wave propagation in a diffusive SEIR epidemic model with nonlocal transmission and a general nonlinear incidence rate

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Abstract

We introduce a diffusive SEIR model with nonlocal delayed transmission between the infected subpopulation and the susceptible subpopulation with a general nonlinear incidence. We show that our results on existence and nonexistence of traveling wave solutions are determined by the basic reproduction number $R_0 = \partial_t F(S_0, 0)/\gamma$ of the corresponding ordinary differential equations and the minimal wave speed c^* . The main difficulties lie in the fact that the semiflow generated here does not admit the order-preserving property. In the present paper, we overcome these difficulties to obtain the threshold dynamics. In view of the numerical simulations, we also obtain that the minimal wave speed is explicitly determined by the time delay and nonlocality in disease transmission and by the spatial movement pattern of the exposed and infected individuals.

MSC: 35K57; 92D30

Keywords: Traveling waves; SEIR model; Nonlinear incidence; Schauder fixed point theorem; Laplace transform

1 Introduction

In this paper, we consider the following diffusive SEIR model with nonlocal delayed transmission and a general nonlinear incidence rate:

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - F\left(S(x, t), \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x-y, t-s) I(y, s) dy ds\right), \quad (1.1)$$

$$\begin{aligned} \partial_t E(x, t) = d_2 \partial_{xx} E(x, t) + F\left(S(x, t), \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x-y, t-s) I(y, s) dy ds\right) \\ - \alpha E(x, t), \end{aligned} \quad (1.2)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + \alpha E(x, t) - \gamma I(x, t), \quad (1.3)$$

$$\partial_t R(x, t) = d_4 \partial_{xx} R(x, t) + \gamma I(x, t), \quad (1.4)$$

where $S(x, t)$, $E(x, t)$, $I(x, t)$, and $R(x, t)$ denote the densities of the susceptible, exposed, infected, and removed individuals at location x and time t , respectively, the parameters $d_i > 0$

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($i = 1, 2, 3, 4$) are diffusion rates for the susceptible, exposed, infected, and removed individuals, respectively, α is the rate of the exposed turning infected, and γ stands for the recovery rate of the infective individuals. Obviously, $1/\alpha$ is the average period for the exposed population to become infected. We use the assumption that the exposed individuals are of no infectiousness and their diffusive rate d_2 is not the same as d_3 . The reaction kernel $K(x - y, t - s) \geq 0$ describes the interaction between the infective and susceptible individuals at spatial location x and present time t , which occurred at location y and at earlier instance s , and is assumed to satisfy the following hypotheses:

(A1) $K(x, t)$ is a nonnegative integrable function such that

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} K(x, t) \, dx \, dt = 1, \quad K(-x, t) = K(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+;$$

(A2) For every $c \geq 0$, there exists $\lambda^c \in (0, +\infty]$ such that

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} K(x, t) e^{-\lambda(x+ct)} \, dx \, dt < +\infty$$

for $\lambda \in [0, \lambda^c)$, and

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} K(x, t) e^{-\lambda(x+ct)} \, dx \, dt \rightarrow +\infty$$

as $\lambda \rightarrow \lambda^c - 0$.

We also make the following assumptions for F :

(H1) $F(S, I)$ is well defined for all $S \geq 0$ and $I \geq 0$ and is continuous and nonnegative with respect to these variables. Furthermore,

$$F(S, 0) = 0, \quad F(0, I) = 0, \quad \partial_S F(S, 0) = 0, \quad \forall S \in [0, +\infty), I \in [0, +\infty);$$

(H2) All partial derivatives of F exist on $(0, +\infty) \times [0, +\infty)$ and are continuous;

(H3) For all $S > 0$ and $I > 0$, $F(S, I)$ is nondecreasing with respect to S and I . For all $S > 0$, $F(S, I)/I$ is nonincreasing with respect to I ;

(H4) For all $S \in (0, +\infty)$ and $I \in [0, +\infty)$, $F(S, I) \leq \partial_I F(S, 0)I$, and $\partial_I F(S, 0)$ is nondecreasing in $S > 0$.

Under hypotheses (H1)–(H4), the function $F(S, I)$ covers many cases that are frequently used in the literature. The case $F(S, I) = \beta S^p I$ with $p > 0$ is studied in [18]. If $F(S, I) = \beta SI/(1 + \zeta I^q)$ with $\zeta > 0$ and $q \geq 1$, then the incidence rate describes the saturated effects of the prevalence of infectious diseases; see [4, 19, 20]. For the case $F(S, I) = \beta SI/(S + I)$, the incidence rate describes the outbreak disease model [26]. For more detail about the nonlinear incidence rates, we refer to [1, 2, 7, 8, 10, 13, 14, 16, 22, 32].

In the study of population dynamics and the spread of infectious diseases, the reaction–diffusion equations with spatio-temporal delay are often used to describe biological and physical evolution processes. The spatial spread of infectious diseases is an important subject in mathematical epidemiology. Compartmental models describing the transmission of infectious diseases have been extensively studied in the literature. Usually, an infectious case is first found at one location, and then the disease spreads to other areas. Consequently, an important question for infectious diseases is what is the spreading speed?

Traveling wave solution is an important tool used in the study of the spreading speed of infectious diseases; see [3, 11, 15]. The existence and nonexistence of nontrivial traveling wave solutions indicate whether or not the disease can spread in the population and how fast a disease invades geographically. Also, theoretical results can help people make decisions on the disease control and prevention.

Some mathematical models may be described by (1.1)–(1.4) with appropriate choices of F and K . If we omit the exposed individuals $E(x, t)$, then (1.1)–(1.4) reduce to

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - F\left(S(x, t), \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x-y, t-s) I(y, s) dy ds\right), \quad (1.5)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + F\left(S(x, t), \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x-y, t-s) I(y, s) dy ds\right) - \gamma I(x, t), \quad (1.6)$$

$$\partial_t R(x, t) = d_4 \partial_{xx} R(x, t) + \gamma I(x, t). \quad (1.7)$$

Taking $F(S, I) = f(S)g(I)$ and $K(x, t) = \delta(x)\delta(t - \tau)$ with δ being the Dirac delta function, (1.5)–(1.7) become to the following model with delay:

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - f(S(x, t))g(I(x, t - \tau)), \quad (1.8)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + f(S(x, t))g(I(x, t - \tau)) - \gamma I(x, t), \quad (1.9)$$

$$\partial_t R(x, t) = d_4 \partial_{xx} R(x, t) + \gamma I(x, t), \quad (1.10)$$

which was studied by Bai and Wu [2]. Choosing $F(S, I) = \beta SI$, (1.5)–(1.7) can be reduced to the following diffusive SIR model with spatio-temporal delay derived by Wang and Wu [27]:

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - \beta S \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x-y, t-s) I(y, s) dy ds, \quad (1.11)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + \beta S \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x-y, t-s) I(y, s) dy ds - \gamma I(x, t), \quad (1.12)$$

$$\partial_t R(x, t) = d_4 \partial_{xx} R(x, t) + \gamma I(x, t). \quad (1.13)$$

The constant β is the transmission rate between the infected and susceptibles. The convolution $\int_{-\infty}^t \int_{-\infty}^{+\infty} K(x-y, t-s) I(y, s) dy ds$ shows the effects of spatial heterogeneity (geographical movement), nonlocal interaction, and time delay such as latent period on the transmission of diseases. They proved the existence and nonexistence of traveling wave solutions for system (1.11)–(1.12) by Schauder's fixed point theorem and Laplace transform.

When $K(x, t) = \delta(x)\delta(t)$, (1.11)–(1.13) can be further reduced to the following diffusive Kermack–McKendrick SIR model proposed and studied by Hosono and Ilyas [12]:

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - \beta S(x, t)I(x, t), \quad (1.14)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + \beta S(x, t)I(x, t) - \gamma I(x, t), \quad (1.15)$$

$$\partial_t R(x, t) = d_4 \partial_{xx} R(x, t) + \gamma I(x, t). \quad (1.16)$$

By using the Schauder fixed point theorem, the authors proved the existence of traveling wave solution of this system when $d_1 = 1$. They also verified that for $\beta S_0 > \gamma$ and $c > c^* := 2\sqrt{d_3(\beta S_0 - \gamma)}$, system (1.14)–(1.16) admits a traveling wave solution $(S(x + ct), I(x + ct), R(x + ct))$ satisfying the boundary conditions $S(\infty) = S_\infty$, $S(-\infty) = S_0$, $S_0 > S_\infty$, and $I(\pm\infty) = 0$.

In particular, in the case $F(S, I) = SI$ and $K(x, t) = \delta(x)\delta(t)$, (1.1)–(1.4) can be reduced to the following SEIR model studied by Tian [23]:

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - \beta S(x, t) I(x, t), \quad (1.17)$$

$$\partial_t E(x, t) = d_2 \partial_{xx} E(x, t) + \beta S(x, t) I(x, t) - \alpha E(x, t), \quad (1.18)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + \alpha E(x, t) - \gamma I(x, t), \quad (1.19)$$

$$\partial_t R(x, t) = d_4 \partial_{xx} R(x, t) + \gamma I(x, t). \quad (1.20)$$

They have shown that the minimal traveling wave speed c^* depends not only on d_3 , but also on d_2 . Moreover, it was proved that if the basic reproduction number $R_0 := \frac{\beta S_0}{\gamma} > 1$ and $c > c^*$, subsystem (1.17)–(1.19) has a nonnegative and nontrivial traveling wave solution $(S(x + ct), E(x + ct), I(x + ct))$ satisfying the boundary conditions $S(-\infty) = S_0$, $S(\infty) = S_\infty < S_0$, and $E(\pm\infty) = I(\pm\infty) = 0$. On the other hand, there is no nonnegative and nontrivial traveling wave solution for subsystem (1.17)–(1.19) if either $0 < c < c^*$ and $R_0 > 1$ or $R_0 \leq 1$. We also refer to Tian and Yuan [24, 25], Xu [30, 31], and references therein for some relevant progress on the existence and nonexistence of traveling wave solutions of SEIR models.

Since equations (1.1)–(1.3) form a closed system, we omit equation (1.4) and study the following system only:

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - F\left(S(x, t), \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x - y, t - s) I(y, s) dy ds\right), \quad (1.21)$$

$$\begin{aligned} \partial_t E(x, t) &= d_2 \partial_{xx} E(x, t) + F\left(S(x, t), \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x - y, t - s) I(y, s) dy ds\right) \\ &\quad - \alpha E(x, t), \end{aligned} \quad (1.22)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + \alpha E(x, t) - \gamma I(x, t). \quad (1.23)$$

Our purpose is to look for the nontrivial and nonnegative traveling wave solutions $(S(x + ct), E(x + ct), I(x + ct))$ of system (1.21)–(1.23) satisfying the following asymptotic boundary conditions at infinity:

$$S(-\infty) = S_0, \quad S(+\infty) = S_\infty < S_0, \quad E(\pm\infty) = I(\pm\infty) = 0, \quad (1.24)$$

where S_0 is a positive constant representing the size of the susceptible individuals before being infected. Let $\xi = x + ct \in \mathbb{R}$. Then the system describing traveling wave solutions is as follows:

$$cS'(\xi) = d_1 S''(\xi) - F\left(S(\xi), \int_0^{+\infty} \int_{-\infty}^{+\infty} I(\xi - y - cs) K(y, s) dy ds\right), \quad (1.25)$$

$$cE'(\xi) = d_2 E''(\xi) + F\left(S(\xi), \int_0^{+\infty} \int_{-\infty}^{+\infty} I(\xi - y - cs)K(y, s) dy ds\right) - \alpha E(\xi), \quad (1.26)$$

$$cI'(\xi) = d_3 I''(\xi) + \alpha E(\xi) - \gamma I(\xi). \quad (1.27)$$

Compared with (1.17)–(1.20), our model (1.1)–(1.4) introduces nonlinear incidence rate and nonlocal delayed transmission and is more complicated. Also, the loss of order-preserving property for system (1.1)–(1.4) makes some classic methods fail to apply, for example, the shooting method [6], connection index theory [9], the general theory of traveling waves for monotone semiflows [17], the geometric singular perturbation method [21], monotone iteration combined with upper–lower solutions [29]. Fortunately, inspired by Zhao and Wang [33] that I can be presented by E and our system can be reduced to a two-dimensional problem, we construct an auxiliary system with parameter κ , use a proper iteration scheme to construct a pair of upper and lower solutions, apply the Schauder fixed point theorem and obtain the existence of traveling wave solutions of the auxiliary system. By using a limit discussion we can see that the solution of the auxiliary system converges to the solution of the original system as $\kappa \rightarrow 0$. Finally, we make use of the two-side Laplace transform in the proof of the nonexistence of traveling wave solutions.

We should point out that the exposed individuals have their own spacial diffusion rate, which plays an important part in the dynamics of transmission of disease. The exposed individuals play an important part in the transmission of diseases for their greater mobility compared to the infected. Therefore it is meaningful to study how the diffusion rate d_2 of the exposed influences the minimal wave speed c^* , and in this work, we will show that the minimal traveling wave speed c^* depends not only on d_2 , but also on d_3 . Moreover, c^* is dependent on the pattern of nonlocal interaction between the infected and susceptible individuals and on the latent period of disease.

The remainder of this paper is organized as follows. In Sect. 2, we first prove some useful lemmas, which will be used in the proof of our main result. Later, we construct an invariant convex closed set, apply the Schauder fixed point theorem to prove the existence of traveling wave solutions for an auxiliary system, and then extend the result to the original system by a limiting argument. In Sect. 3, we prove the nonexistence theorem for two different cases: (i) $\partial_I F(S_0, 0)/\gamma > 1$ and $c < c^*$; (ii) $\partial_I F(S_0, 0)/\gamma \leq 1$. In Sect. 4, we provided some examples to illustrate the main results. Finally, we carry out numerical simulations and give a brief discussion in Sect. 5.

2 Existence of traveling waves

To obtain the existence of traveling wave solutions of system (1.21)–(1.23) satisfying the asymptotic boundary conditions (1.24), we construct the auxiliary system

$$cS'(\xi) = d_1 S''(\xi) - F(S(\xi), [I * K](\xi)), \quad (2.1)$$

$$cE'(\xi) = d_2 E''(\xi) + F(S(\xi), [I * K](\xi)) - \alpha E(\xi) - \kappa E^2(\xi), \quad (2.2)$$

$$cI'(\xi) = d_3 I''(\xi) + \alpha E(\xi) - \gamma I(\xi), \quad (2.3)$$

where κ is a positive constant.

Define

$$C_{\mu^-, \mu^+}(\mathbb{R}) := \left\{ h \in C(\mathbb{R}) : \sup_{x \leq 0} |h(x)e^{-\mu^- x}| + \sup_{x \geq 0} |h(x)e^{-\mu^+ x}| < +\infty \right\}.$$

Integrating (2.3), we obtain

$$I(\xi) = C_1 e^{\lambda^- \xi} + C_2 e^{\lambda^+ \xi} + \frac{\alpha}{\rho} \int_{-\infty}^{\xi} e^{\lambda^- (\xi-x)} E(x) dx + \frac{\alpha}{\rho} \int_{\xi}^{+\infty} e^{\lambda^+ (\xi-x)} E(x) dx, \quad (2.4)$$

where C_1 and C_2 are constants, $\lambda^- < 0 < \lambda^+$ are the two roots of the equation

$$-d_3 \lambda^2 + c\lambda + \gamma = 0,$$

and

$$\rho := d_3(\lambda^+ - \lambda^-) = \sqrt{c^2 + 4d_3\gamma}.$$

Thus, for any given $\epsilon_0 > 0$, if $E(\xi) \in C_{\lambda^-, \lambda^+}(\mathbb{R})$, then the only bounded solution of (2.4) satisfying the asymptotic boundary conditions

$$\lim_{\xi \rightarrow \pm\infty} I(\xi) = 0$$

is

$$[I(E)](\xi) = \frac{\alpha}{\rho} \int_{-\infty}^{\xi} e^{\lambda^- (\xi-x)} E(x) dx + \frac{\alpha}{\rho} \int_{\xi}^{+\infty} e^{\lambda^+ (\xi-x)} E(x) dx. \quad (2.5)$$

Linearizing (2.2) at the initial disease-free point $(S_0, 0, 0)$ yields

$$cE'(\xi) = d_2 E''(\xi) + \partial_I F(S_0, 0)[I(E) * K](\xi) - \alpha E(\xi). \quad (2.6)$$

By using the form $E(\xi) = e^{\lambda \xi}$ in (2.6), where $\lambda \in [0, \min\{\lambda^c, \lambda^+\})$, we have

$$c\lambda e^{\lambda \xi} = d_2 \lambda^2 e^{\lambda \xi} + \frac{\partial_I F(S_0, 0) \alpha \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\lambda(y+cs)} K(y, s) dy ds e^{\lambda \xi}}{-d_3(\lambda - \lambda^-)(\lambda - \lambda^+)} - \alpha e^{\lambda \xi}.$$

The corresponding characteristic equation of (2.6) is

$$\Delta(\lambda, c) := -d_2 \lambda^2 + c\lambda - \frac{\alpha \partial_I F(S_0, 0) \mathcal{R}(\lambda, c)}{-d_3 \lambda^2 + c\lambda + \gamma} + \alpha, \quad (2.7)$$

where

$$\mathcal{R}(\lambda, c) := \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\lambda(y+cs)} K(y, s) dy ds.$$

It is easy to prove the following lemma; see also Tian and Yuan [23, 24].

Lemma 2.1 Assume that $R_0 := \frac{\partial_I F(S_0, 0)}{\gamma} > 1$. Then there exist two positive constants c^* and λ_0 such that

$$\Delta(\lambda_0, c^*) = 0 \quad \text{and} \quad \partial_\lambda \Delta(\lambda_0, c^*) = 0.$$

Furthermore, if $0 < c < c^*$, then $\Delta(\lambda, c) < 0$ for all $\lambda \in [0, \min\{\lambda^c, \lambda^+\})$; if $c > c^*$, then $\Delta(\lambda, c) = 0$ has two positive real roots $\lambda_1 := \lambda_1(c)$ and $\lambda_2 := \lambda_2(c)$ with $0 < \lambda_1 < \lambda_0 < \lambda_2 < \min\{\lambda^c, \lambda^+\}$ and $\Delta(\lambda, c) < 0$ for $\lambda \in (0, \lambda_1) \cup (\lambda_2, \min\{\lambda^c, \lambda^+\})$ and $\Delta(\lambda, c) > 0$ for $\lambda \in (\lambda_1, \lambda_2)$;

Throughout this section, we assume that $R_0 > 1$ and $c > c^*$. For $\xi \in \mathbb{R}$, we define

$$S_+(\xi) := S_0, \quad (2.8)$$

$$S_-(\xi) := \max\{S_0(1 - M_1 e^{\epsilon_1 \xi}), 0\}, \quad (2.9)$$

$$E_+(\xi) := \min\{e^{\lambda_1 \xi}, K_\kappa\}, \quad (2.10)$$

$$E_-(\xi) := \max\{e^{\lambda_1 \xi}(1 - M_2 e^{\epsilon_2 \xi}), 0\}, \quad (2.11)$$

where $M_1, M_2, \epsilon_1, \epsilon_2$ are four positive constants to be determined in the following lemma, and $K_\kappa := \frac{\alpha}{\kappa}(\frac{\partial_t F(S_0, 0)}{\gamma} - 1)$.

Lemma 2.2 *Given sufficiently large $M_1 > 0, M_2 > 0$ and sufficiently small $\epsilon_1 > 0, \epsilon_2 > 0$, we have*

$$-d_1 S_-''(\xi) + c S_-'(\xi) \leq -F(S_-(\xi), [I(E_+) * K](\xi)) \quad (2.12)$$

for $\xi \leq \xi_1 := -\epsilon_1^{-1} \ln M_1$ and

$$-d_2 E_-''(\xi) + c E_-'(\xi) + \alpha E_-(\xi) + \kappa E_-^2(\xi) \leq F(S_-(\xi), [I(E_-) * K](\xi)) \quad (2.13)$$

for $\xi \leq \xi_2 := -\epsilon_2^{-1} \ln M_2$.

Proof We first prove (2.12). Let M_1 be sufficiently large such that $-\epsilon_1^{-1} \ln M_1 \leq \lambda_1^{-1} \ln K_\kappa$. If $\xi \leq \xi_1$, then $S_-(\xi) = S_0(1 - M_1 e^{\epsilon_1 \xi})$ and $E_+(\xi) = e^{\lambda_1 \xi}$. We only need to prove

$$d_1 S_0 M_1 \epsilon_1^2 e^{\epsilon_1 \xi} - c S_0 M_1 \epsilon_1 e^{\epsilon_1 \xi} + \partial_t F(S_0, 0) [I(E_+) * K](\xi) \leq 0.$$

It follows from (2.10) that

$$\begin{aligned} & [I(E_+) * K](\xi) \\ &= \frac{\alpha}{\rho} \int_0^\infty \int_{-\infty}^\infty \left(\int_{-\infty}^{\xi-y-cs} e^{\lambda_1(x-y-cs-x)} E_+(x) dx \right. \\ & \quad \left. + \int_{\xi-y-cs}^\infty e^{\lambda_1(x-y-cs-x)} E_+(x) dx \right) K(y, s) dy ds \\ &\leq \frac{\alpha}{\rho} \int_0^\infty \int_{-\infty}^\infty \left(\int_{-\infty}^{\xi-y-cs} e^{\lambda_1(x-y-cs-x)+\lambda_1 x} dx \right. \\ & \quad \left. + \int_{\xi-y-cs}^\infty e^{\lambda_1(x-y-cs-x)+\lambda_1 x} dx \right) K(y, s) dy ds \\ &= \frac{\alpha}{\rho} \left(\frac{\mathcal{R}(\lambda_1, c) e^{\lambda_1 \xi}}{\lambda_1 - \lambda^-} - \frac{\mathcal{R}(\lambda_1, c) e^{\lambda_1 \xi}}{\lambda_1 - \lambda^+} \right) = \frac{\alpha \mathcal{R}(\lambda_1, c) e^{\lambda_1 \xi}}{-d_3 \lambda_1^2 + c \lambda_1 + \gamma}. \end{aligned} \quad (2.14)$$

Then inequality (2.12) is true if

$$\frac{\alpha \partial_I F(S_0, 0) \mathcal{R}(\lambda_1, c) e^{(\lambda_1 - \epsilon_1)\xi}}{-d_3 \lambda_1^2 + c \lambda_1 + \gamma} \leq (c - d_1 \epsilon_1) S_0 M_1 \epsilon_1.$$

Note that $\xi \leq -\epsilon_1^{-1} \ln M_1$. It suffices to show that

$$\frac{\alpha \partial_I F(S_0, 0) \mathcal{R}(\lambda_1, c)}{-d_3 \lambda_1^2 + c \lambda_1 + \gamma} \left(\frac{1}{M_1} \right)^{\frac{\lambda_1 - \epsilon_1}{\epsilon_1}} \leq (c - d_1 \epsilon_1) S_0 M_1 \epsilon_1,$$

which is obviously true if we choose $\epsilon_1 > 0$ such that $\epsilon_1 < \min\{\lambda_1, c/d_1\}$ and then let M_1 be sufficiently large.

Now we prove that (2.13) holds. Let M_2 be large enough such that $-\epsilon_2^{-1} \ln M_2 \leq -\epsilon_1^{-1} \ln M_1$. If $\xi \leq \xi_2$, then $S_-(\xi) = S_0 - S_0 M_1 e^{\epsilon_1 \xi}$, $E_-(\xi) = e^{\lambda_1 \xi} - M_2 e^{(\lambda_1 + \epsilon_2)\xi}$, and (2.13) is equivalent to

$$\begin{aligned} & -d_2 E_-''(\xi) + c E_-'(\xi) + \alpha E_-(\xi) + \kappa E_-^2(\xi) - \partial_I F(S_0, 0) [I(E_-) * K](\xi) \\ & \leq F(S_-(\xi), [I(E_-) * K](\xi)) - \partial_I F(S_0, 0) [I(E_-) * K](\xi). \end{aligned} \quad (2.15)$$

It follows from (2.11) that

$$\begin{aligned} [I(E_-) * K](\xi) & \geq [I(e^{\lambda_1 \cdot} - M_2 e^{(\lambda_1 + \epsilon_2) \cdot}) * K](\xi) \\ & = \frac{\alpha \mathcal{R}(\lambda_1, c) e^{\lambda_1 \xi}}{-d_3 \lambda_1^2 + c \lambda_1 + \gamma} - \frac{M_2 \alpha \mathcal{R}(\lambda_1 + \epsilon_2, c) e^{(\lambda_1 + \epsilon_2)\xi}}{-d_3 (\lambda_1 + \epsilon_2)^2 + c (\lambda_1 + \epsilon_2) + \gamma}. \end{aligned}$$

To prove inequality (2.15), it suffices to show that

$$\begin{aligned} & \partial_I F(S_0, 0) [I(E_-) * K](\xi) - F(S_-(\xi), [I(E_-) * K](\xi)) \\ & \leq M_2 \Delta(\lambda_1 + \epsilon_2, c) e^{(\lambda_1 + \epsilon_2)\xi} - \kappa E_-^2(\xi). \end{aligned} \quad (2.16)$$

By (H3) and (H4), for any $\sigma \in (0, \partial_I F(S_0, 0))$, there exists a small positive constant δ_0 such that

$$\partial_I F(S_0, 0) \geq \frac{F(S, I)}{I} \geq \partial_I F(S_0, 0) - \sigma, \quad 0 < I < \delta_0, S_0 - \delta_0 \leq S \leq S_0.$$

Since

$$[I(E_-) * K](\xi) \leq [I(e^{\lambda_1 \cdot}) * K](\xi) = \frac{\alpha \mathcal{R}(\lambda_1, c) e^{\lambda_1 \xi}}{-d_3 \lambda_1^2 + c \lambda_1 + \gamma}, \quad S_-(\xi) = S_0 - S_0 M_1 e^{\epsilon_1 \xi}$$

and $\xi \leq \xi_2 = -\epsilon_2^{-1} \ln M_2$, we can choose M_2 large enough such that

$$0 < [I(E_-) * K](\xi) < \delta_0, \quad S_0 - \delta_0 \leq S_-(\xi) \leq S_0 \quad \text{for all } \xi \leq \xi_2.$$

It follows that

$$0 \leq \partial_I F(S_0, 0) - \frac{F(S_-(\xi), [I(E_-) * K](\xi))}{[I(E_-) * K](\xi)} \leq \sigma$$

for all $\xi \leq \xi_2$. Note that $\frac{F(S,I)}{I}$ is nonincreasing in I for all $S > 0$. Then we obtain that

$$\begin{aligned} & \partial_I F(S_0, 0)[I(E_-) * K](\xi) - F(S_-(\xi), [I(E_-) * K](\xi)) \\ &= \left\{ \partial_I F(S_0, 0) - \frac{F(S_-(\xi), [I(E_-) * K](\xi))}{[I(E_-) * K](\xi)} \right\} [I(E_-) * K](\xi) \\ &\leq \left(\frac{\partial_I F(S_0, 0) - \frac{F(S_-(\xi), [I(E_-) * K](\xi))}{[I(E_-) * K](\xi)}}{2} + [I(E_-) * K](\xi) \right)^2 \\ &\leq \left(\partial_I F(S_0, 0) - \frac{F(S_-(\xi), [I(E_-) * K](\xi))}{[I(E_-) * K](\xi)} + [I(E_-) * K](\xi) \right)^2 \\ &\leq (\sigma + [I(E_-) * K](\xi))^2. \end{aligned} \quad (2.17)$$

Letting $\sigma \rightarrow 0$, we have

$$\partial_I F(S_0, 0)[I(E_-) * K](\xi) - F(S_-(\xi), [I(E_-) * K](\xi)) \leq [I(E_-) * K]^2(\xi).$$

Therefore, to prove inequality (2.16), we need only to show that

$$\frac{\alpha^2 \mathcal{R}^2(\lambda_1, c) e^{2\lambda_1 \xi}}{(-d_3 \lambda_1^2 + c \lambda_1 + \gamma)^2} + \kappa e^{2\lambda_1 \xi} \leq M_2 \Delta(\lambda_1 + \epsilon_2, c) e^{(\lambda_1 + \epsilon_2) \xi}.$$

Noting that $\xi \leq \xi_2 = -\epsilon_2^{-1} \ln M_2$, it suffices to prove the above inequality for $\xi = \xi_2$:

$$\left[\frac{\alpha^2 \mathcal{R}^2(\lambda_1, c)}{(-d_3 \lambda_1^2 + c \lambda_1 + \gamma)^2} + \kappa \right] \left(\frac{1}{M_2} \right)^{\frac{\lambda_1 - \epsilon_2}{\epsilon_2}} \leq M_2 \Delta(\lambda_1 + \epsilon_2, c).$$

By taking $M_2 = 1/\Delta(\lambda_1 + \epsilon_2, c)$ and $\epsilon_2 \rightarrow 0$ we see that the above inequality holds. $\square$

For $i = 1, 2$, we will give the definitions of a second-order linear differential operator $\mathcal{D}_i$ and its inverse $\mathcal{D}_i^{-1}$. Choose $\beta_1 > \max_{0 \leq S \leq S_0, 0 \leq I \leq \frac{\alpha K_K}{\gamma}} \{\partial_S F(S, I)\}$ and $\beta_2 > \alpha(\frac{2\partial_I F(S_0, 0)}{\gamma} - 1)$. The roots of the equation

$$-d_i \Lambda^2 + c \Lambda + \beta_i = 0$$

are

$$\Lambda_i^\pm := \frac{c \pm \sqrt{c^2 + 4d_i \beta_i}}{2d_i}.$$

Furthermore, choose β_i so large that

$$\min\{-\Lambda_1^-, -\Lambda_2^-\} > \lambda_1.$$

We introduce the new symbol

$$R_i := d_i(\Lambda_i^+ - \Lambda_i^-) = \sqrt{c^2 + 4d_i \beta_i}.$$

The differential operator $\mathcal{D}_i$, $i = 1, 2$, is defined by

$$\mathcal{D}_i(\phi) := -d_i\phi'' + c\phi' + \beta_i\phi$$

for $\phi \in C^2(\mathbb{R})$. The inverse operator $\mathcal{D}_i^{-1}$ is given by the integral representation

$$(\mathcal{D}_i^{-1}\phi)(\xi) := \frac{1}{R_i} \int_{-\infty}^{\xi} e^{\Lambda_i^-(\xi-x)} \phi(x) dx + \frac{1}{R_i} \int_{\xi}^{+\infty} e^{\Lambda_i^+(\xi-x)} \phi(x) dx$$

for $\phi \in C_{\mu^-, \mu^+}(\mathbb{R})$, where $\mu^- > \max\{\Lambda_1^-, \Lambda_2^-\}$ and $\mu^+ < \min\{\Lambda_1^+, \Lambda_2^+\}$. Furthermore, by a simple calculation we get

$$\begin{aligned} (\mathcal{D}_i^{-1}\phi)'(\xi) &:= \frac{\Lambda_i^-}{R_i} \int_{-\infty}^{\xi} e^{\Lambda_i^-(\xi-x)} \phi(x) dx + \frac{\Lambda_i^+}{R_i} \int_{\xi}^{+\infty} e^{\Lambda_i^+(\xi-x)} \phi(x) dx, \\ (\mathcal{D}_i^{-1}\phi)''(\xi) &:= \frac{(\Lambda_i^-)^2}{R_i} \int_{-\infty}^{\xi} e^{\Lambda_i^-(\xi-x)} \phi(x) dx + \frac{(\Lambda_i^+)^2}{R_i} \int_{\xi}^{+\infty} e^{\Lambda_i^+(\xi-x)} \phi(x) dx - \frac{\phi(\xi)}{d_i}. \end{aligned}$$

Now we state some properties of the operators $\mathcal{D}_i$ and $\mathcal{D}_i^{-1}$ proved in [26].

Lemma 2.3 ([26]) *Let $i = 1, 2$. We have*

$$\mathcal{D}_i^{-1}(\mathcal{D}_i\phi) = \phi$$

for any $\phi \in C^2(\mathbb{R})$ such that $\phi, \phi', \phi'' \in C_{\mu^-, \mu^+}(\mathbb{R})$. Moreover,

$$\mathcal{D}_i(\mathcal{D}_i^{-1}\phi) = \phi$$

for $\phi \in C_{\mu^-, \mu^+}(\mathbb{R})$, where $\mu^- > \max\{\Lambda_1^-, \Lambda_2^-\}$ and $\mu^+ < \min\{\Lambda_1^+, \Lambda_2^+\}$. Let

$$\varphi := \max\{e^{\lambda\xi}(1 - Me^{\epsilon\xi}), 0\}$$

for any $M > 0$, $\epsilon > 0$, and λ such that $\Lambda_i^- < \lambda < \lambda + \epsilon < \Lambda_i^+$. Then we have

$$\mathcal{D}_i^{-1}(\mathcal{D}_i\varphi) \geq \varphi.$$

Given any $A > 0$, we have

$$\mathcal{D}_i^{-1}(\mathcal{D}_i\psi) \leq \psi$$

for $\psi := \min\{e^{\lambda\xi}, A\}$, where $0 < \lambda < \Lambda_i^+$.

Remark 2.1 Although $\varphi(\xi)$ and $\psi(\xi)$ have certain points at which they are not differentiable, the integrals $\mathcal{D}_i^{-1}(\mathcal{D}_i\varphi)$ and $\mathcal{D}_i^{-1}(\mathcal{D}_i\psi)$ are well defined in the sense of distribution.

Define the convex closed set

$$\Omega := \{(S, E) \in C(\mathbb{R}, \mathbb{R}^2) : S_- \leq S \leq S_+, E_- \leq E \leq E_+\}$$

and the map $\mathcal{G} : \Omega \rightarrow C(\mathbb{R}, \mathbb{R}^2)$ by

$$\mathcal{G}[S(\cdot), E(\cdot)](\xi) = \begin{pmatrix} \mathcal{G}_1[S(\cdot), E(\cdot)](\xi) \\ \mathcal{G}_2[S(\cdot), E(\cdot)](\xi) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{G}_1[S(\cdot), E(\cdot)](\xi) &= \mathcal{D}_1^{-1}[\beta_1 S - F(S, [I(E) * K])](\xi), \\ \mathcal{G}_2[S(\cdot), E(\cdot)](\xi) &= \mathcal{D}_2^{-1}[\beta_2 E + F(S, [I(E) * K]) - \alpha E - \kappa E^2](\xi). \end{aligned}$$

Lemma 2.4 *The operator $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ maps Ω into Ω .*

Proof For any given $(S, E) \in \Omega$, we first show that

$$S_- \leq \mathcal{G}_1[S(\cdot), E(\cdot)] \leq S_+.$$

Since $\beta_1 S - F(S, [I(E) * K]) \leq \beta_1 S_0 = \mathcal{D}_1 S_+$, we have

$$\mathcal{G}_1[S(\cdot), E(\cdot)] \leq \mathcal{D}_1^{-1}(\mathcal{D}_1 S_+) = S_+.$$

Due to $\beta_1 > \max_{0 \leq S \leq S_0, 0 \leq I \leq \frac{\alpha K \kappa}{\gamma}} \{\partial_S F(S, I)\}$, we have that $\beta_1 S - F(S, [I(E) * K])$ is increasing in $S \in \mathbb{R}_+$. From Lemma 2.2 and $[I(E_+) * K] \geq [I(E) * K]$ we have for $\xi < \xi_1$,

$$\beta_1 S - F(S, [I(E) * K]) \geq \beta_1 S_- - F(S_-, [I(E) * K]) \geq \beta_1 S_- - F(S_-, [I(E_+) * K]) \geq \mathcal{D}_1 S_-.$$

Moreover,

$$\beta_1 S - F(S, [I(E) * K]) \geq 0 = \mathcal{D}_1 S_-$$

for $\xi > \xi_1$. Coupling the above two inequalities and using Lemma 2.3 yield

$$\mathcal{G}_1[S(\cdot), E(\cdot)] \geq \mathcal{D}_1^{-1}(\mathcal{D}_1 S_-) \geq S_-.$$

Next, we consider $\mathcal{G}_2[S(\cdot), E(\cdot)](\xi)$. Since $\beta_2 > \alpha(\frac{2\partial_I F(S_0, 0)}{\gamma} - 1)$, the function $(\beta_2 - \alpha)E - \kappa E^2$ is increasing with respect to E , which implies

$$\beta_2 E + F(S, [I(E) * K]) - \alpha E - \kappa E^2 \geq 0 = \mathcal{D}_2 E_-$$

for $\xi > \xi_2$. Moreover, by Lemma 2.2 we get

$$\beta_2 E + F(S, [I(E) * K]) - \alpha E - \kappa E^2 \geq \beta_2 E_- + F(S_-, [I(E_-) * K]) - \alpha E_- - \kappa E_-^2 = \mathcal{D}_2 E_-$$

for $\xi < \xi_2$. In view of Lemma 2.3, we obtain from the above two inequalities that

$$\mathcal{G}_2[S(\cdot), E(\cdot)] \geq \mathcal{D}_2^{-1}(\mathcal{D}_2 E_-) \geq E_-.$$

Since $E_+(\xi) = \min\{e^{\lambda_1 \xi}, K_\kappa\}$, it follows from $F(S, [I(E) * K]) \leq \partial_I F(S_0, 0)[I(E_+) * K]$ that

$$\beta_2 E + F(S, [I(E) * K]) - \alpha E - \kappa E^2 \leq \beta_2 E_+ + \partial_I F(S_0, 0)[I(E_+) * K] - \alpha E_+ - \kappa E_+^2 = \mathcal{D}_2 E_+$$

for $\xi \leq \xi_3 := \lambda_1^{-1} \ln K_\kappa$ and

$$\beta_2 E + F(S, [I(E) * K]) - \alpha E - \kappa E^2 \leq \beta_2 K_\kappa + \partial_I F(S_0, 0) \frac{K_\kappa \alpha}{\gamma'} - \alpha K_\kappa - \kappa K_\kappa^2 = \mathcal{D}_2 E_+$$

for $\xi > \xi_3$. A combination of the above two inequalities and Lemma 2.3 yields

$$\mathcal{G}_2[S(\cdot), E(\cdot)] \leq \mathcal{D}_2^{-1}(\mathcal{D}_2 E_+) \leq E_+.$$

This ends the proof. $\square$

Choose μ satisfying

$$\lambda_1 < \mu < -\Lambda_i^- < \Lambda_i^+, \quad i = 1, 2.$$

We set the functional space

$$\mathcal{B}_\mu(\mathbb{R}, \mathbb{R}^2) := \{\phi = (\phi_1, \phi_2) : \phi_i \in C(\mathbb{R}) \text{ and } |\phi_i|_\mu < +\infty, i = 1, 2\}$$

equipped with the norm

$$|\phi|_\mu := \max\{|\phi_i|_\mu : i = 1, 2\}, \quad (2.18)$$

where

$$|\phi_i|_\mu := \sup_{\xi \in \mathbb{R}} e^{-\mu|\xi|} |\phi_i(\xi)|. \quad (2.19)$$

Since $\mu > \lambda_1 > 0$, it is easy to see that Ω is uniformly bounded under the norm $|\cdot|_\mu$ in $\mathcal{B}_\mu(\mathbb{R}, \mathbb{R}^2)$. Before applying the Schauder fixed point theorem, we should verify that $\mathcal{G}$ is continuous and compact on Ω .

Lemma 2.5 *Suppose that $(S_1, E_1) \in \Omega \cap \mathcal{B}_\mu(\mathbb{R}, \mathbb{R}^2)$ and $(S_2, E_2) \in \Omega \cap \mathcal{B}_\mu(\mathbb{R}, \mathbb{R}^2)$. Then $|[I(E_1) * K](\cdot) - [I(E_2) * K](\cdot)|_\mu \rightarrow 0$ if $|S_1(\cdot) - S_2(\cdot)|_\mu \rightarrow 0$ and $|E_1(\cdot) - E_2(\cdot)|_\mu \rightarrow 0$.*

Proof Recall the integral form of $I(E)$:

$$I(E)(\xi) = \frac{\alpha}{\rho} \int_{-\infty}^{\xi} e^{\lambda^-(\xi-x)} E(x) dx + \frac{\alpha}{\rho} \int_{\xi}^{+\infty} e^{\lambda^+(\xi-x)} E(x) dx.$$

For any $\xi \in \mathbb{R}$, we have

$$|I(E)(\cdot)|_\mu = \frac{\alpha}{\rho} \int_{-\infty}^{\xi} e^{\lambda^-(\xi-x)} E(x) e^{-\mu|\xi|} dx + \frac{\alpha}{\rho} \int_{\xi}^{+\infty} e^{\lambda^+(\xi-x)} E(x) e^{-\mu|\xi|} dx \leq \frac{\alpha}{\rho} \overline{C}(\xi) |E(\cdot)|_\mu$$

with

$$\overline{C}(\xi) = e^{-\mu|\xi|} \left\{ \int_{-\infty}^{\xi} e^{\lambda^-(\xi-x)+\mu|x|} dx + \int_{\xi}^{+\infty} e^{\lambda^+(\xi-x)+\mu|x|} dx \right\}.$$

Note that $\lambda^- < -\mu < \mu < \lambda^+$. A simple application of L'Hospital rule yields

$$\overline{C}(-\infty) = \frac{1}{\mu + \lambda^+} - \frac{1}{\mu + \lambda^-}, \quad \overline{C}(+\infty) = \frac{1}{\lambda^+ - \mu} + \frac{1}{\mu - \lambda^-}.$$

Consequently, there exists a constant $\overline{L} > 0$ such that $|\overline{C}(\xi)| \leq \overline{L}$ for any $\xi \in \mathbb{R}$. Thanks to $E_1, E_2 \in \Omega$, we obtain $|E_1(x) - E_2(x)| \leq e^{\lambda_1 x}$. Then it follows that

$$\begin{aligned} & |I(E_1)(\xi) - I(E_2)(\xi)| \\ & \leq \frac{\alpha}{\rho} \left[\int_{-\infty}^{\xi} e^{\lambda^-(\xi-x)} |E_1(x) - E_2(x)| dx + \int_{\xi}^{+\infty} e^{\lambda^+(\xi-x)} |E_1(x) - E_2(x)| dx \right] \\ & \leq \frac{\alpha}{\rho} \left[\int_{-\infty}^{\xi} e^{\lambda^-(\xi-x)+\lambda_1 x} dx + \int_{\xi}^{+\infty} e^{\lambda^+(\xi-x)+\lambda_1 x} dx \right] \\ & = \frac{\alpha e^{\lambda_1 \xi}}{-d_3 \lambda_1^2 + c \lambda_1 + \gamma}. \end{aligned}$$

Given $\varepsilon > 0$ sufficiently small, since $e^{-\lambda_1(y+cs)} K(y, s) \in L^1(\mathbb{R} \times \mathbb{R}^+)$, there exists $N^* > 0$ such that

$$\int \int_{\mathbb{R} \times \mathbb{R}^+ \setminus [-N^*, N^*] \times [0, N^*]} e^{-\lambda_1(y+cs)} K(y, s) dy ds < \frac{(-d_3 \lambda_1^2 + c \lambda_1 + \gamma) \varepsilon}{2\alpha}.$$

Furthermore, from the above argument we obtain

$$\begin{aligned} & |[I(E_1) * K](\cdot) - [I(E_2) * K](\cdot)|_{\mu} \\ & \leq \frac{\varepsilon}{2} + e^{-\mu|\xi|} \int_0^{N^*} \int_{-N^*}^{N^*} |I(E_1)(\xi - y - cs) - I(E_2)(\xi - y - cs)| K(y, s) dy ds \\ & \leq \frac{\varepsilon}{2} + \frac{\alpha \overline{L}}{\rho} |E_1(\cdot) - E_2(\cdot)|_{\mu} e^{-\mu|\xi|} e^{\mu(|\xi| + N^* + cN^*)} \int_0^{N^*} \int_{-N^*}^{N^*} K(y, s) dy ds \\ & \leq \varepsilon \end{aligned}$$

if $|E_1(\cdot) - E_2(\cdot)|_{\mu} \leq \varepsilon \rho e^{-\mu(1+c)N^*} / 2\alpha \overline{L}$. The proof is finished. $\square$

Lemma 2.6 *The map $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : \Omega \rightarrow \Omega$ is continuous and compact with respect to the norm $|\cdot|_{\mu}$ in $\mathcal{B}_{\mu}(\mathbb{R}, \mathbb{R}^2)$.*

Proof For any $(S_1, E_1) \in \Omega$ and $(S_2, E_2) \in \Omega$, by the mean-value theorem we have

$$\begin{aligned} & |F(S_1, [I(E_1) * K]) - F(S_2, [I(E_2) * K])| \\ & = |F(S_1, [I(E_1) * K]) - F(S_1, [I(E_2) * K]) + F(S_1, [I(E_2) * K]) - F(S_2, [I(E_2) * K])| \\ & \leq \partial_I F(S_1, \tilde{I}) |[I(E_1) * K] - [I(E_2) * K]| + \partial_S F(\tilde{S}, [I(E_2) * K]) |S_1 - S_2| \end{aligned}$$

$$\leq m|[I(E_1) * K] - [I(E_2) * K]| + \hat{m}|S_1 - S_2|,$$

where $\tilde{I}$ lies between $[I(E_1) * K]$ and $[I(E_2) * K]$, $\tilde{S}$ lies between S_1 and S_2 , and

$$m = \max \left\{ \partial_I F(S, I) : \begin{array}{l} 0 \leq S \leq S_0 \\ 0 \leq I \leq \frac{\alpha K_K}{\gamma} \end{array} \right\} \quad \text{and} \quad \hat{m} = \max \left\{ \partial_S F(S, I) : \begin{array}{l} 0 \leq S \leq S_0 \\ 0 \leq I \leq \frac{\alpha K_K}{\gamma} \end{array} \right\}.$$

Then

$$\begin{aligned} & |(\beta_1 S_1 - F(S_1, [I(E_1) * K])) - (\beta_1 S_2 - F(S_2, [I(E_2) * K]))| \\ & \leq (\beta_1 + \hat{m})|S_1 - S_2| + m|[I(E_1) * K] - [I(E_2) * K]| \\ & \leq L_K(|S_1 - S_2| + |[I(E_1) * K] - [I(E_2) * K]|), \end{aligned}$$

where $L_K = \max\{\beta_1 + \hat{m}, m\}$. Thus we obtain that

$$\begin{aligned} & |\mathcal{G}_1[S_1(\cdot), E_1(\cdot)](\xi) - \mathcal{G}_1[S_2(\cdot), E_2(\cdot)](\xi)| e^{-\mu|\xi|} \\ & \leq \frac{L_K}{R_1} C(\xi) [|S_1(\cdot) - S_2(\cdot)|_\mu + |[I(E_1) * K](\cdot) - [I(E_2) * K](\cdot)|_\mu] \end{aligned}$$

with

$$C(\xi) = e^{-\mu|\xi|} \left\{ \int_{-\infty}^{\xi} e^{\Lambda_1^-(\xi-x)+\mu|x|} dx + \int_{\xi}^{+\infty} e^{\Lambda_1^+(\xi-x)+\mu|x|} dx \right\}.$$

Since $\lambda_1 < \mu < \min\{-\Lambda_1^-, -\Lambda_2^-\}$, applying L'Hospital rule to the above formula gives

$$C(-\infty) = \frac{1}{\mu + \Lambda_1^+} - \frac{1}{\mu + \Lambda_1^-}, \quad C(+\infty) = \frac{1}{\Lambda_1^+ - \mu} + \frac{1}{\mu - \Lambda_1^-}.$$

Thus $C(\xi)$ is uniformly bounded on $\mathbb{R}$. By Lemma 2.5 we conclude that $\mathcal{G}_1$ is continuous with respect to the norm $|\cdot|_\mu$. Similarly, we can show that $\mathcal{G}_2$ is also continuous.

To prove the compactness of $\mathcal{G}$, we will use the Ascoli–Arzelà theorem and the diagonal process. Denote $I_k := [-k, k]$, $k \in \mathbb{N}$, and consider Ω as a bounded subset of $C(I_k, \mathbb{R}^2)$ with the maximum norm. It is easy to see that $\mathcal{G}(\Omega)$ is uniformly bounded.

Next, we will show that $\mathcal{G}(\Omega)$ is equicontinuous.

Note that $\beta_1 > \max_{0 \leq S \leq S_0, 0 \leq I \leq \frac{\alpha K_K}{\gamma}} \{\partial_S F(S, I)\}$ and $S(x) \leq S_0$. For any $(S, E) \in \Omega$, we have

$$\begin{aligned} \left| \frac{d}{d\xi} \mathcal{G}_1[S(\cdot), E(\cdot)](\xi) \right| &= \frac{1}{R_1} \left| \left\{ \Lambda_1^- \int_{-\infty}^{\xi} e^{\Lambda_1^-(\xi-x)} [\beta_1 S(x) - F(S(x), [I(E) * K](x))] dx \right. \right. \\ & \quad \left. \left. + \Lambda_1^+ \int_{\xi}^{+\infty} e^{\Lambda_1^+(\xi-x)} [\beta_1 S(x) - F(S(x), [I(E) * K](x))] dx \right\} \right| \\ &\leq \frac{-\beta_1 S_0 \Lambda_1^-}{R_1} \int_{-\infty}^{\xi} e^{\Lambda_1^-(\xi-x)} dx + \frac{\beta_1 S_0 \Lambda_1^+}{R_1} \int_{\xi}^{+\infty} e^{\Lambda_1^+(\xi-x)} dx \\ &= \frac{2\beta_1 S_0}{R_1}. \end{aligned}$$

Since $\beta_2 > \alpha \left[\frac{2\partial_I F(S_0, 0)}{\gamma} - 1 \right]$ and $F(S(x), [I(E) * K](x)) \leq \partial_I F(S_0, 0)[I(E) * K](x) \leq \frac{\alpha \partial_I F(S_0, 0) K_\kappa}{\gamma}$, we have

$$\begin{aligned} & \left| \frac{d}{d\xi} \mathcal{G}_2[S(\cdot), E(\cdot)](\xi) \right| \\ &= \frac{1}{R_2} \left| \left\{ \Lambda_2^- \int_{-\infty}^{\xi} e^{\Lambda_2^-(\xi-x)} [F(S(x), [I(E) * K](x)) + (\beta_2 - \alpha)E(x) - \epsilon E^2(x)] dx \right. \right. \\ & \quad \left. \left. + \Lambda_2^+ \int_{\xi}^{+\infty} e^{\Lambda_2^+(\xi-x)} [F(S(x), [I(E) * K](x)) + (\beta_2 - \alpha)E(x) - \epsilon E^2(x)] dx \right\} \right| \\ &\leq \frac{-\beta_2 K_\kappa \Lambda_2^-}{\rho_2} \int_{-\infty}^{\xi} e^{\Lambda_2^-(\xi-x)} dx + \frac{\beta_2 K_\kappa \Lambda_2^+}{R_2} \int_{\xi}^{+\infty} e^{\Lambda_2^+(\xi-x)} dx \\ &= \frac{2\beta_2 K_\kappa}{R_2}. \end{aligned}$$

Let $\{u_n\}$ be a sequence of Ω viewed as a bounded subset of $C(I_k)$. Since $\{\mathcal{G}(u_n)\}$ is uniformly bounded and equicontinuous on I_k , by the Ascoli–Arzelà theorem and the diagonal process we can choose a subsequence $\{u_{n_k}\}$ such that $v_{n_k} := \mathcal{G}u_{n_k}$ converges in $C(I_k)$ for all $k \in \mathbb{N}$. Let $v = \lim_{k \rightarrow +\infty} v_{n_k}$. Obviously, $v \in C(\mathbb{R}, \mathbb{R}^2)$. Since $\mathcal{G}(\Omega) \subseteq \Omega$ and Ω is closed, we have $v \in \Omega$. It follows from $\mu > \lambda_1 > 0$ that $|E_+(\cdot)|_\mu$ is bounded and Ω is uniformly bounded with respect to the norm $|\cdot|_\mu$. Thus $|v_{n_k} - v|_\mu$ is uniformly bounded for all $k \in \mathbb{N}$. Given any $\epsilon > 0$, we can choose a constant $M > 0$ independent of v_{n_k} such that $e^{-\mu|\xi|} |v_{n_k}(\xi) - v(\xi)| < \epsilon$ for any $|\xi| > M$ and $k \in \mathbb{N}$. On the other hand, v_{n_k} converges to v on the compact interval $[-M, M]$ with respect to the maximum norm, and thus there exists $K \in \mathbb{N}$ such that $e^{-\mu|\xi|} |v_{n_k}(\xi) - v(\xi)| < \epsilon$ for all $|\xi| \leq M$ and $k > K$. Hence v_{n_k} converges to v with respect to the norm $|\cdot|_\mu$. This proves the compactness of the map $\mathcal{G}$. $\square$

Theorem 2.1 *Let $c > c^*$. Then system (2.1)–(2.3) has a traveling wave solution $(S(x + ct), E(x + ct), I(x + ct))$ satisfying the asymptotic boundary conditions (1.24). Furthermore, $S(\xi)$ is nonincreasing in $\mathbb{R}$, $0 \leq E(\xi) \leq S_0 - S_\infty$ and $0 \leq I(\xi) < S_0 - S_\infty$ for all $\xi \in \mathbb{R}$, and*

$$\begin{aligned} \int_{-\infty}^{+\infty} \alpha E(\xi) d\xi &= \int_{-\infty}^{+\infty} \gamma I(\xi) d\xi \\ &< \int_{-\infty}^{+\infty} [\alpha E(\xi) + \kappa E^2(\xi)] d\xi \\ &= \int_{-\infty}^{+\infty} F(S(\xi), [I * K](\xi)) d\xi = c(S_0 - S_\infty). \end{aligned} \quad (2.20)$$

Proof Since $\mathcal{G}$ is continuous and compact on Ω , by the Schauder fixed point theorem, $\mathcal{G}$ has a fixed point $(S, E) \in \Omega$ such that

$$\begin{aligned} S &= \mathcal{G}_1[S(\cdot), E(\cdot)] = \mathcal{D}_1^{-1}[\beta_1 S - F(S, [I(E) * K])], \\ E &= \mathcal{G}_2[S(\cdot), E(\cdot)] = \mathcal{D}_2^{-1}[\beta_2 E + F(S, [I(E) * K]) - \alpha E - \kappa E^2]. \end{aligned}$$

Due to $S, E \in C_{-\mu, \mu}(\mathbb{R})$ and $\Lambda_i^- < -\mu < \mu < \Lambda_i^+$ for $i = 1, 2$, we have

$$\mathcal{D}_1 S = \beta_1 S - F(S, [I(E) * K]), \quad \mathcal{D}_2 E = \beta_2 E + F(S, [I(E) * K]) - \alpha E - \kappa E^2.$$

Next, we will verify the asymptotic boundary conditions (1.24). Note that $S_-(\xi) \leq S(\xi) \leq S_0$ and $E_-(\xi) \leq E(\xi) \leq E_+(\xi)$. It follows from the squeeze theorem that $S(\xi) \rightarrow S_0$ and $E(\xi) \sim e^{\lambda_1 \xi}$ as $\xi \rightarrow -\infty$. Recall the integral form of $I(E)$:

$$I(E)(\xi) = \frac{\alpha}{\rho} \int_{-\infty}^{\xi} e^{\lambda^-(\xi-x)} E(x) dx + \frac{\alpha}{\rho} \int_{\xi}^{+\infty} e^{\lambda^+(\xi-x)} E(x) dx. \quad (2.21)$$

By the L'Hospital rule we have that

$$\lim_{\xi \rightarrow -\infty} I(E)(\xi) = \lim_{\xi \rightarrow -\infty} \frac{\alpha}{\rho} \left(\frac{e^{-\lambda^-\xi} E(\xi)}{-\lambda^- e^{-\lambda^-\xi}} \right) + \lim_{\xi \rightarrow -\infty} \frac{\alpha}{\rho} \left(\frac{-e^{-\lambda^+\xi} E(\xi)}{-\lambda^+ e^{-\lambda^+\xi}} \right) = 0.$$

Applying the L'Hospital rule to the maps $\mathcal{G}_1$ and $\mathcal{G}_2$, it is easy to show that $S'(-\infty) = 0$ and $E'(-\infty) = 0$. From (2.21) we obtain

$$I(E)'(\xi) = \frac{\alpha \lambda^-}{\rho} \int_{-\infty}^{\xi} e^{\lambda^-(\xi-x)} E(x) dx + \frac{\alpha \lambda^+}{\rho} \int_{\xi}^{+\infty} e^{\lambda^+(\xi-x)} E(x) dx.$$

Using the L'Hospital rule again, we get

$$\lim_{\xi \rightarrow -\infty} I(E)'(\xi) = \lim_{\xi \rightarrow -\infty} \frac{\alpha \lambda^-}{\rho} \left(\frac{e^{-\lambda^-\xi} E(\xi)}{-\lambda^- e^{-\lambda^-\xi}} \right) + \lim_{\xi \rightarrow -\infty} \frac{\alpha \lambda^+}{\rho} \left(\frac{-e^{-\lambda^+\xi} E(\xi)}{-\lambda^+ e^{-\lambda^+\xi}} \right) = 0.$$

Finally, it follows from (2.1)–(2.3) that $S''(-\infty) = 0$, $E''(-\infty) = 0$, and $I''(-\infty) = 0$.

Now we are ready to investigate the asymptotic behavior of $S(\xi)$, $E(\xi)$, and $I(\xi)$ as $\xi \rightarrow +\infty$. An integration of (2.1) from $-\infty$ to ξ gives

$$d_1 S'(\xi) = c[S(\xi) - S_0] + \int_{-\infty}^{\xi} F(S(x), [I(E) * K](x)) dx. \quad (2.22)$$

Since $S(\xi)$ and $S'(\xi)$ are uniformly bounded, the integral on the right-hand side should be uniformly bounded. Thus we obtain that

$$\int_{-\infty}^{+\infty} F(S(\xi), [I(E) * K](\xi)) dx < +\infty.$$

Clearly, (2.1) implies

$$(e^{-\frac{c\xi}{d_1}} S'(\xi))' = e^{-\frac{c\xi}{d_1}} \left(S''(\xi) - \frac{c}{d_1} S'(\xi) \right) = \frac{1}{d_1} e^{-\frac{c\xi}{d_1}} F(S(\xi), [I(E) * K](\xi)).$$

Integrating the above equality from ξ to $+\infty$ gives

$$e^{-\frac{c\xi}{d_1}} S'(\xi) = -\frac{1}{d_1} \int_{\xi}^{+\infty} e^{-\frac{cx}{d_1}} F(S(x), [I(E) * K](x)) dx, \quad \forall \xi \in \mathbb{R}. \quad (2.23)$$

Hence S is nonincreasing. Since $(S, E) \in \Omega$, for $\xi < 0$ with $|\xi|$ sufficiently large, we have

$$\int_{\xi}^{+\infty} e^{-\frac{cx}{d_1}} F(S(x), [I(E) * K](x)) dx > 0.$$

Thus there exists $\xi^* < 0$ such that $S'(\xi) < 0$ for all $\xi < \xi^*$, which implies $0 \leq S(+\infty) = S_\infty < S_0$. Integrating (2.2) from $-\infty$ to ξ yields

$$\begin{aligned} cE(\xi) &= d_2 E'(\xi) + \int_{-\infty}^{\xi} F(S(x), [I(E) * K](x)) \, dx - \alpha \int_{-\infty}^{\xi} E(x) \, dx \\ &\quad - \kappa \int_{-\infty}^{\xi} E^2(x) \, dx. \end{aligned} \quad (2.24)$$

By the boundedness of E and E' on $\xi \in \mathbb{R}$ we obtain

$$\int_{-\infty}^{+\infty} E(\xi) \, d\xi < +\infty \quad \text{and} \quad E(+\infty) = 0,$$

which implies $I(+\infty) = 0$.

Similarly to the argument of (2.23), we have

$$E'(\xi) = \frac{1}{d_2} e^{\frac{c\xi}{d_2}} \int_{\xi}^{+\infty} e^{-\frac{cx}{d_2}} [F(S(x), [I(E) * K](x)) - \alpha E(x) - \kappa E^2(x)] \, dx.$$

By applying the L'Hospital rule we obtain $E'(+\infty) = 0$, which yields $I'(+\infty) = 0$. Then from (2.2) we have $E''(+\infty) = 0$. Applying the L'Hospital rule to S' again, we have that $S'(+\infty) = 0$. Therefore from (2.1) we obtain $S''(+\infty) = 0$. As a consequence, from (2.3), (2.22), and (2.24) we have

$$\int_{-\infty}^{+\infty} \alpha E(\xi) \, d\xi = \int_{-\infty}^{+\infty} \gamma I(\xi) \, d\xi$$

and

$$\int_{-\infty}^{+\infty} [\alpha E(\xi) + \kappa E^2(\xi)] \, d\xi = \int_{-\infty}^{+\infty} F(S(\xi), [I * K](\xi)) \, d\xi = c(S_0 - S_\infty).$$

Finally, we intend to prove the inequalities $E(\xi) \leq S_0 - S_\infty$ and $I(\xi) < S_0 - S_\infty$ for all $\xi \in \mathbb{R}$. Since $E(\xi) \sim e^{\lambda_1 \xi}$ as $\xi \rightarrow -\infty$ and $E(\xi) \rightarrow 0$ as $\xi \rightarrow +\infty$, we can define

$$\mathcal{H}(\xi) := E(\xi) + \frac{1}{c} \int_{-\infty}^{\xi} [\alpha E(x) + \kappa E^2(x)] \, dx + \frac{1}{c} \int_{\xi}^{+\infty} e^{\frac{c(\xi-x)}{d_2}} [\alpha E(x) + \kappa E^2(x)] \, dx. \quad (2.25)$$

By the properties of $E(\xi)$ and the L'Hospital rule we obtain

$$\lim_{\xi \rightarrow -\infty} \mathcal{H}(\xi) = 0 \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} \mathcal{H}(\xi) = \frac{1}{c} \int_{-\infty}^{+\infty} [\alpha E(x) + \kappa E^2(x)] \, dx = S_0 - S_\infty.$$

By differentiating (2.25) we get

$$\mathcal{H}'(\xi) = E'(\xi) + \frac{1}{d_2} \int_{\xi}^{+\infty} e^{\frac{c(\xi-x)}{d_2}} [\alpha E(x) + \kappa E^2(x)] \, dx.$$

It is easy to see that

$$\lim_{\xi \rightarrow -\infty} \mathcal{H}'(\xi) = \frac{1}{d_2} \lim_{\xi \rightarrow -\infty} \int_{\xi}^{+\infty} e^{\frac{c(\xi-x)}{d_2}} [\alpha E(x) + \kappa E^2(x)] \, dx = 0 \quad \text{and} \quad \lim_{\xi \rightarrow +\infty} \mathcal{H}'(\xi) = 0$$

from the above equality. By differentiating (2.25) twice we get

$$-d_2 \mathcal{H}''(\xi) + c \mathcal{H}'(\xi) = -d_2 E''(\xi) + c E'(\xi) + \alpha E(\xi) + \kappa E^2(\xi) = F(S(\xi), [I * K](\xi)).$$

An integration of the above equation from ξ to $+\infty$ gives

$$\mathcal{H}'(\xi) = \frac{1}{d_2} e^{\frac{c\xi}{d_2}} \int_{\xi}^{+\infty} e^{-\frac{cx}{d_2}} F(S(x), [I * K](x)) dx \geq 0.$$

Here we have used the fact that $\mathcal{H}'(+\infty) = 0$. Hence $\mathcal{H}(\xi)$ is nondecreasing on $\mathbb{R}$. Since $\mathcal{H}(+\infty) = S_0 - S_{\infty}$ by the asymptotic formula obtained from equation (2.25), we obtain from the above equality that $\mathcal{H}(\xi) \leq S_0 - S_{\infty}$ for all $\xi \in \mathbb{R}$. Since $E(\xi) \leq \mathcal{H}(\xi)$ by definition (2.25), it follows that $E(\xi) \leq S_0 - S_{\infty}$ for all $\xi \in \mathbb{R}$.

To show $0 \leq I(\xi) < S_0 - S_{\infty}$, we define the function

$$\hat{\mathcal{H}}(\xi) := I(\xi) + \frac{\gamma}{c} \int_{-\infty}^{\xi} I(x) dx + \frac{\gamma}{c} \int_{\xi}^{+\infty} e^{\frac{c(\xi-x)}{d_3}} I(x) dx$$

for $\xi \in \mathbb{R}$. We can check that it satisfies the equation

$$-d_3 \hat{\mathcal{H}}''(\xi) + c \hat{\mathcal{H}}'(\xi) = \alpha E(\xi), \quad \forall \xi \in \mathbb{R}.$$

Obviously,

$$\begin{aligned} \lim_{\xi \rightarrow -\infty} \hat{\mathcal{H}}(\xi) &= 0, & \lim_{\xi \rightarrow +\infty} \hat{\mathcal{H}}(\xi) &= \frac{\gamma}{c} \int_{-\infty}^{+\infty} I(\xi) d\xi < S_0 - S_{\infty} \quad \text{and} \\ \lim_{\xi \rightarrow +\infty} \hat{\mathcal{H}}'(\xi) &= \lim_{\xi \rightarrow -\infty} \hat{\mathcal{H}}'(\xi) = 0. \end{aligned}$$

Furthermore, we have that

$$\hat{\mathcal{H}}'(\xi) = \frac{\alpha}{d_3} e^{\frac{c\xi}{d_3}} \int_{\xi}^{+\infty} e^{-\frac{cx}{d_3}} E(x) dx \geq 0.$$

Hence we get that $I(\xi) \leq \hat{\mathcal{H}}(\xi) < S_0 - S_{\infty}$ for all $\xi \in \mathbb{R}$. □

Theorem 2.2 *There exists a positive constant c^* such that if $R_0 := \frac{\partial_I F(S_0, 0)}{\gamma} > 1$ and $c > c^*$, then system (1.21)–(1.23) has a nontrivial and nonnegative traveling wave solution (S, E, I) satisfying the asymptotic boundary conditions (1.24). Furthermore, S is decreasing, $0 \leq E(\xi) \leq S_0 - S_{\infty}$ and $0 \leq I(\xi) \leq S_0 - S_{\infty}$ for all $\xi \in \mathbb{R}$, and*

$$\int_{-\infty}^{+\infty} \alpha E(\xi) d\xi = \int_{-\infty}^{+\infty} \gamma I(\xi) d\xi = \int_{-\infty}^{+\infty} F(S(\xi), [I * K](\xi)) d\xi = c(S_0 - S_{\infty}). \quad (2.26)$$

Proof For $c > c^*$, let $\{\tau_k\}_{k=1}^{+\infty}$ be a sequence such that $0 < \tau_{k+1} < \tau_k < 1$ and $\tau_k \rightarrow 0$ as $k \rightarrow +\infty$. By Theorem 2.1, for any $\kappa = \tau_k$, there exists a solution $(S_k(\xi), E_k(\xi), I_k(\xi))$ of (2.1)–(2.3) satisfying the results of Theorem 2.1. From (2.23) we have

$$|S'_k(\xi)| = \frac{1}{d_1} e^{\frac{c\xi}{d_1}} \int_{\xi}^{+\infty} e^{-\frac{cx}{d_1}} F(S_k(x), [I_k(E) * K](x)) dx$$

$$\begin{aligned} &\leq \frac{\partial_I F(S_0, 0)(S_0 - S_\infty)}{d_1} e^{\frac{c\xi}{d_1}} \int_{\xi}^{+\infty} e^{-\frac{cx}{d_1}} dx \\ &= \frac{\partial_I F(S_0, 0)(S_0 - S_\infty)}{c}. \end{aligned}$$

Similarly, we can show that

$$|E'_k(\xi)| \leq \frac{(\partial_I F(S_0, 0) + \alpha + S_0 - S_\infty)(S_0 - S_\infty)}{c} \quad \text{and} \quad |I'_k(\xi)| \leq \frac{(\alpha + \gamma)(S_0 - S_\infty)}{c}.$$

By (2.1)–(2.3) there exists a positive constant M , independent of ξ and k , such that

$$|S''_k(\xi)|, |E''_k(\xi)|, |I''_k(\xi)|, |S'''_k(\xi)|, |E'''_k(\xi)|, |I'''_k(\xi)| \leq M, \quad \forall \xi \in \mathbb{R}.$$

Thus $(S_k, E_k, I_k), (S'_k, E'_k, I'_k)$ and (S''_k, E''_k, I''_k) are equicontinuous and uniformly bounded on $\mathbb{R}$. By the Ascoli–Arzelà theorem it follows that there exists a subsequence of $\{\tau_k\}$, still denoted by $\{\tau_k\}$, such that, as $k \rightarrow +\infty$,

$$(S_k, E_k, I_k) \rightarrow (S, E, I), \quad (S'_k, E'_k, I'_k) \rightarrow (S', E', I'), \quad (S''_k, E''_k, I''_k) \rightarrow (S'', E'', I'')$$

uniformly on every bounded closed interval and pointwise on $\mathbb{R}$. Noting that $I_k(\xi)$ is bounded on $\mathbb{R}$, we have

$$I_k(\xi - x - ct, t)K(x, t) \leq (S_0 - S_\infty)K(x, t), \quad \forall \xi \in \mathbb{R}, \forall x \in \mathbb{R}, \forall t \in \mathbb{R}^+, \forall k \in \mathbb{N}.$$

By Lebesgue's dominated convergence theorem and the continuity of F we get that

$$\lim_{k \rightarrow +\infty} F(S_k(\xi), [I_k * K](\xi)) = F(S(\xi), [I * K](\xi)).$$

Letting $k \rightarrow +\infty$ in (2.1)–(2.3), we have

$$\begin{aligned} cS'(\xi) &= d_1 S''(\xi) - F(S(\xi), [I * K](\xi)), \\ cE'(\xi) &= d_2 E''(\xi) + F(S(\xi), [I * K](\xi)) - \alpha E(\xi), \\ cI'(\xi) &= d_3 I''(\xi) + \alpha E(\xi) - \gamma I(\xi). \end{aligned}$$

Hence (S, E, I) is a solution of system (1.25)–(1.27) satisfying the asymptotic boundary conditions (1.24) and satisfies

$$\int_{-\infty}^{+\infty} \alpha E(\xi) d\xi = \int_{-\infty}^{+\infty} \gamma I(\xi) d\xi = \int_{-\infty}^{+\infty} F(S(\xi), [I * K](\xi)) d\xi = c(S_0 - S_\infty).$$

Finally, we show the nontriviality of solution (S, E, I) . We notice that there exist $M_1, M_2, \epsilon_1, \epsilon_2$ as in Lemma 2.2, independent of k . Applying Lemma 2.2 to the auxiliary system (2.1)–(2.3), we get that there exists a nonnegative uniform lower bound for $E_k(x)$, $k \in \mathbb{N}$, which implies the nontriviality of $E(\xi)$. The nontriviality of E ensures the nontriviality of (S, E, I) . This ends our proof. $\square$

3 Nonexistence of traveling waves

Theorem 3.1 *If $R_0 = \frac{\partial_t F(S_0, 0)}{\gamma} \leq 1$, then there does not exist a nontrivial nonnegative traveling wave solution (S, E, I) of system (1.21)–(1.23) satisfying the asymptotic boundary conditions (1.24).*

Proof Integrating (1.26) and (1.27) on $\mathbb{R}$ yields

$$c \int_{-\infty}^{+\infty} E'(\xi) d\xi = d_2 \int_{-\infty}^{+\infty} E''(\xi) d\xi + \int_{-\infty}^{+\infty} F(S(\xi), [I(E) * K](\xi)) d\xi - \alpha \int_{-\infty}^{+\infty} E(\xi) d\xi$$

and

$$c \int_{-\infty}^{+\infty} I'(\xi) d\xi = d_3 \int_{-\infty}^{+\infty} I''(\xi) d\xi + \alpha \int_{-\infty}^{+\infty} E(\xi) d\xi - \gamma \int_{-\infty}^{+\infty} I(\xi) d\xi.$$

From the asymptotic boundary conditions

$$E(\pm\infty) = I(\pm\infty) = E'(\pm\infty) = I'(\pm\infty) = 0$$

we have

$$\begin{aligned} \gamma \int_{-\infty}^{+\infty} I(\xi) d\xi &= \int_{-\infty}^{+\infty} F(S(\xi), [I(E) * K](\xi)) d\xi < \partial_t F(S_0, 0) \int_{-\infty}^{+\infty} [I(E) * K](\xi) d\xi \\ &\leq \gamma \int_{-\infty}^{+\infty} I(\xi) d\xi. \end{aligned}$$

Here we used the fact that $I(\xi)$ is a nontrivial nonnegative function. We get a contradiction. $\square$

Theorem 3.2 *If $R_0 = \frac{\partial_t F(S_0, 0)}{\gamma} > 1$ and $c \in (0, c^*)$, then there does not exist a nontrivial and nonnegative traveling wave solution (S, E, I) of system (1.21)–(1.23) satisfying the asymptotic boundary conditions (1.24).*

Proof Suppose on the contrary that $(S(x+ct), E(x+ct), I(x+ct))$ is a traveling wave solution of (1.21)–(1.23) satisfying the asymptotic boundary conditions (1.24). Since $\partial_t F(S_0, 0)/\gamma > 1$, there exists a small constant $\delta_0 > 0$ such that

$$\frac{F(S, I)}{I} \geq \frac{\partial_t F(S_0, 0) + \gamma}{2}, \quad 0 < I < \delta_0, S_0 - \delta_0 < S < S_0 + \delta_0.$$

In view of

$$S(\xi) \rightarrow S_0, \quad [I * K](\xi) \rightarrow 0 \quad \text{as } \xi \rightarrow -\infty,$$

there exists $\tilde{\xi} < 0$ such that

$$S_0 - \delta_0 < S(\xi) < S_0 + \delta_0 \quad \text{and} \quad 0 \leq [I * K](\xi) < \delta_0$$

for all $\xi \leq \tilde{\xi}$. Then, for any $\xi \leq \tilde{\xi}$, we have

$$cE'(\xi) - d_2E''(\xi) = F(S(\xi), [I(E) * K](\xi)) - \alpha E(\xi)$$

$$> \frac{\partial_I F(S_0, 0) + \gamma}{2} [I(E) * K](\xi) - \alpha E(\xi). \quad (3.1)$$

From $\lim_{\xi \rightarrow -\infty} I(\xi)/E(\xi) = \alpha/\gamma$ we obtain

$$\lim_{\xi \rightarrow -\infty} \frac{\frac{\partial_I F(S_0, 0) + \gamma}{2} I(\xi) - \alpha E(\xi)}{E(\xi)} = \frac{(\partial_I F(S_0, 0) - \gamma)\alpha}{2\gamma}.$$

Hence we can choose $\tilde{\xi}$ sufficiently small such that for all $\xi \leq \tilde{\xi}$,

$$\begin{aligned} cE'(\xi) &> d_2 E''(\xi) + \frac{\partial_I F(S_0, 0) + \gamma}{2} \{[I(E) * K](\xi) - [I(E)](\xi)\} \\ &\quad + \frac{(\partial_I F(S_0, 0) - \gamma)\alpha}{4\gamma} E(\xi). \end{aligned} \quad (3.2)$$

An integration of (3.2) gives

$$\begin{aligned} &\frac{(\partial_I F(S_0, 0) - \gamma)\alpha}{4\gamma} \int_{-\infty}^{\xi} E(x) \, dx \\ &\leq cE(\xi) - d_2 E'(\xi) - \frac{\partial_I F(S_0, 0) + \gamma}{2} \left[\int_{-\infty}^{\xi} [I(E) * K](x) \, dx - \int_{-\infty}^{\xi} I(E)(x) \, dx \right] \end{aligned} \quad (3.3)$$

for all $\xi \leq \tilde{\xi}$. Let

$$\mathcal{J}(\xi) := \int_{-\infty}^{\xi} E(x) \, dx, \quad \mathcal{L}(\xi) := \int_{-\infty}^{\xi} I(x) \, dx$$

for $\xi \in \mathbb{R}$. It is easy to see that $\mathcal{J}(\xi)$ and $\mathcal{L}(\xi)$ are nondecreasing and $\lim_{\xi \rightarrow -\infty} \mathcal{J}(\xi) = \lim_{\xi \rightarrow -\infty} \mathcal{L}(\xi) = 0$. By computation we obtain

$$\begin{aligned} \int_{-\infty}^{\xi} [I(E) * K](x) \, dx &= \int_{-\infty}^{\xi} \int_0^{+\infty} \int_{-\infty}^{+\infty} I(x - y - cs) K(y, s) \, dy \, ds \, dx \\ &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{\xi} I(x - y - cs) K(y, s) \, dx \, dy \, ds \\ &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \mathcal{L}(\xi - y - cs) K(y, s) \, dy \, ds = (\mathcal{L} * K)(\xi). \end{aligned}$$

Then (3.3) becomes

$$\frac{(\partial_I F(S_0, 0) - \gamma)\alpha}{4\gamma} \mathcal{J}(\xi) \leq cE(\xi) - d_2 E'(\xi) - \frac{\partial_I F(S_0, 0) + \gamma}{2} [(\mathcal{L} * K)(\xi) - \mathcal{L}(\xi)]. \quad (3.4)$$

Integrating (3.4) from $-\infty$ to $\xi \leq \tilde{\xi}$, we have

$$\begin{aligned} &\frac{(\partial_I F(S_0, 0) - \gamma)\alpha}{4\gamma} \int_{-\infty}^{\xi} \mathcal{J}(x) \, dx \\ &\leq c\mathcal{J}(\xi) - d_2 E(\xi) - \frac{\partial_I F(S_0, 0) + \gamma}{2} \int_{-\infty}^{\xi} [(\mathcal{L} * K)(x) - \mathcal{L}(x)] \, dx. \end{aligned} \quad (3.5)$$

By calculation we get

$$\begin{aligned}
 & \int_{-\infty}^{\xi} [(\mathcal{L} * K)(x) - \mathcal{L}(x)] dx \\
 &= \lim_{z \rightarrow -\infty} \int_z^{\xi} [(\mathcal{L} * K)(x) - \mathcal{L}(x)] dx \\
 &= \lim_{z \rightarrow -\infty} \int_z^{\xi} \int_0^{+\infty} \int_{-\infty}^{+\infty} K(y, s) [\mathcal{L}(x - y - cs) - \mathcal{L}(x)] dy ds dx \\
 &= - \lim_{z \rightarrow -\infty} \int_z^{\xi} \int_0^{+\infty} \int_{-\infty}^{+\infty} (y + cs) K(y, s) \int_0^1 \mathcal{L}'(x - (y + cs)\theta) d\theta dy ds dx \\
 &= - \int_0^{+\infty} \int_{-\infty}^{+\infty} (y + cs) K(y, s) \int_0^1 \mathcal{L}(\xi - (y + cs)\theta) d\theta dy ds.
 \end{aligned}$$

Thus (3.5) is equivalent to

$$\begin{aligned}
 & \frac{(\partial_t F(S_0, 0) - \gamma)\alpha}{4\gamma} \int_{-\infty}^{\xi} \mathcal{J}(x) dx \\
 & \leq c\mathcal{J}(\xi) - d_2 E(\xi) \\
 & \quad + \frac{\partial_t F(S_0, 0) + \gamma}{2} \int_0^{+\infty} \int_{-\infty}^{+\infty} (y + cs) K(y, s) \int_0^1 \mathcal{L}(\xi - (y + cs)\theta) d\theta dy ds.
 \end{aligned}$$

Since $(y + cs)\mathcal{L}(\xi - (y + cs)\theta)$ is nonincreasing on $\theta \in [0, 1]$ and $K(-x, t) = K(x, t)$, we have

$$\begin{aligned}
 & \frac{(\partial_t F(S_0, 0) - \gamma)\alpha}{4\gamma} \int_{-\infty}^{\xi} \mathcal{J}(x) dx + d_2 E(\xi) \\
 & \leq c\mathcal{J}(\xi) + \frac{\partial_t F(S_0, 0) + \gamma}{2} \int_0^{+\infty} \int_{-\infty}^{+\infty} (y + cs) K(y, s) \mathcal{L}(\xi) dy ds \\
 & = c\mathcal{J}(\xi) + \frac{\partial_t F(S_0, 0) + \gamma}{2} \int_0^{+\infty} \int_{-\infty}^{+\infty} cs K(y, s) dy ds \mathcal{L}(\xi) \\
 & \leq c \left(1 + \frac{\alpha(\partial_t F(S_0, 0) + \gamma) \int_0^{+\infty} \int_{-\infty}^{+\infty} s K(y, s) dy ds}{\gamma} \right) \mathcal{J}(\xi) := M\mathcal{J}(\xi) \tag{3.6}
 \end{aligned}$$

for all $\xi \leq \tilde{\xi}$. Here we used the inequality $\mathcal{L}(\xi) \leq 2\alpha\mathcal{J}(\xi)/\gamma$.

In fact, by (2.5) we obtain that $\lim_{\xi \rightarrow -\infty} I(\xi)/E(\xi) = \alpha/\gamma$. Thus we can choose $\tilde{\xi}$ small enough such that $I(\xi)/E(\xi) \leq 2\alpha/\gamma$ for all $\xi \leq \tilde{\xi}$. It follows that

$$\mathcal{L}(\xi) \leq \int_{-\infty}^{\xi} I(x) dx \leq \frac{2\alpha}{\gamma} \int_{-\infty}^{\xi} E(x) dx = \frac{2\alpha}{\gamma} \mathcal{J}(\xi)$$

for all $\xi \leq \tilde{\xi}$. Since $\mathcal{J}$ is nondecreasing, we have

$$\frac{(\partial_t F(S_0, 0) - \gamma)\alpha}{4\gamma} \chi \mathcal{J}(\xi - \chi) + d_2 E(\xi) \leq M\mathcal{J}(\xi)$$

for all $\chi > 0$ and $\xi \leq \tilde{\xi}$. We choose $\chi > \frac{8\gamma M}{\alpha(\partial_t F(S_0, 0) - \gamma)}$. Then

$$\mathcal{J}(\xi - \chi) < \frac{1}{2} \mathcal{J}(\xi)$$

for all $\xi \leq \tilde{\xi}$. Define $\mu_0 := \min\{\frac{\ln 2}{\eta}, \frac{\lambda^+}{2}, \frac{\lambda^*}{2}\}$ and $\tilde{\mathcal{J}}(\xi) := \mathcal{J}(\xi)e^{-\mu_0\xi}$. Then

$$\tilde{\mathcal{J}}(\xi - \chi) = \mathcal{J}(\xi - \chi)e^{-\mu_0(\xi - \chi)} < \frac{1}{2}e^{\mu_0\chi} \mathcal{J}(\xi)e^{-\mu_0\xi} = \tilde{\mathcal{J}}(\xi)$$

for all $\xi \leq \tilde{\xi}$, which implies that $\tilde{\mathcal{J}}(\xi)$ is bounded as $\xi \rightarrow -\infty$. Since $\tilde{\mathcal{J}}(\xi) \rightarrow 0$ as $\xi \rightarrow +\infty$, there exists a constant $\tilde{C} > 0$ such that

$$\tilde{\mathcal{J}}(\xi) \leq \tilde{C}, \quad \forall \xi \in \mathbb{R},$$

which yields

$$\mathcal{J}(\xi) \leq \tilde{C}e^{\mu_0\xi}, \quad \forall \xi \in \mathbb{R}.$$

Consequently, there exists a constant $\hat{C}$ such that $\int_{-\infty}^{\xi} \mathcal{J}(x) dx \leq \hat{C}e^{\mu_0\xi}$ for all $\xi \in \mathbb{R}$. Furthermore, from (3.2), (3.4), and (3.6) we get

$$\sup_{\xi \in \mathbb{R}} \{E(\xi)e^{-\mu_0\xi}\} < +\infty, \quad \sup_{\xi \in \mathbb{R}} \{|E'(\xi)|e^{-\mu_0\xi}\} < +\infty, \quad \sup_{\xi \in \mathbb{R}} \{|E''(\xi)|e^{-\mu_0\xi}\} < +\infty.$$

It follows from (2.5) that $\lim_{\xi \rightarrow \pm\infty} I(\xi)/E(\xi) = \alpha/\gamma$. Thus we have

$$\sup_{\xi \in \mathbb{R}} \{I(\xi)e^{-\mu_0\xi}\} < +\infty \quad \text{and} \quad \sup_{\xi \in \mathbb{R}} \{[I * K](\xi)e^{-\mu_0\xi}\} < +\infty.$$

Since

$$\begin{aligned} & -d_2 E''(\xi) + cE'(\xi) + \alpha E(\xi) - \partial_I F(S_0, 0)[I(E) * K](\xi) \\ & = F(S(\xi), [I(E) * K](\xi)) - \partial_I F(S_0, 0)[I(E) * K](\xi), \end{aligned}$$

applying the two-sided Laplace transform to both sides of the above equality yields

$$\begin{aligned} & \Delta(\mu, c) \int_{-\infty}^{+\infty} e^{-\mu\xi} E(\xi) d\xi \\ & = - \int_{-\infty}^{+\infty} e^{-\mu\xi} [\partial_I F(S_0, 0)[I(E) * K](\xi) - F(S(\xi), [I(E) * K](\xi))] d\xi. \end{aligned} \quad (3.7)$$

Here we used the fact that

$$\begin{aligned} & \int_{-\infty}^{+\infty} e^{-\mu\xi} [I * K](\xi) d\xi \\ & = \int_{-\infty}^{+\infty} e^{-\mu\xi} \int_0^{+\infty} \int_{-\infty}^{+\infty} K(y, s) I(\xi - y - cs) dy ds d\xi \\ & = \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\mu(y+cs)} K(y, s) \int_{-\infty}^{+\infty} e^{-\mu(\xi-y-cs)} I(\xi - y - cs) d\xi dy ds \\ & = \mathcal{R}(\mu, c) \int_{-\infty}^{+\infty} e^{-\mu\xi} I(\xi) d\xi, \end{aligned}$$

where

$$\begin{aligned}
 \int_{-\infty}^{+\infty} e^{-\mu\xi} I(\xi) d\xi &= \int_{-\infty}^{+\infty} e^{-\mu\xi} I(E)(\xi) d\xi \\
 &= \frac{\alpha}{\rho} \int_{-\infty}^{+\infty} e^{-\mu\xi} \left(\int_{-\infty}^{\xi} e^{\lambda^-(\xi-x)} E(x) dx + \int_{\xi}^{+\infty} e^{\lambda^+(\xi-x)} E(x) dx \right) d\xi \\
 &= \frac{\alpha}{\rho} \int_{-\infty}^{+\infty} e^{-\mu\xi} \left(\int_0^{+\infty} e^{\lambda^-x} E(\xi-x) dx + \int_{-\infty}^0 e^{\lambda^+x} E(\xi-x) dx \right) d\xi \\
 &= \frac{\alpha}{\rho} \int_{-\infty}^{+\infty} e^{-\mu\xi} E(\xi) d\xi \left(\int_0^{+\infty} e^{(\lambda^--\mu)x} dx + \int_{-\infty}^0 e^{(\lambda^+-\mu)x} dx \right) \\
 &= \frac{\alpha}{-d_3\mu^2 + c\mu + \gamma} \int_{-\infty}^{+\infty} e^{-\mu\xi} E(\xi) d\xi.
 \end{aligned}$$

The integrals on both sides of (3.7) are well defined for any $\mu \in (0, \mu_0)$. By the assumption that $0 < c < c^*$, $\Delta(\mu, c)$ is always negative for all $\mu \in (0, \min\{\lambda^+, \lambda^c\})$. The integrals in (3.7) can be analytically continued to the interval $[0, \min\{\lambda^+, \lambda^c\})$. Otherwise, by the theory of convergence region of the two-side Laplace transform (see [5, 27, 28]) the integral $\int_{-\infty}^{+\infty} e^{-\mu\xi} E(\xi) d\xi$ has a singularity at $\mu = \mu^* \in (0, \min\{\lambda^+, \lambda^c\})$ and is analytic for all $\mu < \mu^*$. On the other hand, since $S(-\infty) = S_0$ and $[I(E) * K](-\infty) = 0$, from Lemma 2.2 we know that, as $\xi \rightarrow -\infty$,

$$\partial_I F(S_0, 0)[I(E) * K](\xi) - F(S(\xi), [I(E) * K](\xi)) \leq [I(E) * K]^2(\xi).$$

Then, as $\xi \rightarrow -\infty$, we have

$$\begin{aligned}
 &e^{-2\mu_0\xi} [\partial_I F(S_0, 0)[I(E) * K](\xi) - F(S(\xi), [I(E) * K](\xi))] \\
 &\leq e^{-2\mu_0\xi} [I(E) * K]^2(\xi) \leq \left[\sup_{\xi \in \mathbb{R}} e^{-\mu_0\xi} [I(E) * K](\xi) \right]^2 < +\infty.
 \end{aligned}$$

Note that $e^{-2\mu_0\xi} [\partial_I F(S_0, 0)[I(E) * K](\xi) - F(S(\xi), [I(E) * K](\xi))]$ is bounded as $\xi \rightarrow +\infty$. Hence we obtain

$$\sup_{\xi \in \mathbb{R}} \{ \partial_I F(S_0, 0)[I(E) * K](\xi) - F(S(\xi), [I(E) * K](\xi)) \} < +\infty.$$

It follows that the integrals in (3.7) are analytic for any $\mu < \min\{\mu_0 + \mu^*, \lambda^+, \lambda^c\}$, which is a contradiction. We rewrite equality (3.7) as

$$\int_{-\infty}^{+\infty} e^{-\mu\xi} [\Delta(\mu, c)E(\xi) + \partial_I F(S_0, 0)[I(E) * K](\xi) - F(S(\xi), [I(E) * K](\xi))] d\xi = 0.$$

However, for $c \in (0, c^*)$, we have that

$$\Delta(\mu, c) \rightarrow -\infty \quad \text{as } \mu \rightarrow \min\{\lambda^+, \lambda^c\} - 0,$$

which implies that the last equality is false, a contradiction. Thus we get the desired results. $\square$

4 Examples

Example 4.1 If $F(S, I) = \beta S^p I$ where $p > 0$, system (1.1)–(1.4) is rewritten as

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - \beta S^p(x, t) \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x - y, t - s) I(y, s) dy ds, \quad (4.1)$$

$$\partial_t E(x, t) = d_2 \partial_{xx} E(x, t) + \beta S^p(x, t) \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x - y, t - s) I(y, s) dy ds - \alpha E(x, t), \quad (4.2)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + \alpha E(x, t) - \gamma I(x, t), \quad (4.3)$$

$$\partial_t R(x, t) = d_4 \partial_{xx} R(x, t) + \gamma I(x, t). \quad (4.4)$$

It is easy to see that (A1)–(A2) and (H1)–(H4) hold. By Theorems 2.2, 3.1, and 3.2 we obtain the following results.

Theorem 4.1 *There exists a positive constant c^* such that*

(i) *if $R_0 = \frac{\beta S_0^p}{\gamma} > 1$ and $c > c^*$, then system (4.1)–(4.3) has a nontrivial nonnegative traveling wave solution (S, E, I) satisfying the asymptotic boundary conditions*

$$S(-\infty) = S_0, \quad S(+\infty) = S_\infty < S_0, \quad E(\pm\infty) = I(\pm\infty) = 0. \quad (4.5)$$

Furthermore, S is decreasing, $0 \leq E(\xi) \leq S_0 - S_\infty$ and $0 \leq I(\xi) \leq S_0 - S_\infty$ for all $\xi \in \mathbb{R}$, and

$$\int_{-\infty}^{+\infty} \alpha E(\xi) d\xi = \int_{-\infty}^{+\infty} \gamma I(\xi) d\xi = \int_{-\infty}^{+\infty} \beta S^p(\xi) [I * K](\xi) d\xi = c(S_0 - S_\infty),$$

where $[I * K](\xi) = \int_0^{+\infty} \int_{-\infty}^{+\infty} I(\xi - y - cs) K(y, s) dy ds$.

(ii) *if either $R_0 = \frac{\beta S_0^p}{\gamma} > 1$ and $c \in (0, c^*)$ or $R_0 \leq 1$, then there exists no nontrivial nonnegative traveling wave solution (S, E, I) satisfying the asymptotic boundary conditions (4.5).*

Remark 4.1 For the case $p = 1$ and $K(x, t) = \hat{K}(x)\delta(t)$, Tian and Yuan [23] established the traveling wave solutions of system (4.1)–(4.4) with Laplacian diffusion.

Example 4.2 If $F(S, I) = \frac{\beta SI}{1 + \zeta I^q}$, then system (1.1)–(1.4) can be rewritten as

$$\partial_t S(x, t) = d_1 \partial_{xx} S(x, t) - \frac{\beta S \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x - y, t - s) I(y, s) dy ds}{1 + \zeta [\int_{-\infty}^t \int_{-\infty}^{+\infty} K(x - y, t - s) I(y, s) dy ds]^q}, \quad (4.6)$$

$$\partial_t E(x, t) = d_2 \partial_{xx} E(x, t) + \frac{\beta S \int_{-\infty}^t \int_{-\infty}^{+\infty} K(x - y, t - s) I(y, s) dy ds}{1 + \zeta [\int_{-\infty}^t \int_{-\infty}^{+\infty} K(x - y, t - s) I(y, s) dy ds]^q} - \alpha E(x, t), \quad (4.7)$$

$$\partial_t I(x, t) = d_3 \partial_{xx} I(x, t) + \alpha E(x, t) - \gamma I(x, t), \quad (4.8)$$

$$\partial_t R(x, t) = d_4 \partial_{xx} R(x, t) + \gamma I(x, t). \quad (4.9)$$

Obviously, (A1)–(A2) and (H1)–(H4) hold when $\zeta > 0$ and $q > 1$. Applying Theorems 2.2, 3.1, and 3.2 we have the following theorem.

Theorem 4.2 *There exists a positive constant c^* such that*

(i) if $R_0 = \frac{\beta S_0}{\gamma} > 1$ and $c > c^*$, then system (4.6)–(4.9) has a nontrivial and nonnegative traveling wave solution (S, E, I, R) satisfying the asymptotic boundary conditions (4.5). Furthermore, S is decreasing, $0 \leq E(\xi) \leq S_0 - S_\infty$ and $0 \leq I(\xi) \leq S_0 - S_\infty$ for all $\xi \in \mathbb{R}$, and

$$\int_{-\infty}^{+\infty} \alpha E(\xi) d\xi = \int_{-\infty}^{+\infty} \gamma I(\xi) d\xi = \int_{-\infty}^{+\infty} \frac{\beta S(\xi)[I * K](\xi)}{1 + \zeta[I * K]^q(\xi)} d\xi = c(S_0 - S_\infty),$$

where $[I * K](\xi) = \int_0^{+\infty} \int_{-\infty}^{+\infty} I(\xi - y - cs)K(y, s) dy ds$.

(ii) if either $R_0 = \frac{\beta S_0}{\gamma} > 1$ and $c \in (0, c^*)$ or $R_0 \leq 1$, then there exists no nontrivial nonnegative traveling wave solution (S, E, I) satisfying the asymptotic boundary conditions (4.5).

Remark 4.2 For the case $K(x, t) = \delta(x)\delta(t)$ and $q = 1$, that is, saturated incidence, Xu [30] considered the traveling wave solutions of system (4.6)–(4.9) with the assumption $d_2 = d_3 = d$ and obtained similar results as in Theorem 4.2.

5 Numerical simulations and discussion

In this paper, we consider a diffusive SEIR model with nonlocal delayed transmission and a general nonlinear incidence rate. By using the Schauder fixed point theorem and two-side Laplace transform we prove the existence and nonexistence of traveling wave solutions in terms of R_0 and c^* . The results show that c^* is the minimal wave speed, but it requires further research to show that c^* is the asymptotic speed of propagation. This minimal wave speed c^* is defined by Lemma 2.1, from which it is easy to see that c^* depends on the diffusion rate d_2 of the exposed individuals, the diffusion rate d_3 of the infected individuals, the pattern of nonlocal interaction between the susceptible and infected individuals, and the latent period of disease. For $c > 0$ and $\lambda \in [0, \min\{\lambda^c, \lambda^+\})$, c^* is determined by the following equations:

$$\begin{aligned} \Delta(\lambda, c) &= -d_2\lambda^2 + c\lambda - \frac{\alpha \partial_I F(S_0, 0) \mathcal{R}(\lambda, c)}{-d_3\lambda^2 + c\lambda + \gamma} + \alpha, \\ \partial_c \Delta(\lambda, c) &= \lambda + \frac{\alpha \lambda \partial_I F(S_0, 0) \int_0^{+\infty} \int_{-\infty}^{+\infty} s e^{-\lambda(y+cs)} K(y, s) dy ds}{-d_3\lambda^2 + c\lambda + \gamma} + \frac{\alpha \lambda \partial_I F(S_0, 0) \mathcal{R}(\lambda, c)}{(-d_3\lambda^2 + c\lambda + \gamma)^2} > 0, \end{aligned}$$

where

$$\mathcal{R}(\lambda, c) := \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{-\lambda(y+cs)} K(y, s) dy ds.$$

Choosing $K(x, t) = \delta(x)\delta(t - \tau)$ with $\tau > 0$, we have

$$\begin{aligned} \Delta_1(\lambda, c, d_2, d_3, \tau) &= -d_2\lambda^2 + c\lambda - \frac{\alpha \partial_I F(S_0, 0) e^{-\lambda c \tau}}{-d_3\lambda^2 + c\lambda + \gamma} + \alpha, \\ \partial_\tau \Delta_1(\lambda, c, d_2, d_3, \tau) &= \frac{\alpha \lambda c \partial_I F(S_0, 0) e^{-\lambda c \tau}}{-d_3\lambda^2 + c\lambda + \gamma} > 0, \end{aligned}$$

and

$$\partial_{d_2} \Delta_1(\lambda, c, d_2, d_3, \tau) = -\lambda^2 < 0, \quad \partial_{d_3} \Delta_1(\lambda, c, d_2, d_3, \tau) = -\frac{\alpha \lambda^2 \partial_I F(S_0, 0) e^{-\lambda c \tau}}{(-d_3\lambda^2 + c\lambda + \gamma)^2} < 0.$$

Taking $K(x, t) = \frac{1}{\sqrt{4\pi\rho}} e^{-\frac{x^2}{4\rho}} \delta(t)$, we have

$$\Delta_2(\lambda, c, d_2, d_3, \rho) = -d_2\lambda^2 + c\lambda - \frac{\alpha \partial_I F(S_0, 0) e^{\rho\lambda^2}}{-d_3\lambda^2 + c\lambda + \gamma} + \alpha,$$

$$\partial_\rho \Delta_2(\lambda, c, d_2, d_3, \rho) = -\frac{\alpha \lambda^2 \partial_I F(S_0, 0) e^{\rho\lambda^2}}{-d_3\lambda^2 + c\lambda + \gamma} < 0,$$

and

$$\partial_{d_2} \Delta_2(\lambda, c, d_2, d_3, \rho) = -\lambda^2 < 0, \quad \partial_{d_3} \Delta_2(\lambda, c, d_2, d_3, \tau) = -\frac{\alpha \lambda^2 \partial_I F(S_0, 0) e^{\rho\lambda^2}}{(-d_3\lambda^2 + c\lambda + \gamma)^2} < 0.$$

By calculation we get

$$\begin{aligned} & \frac{dc_1^*(d_2, d_3, \tau)}{d\tau} \\ &= \frac{\alpha \lambda_0 c_1^* \partial_I F(S_0, 0) e^{-\lambda_0 c_1^* \tau} \mathcal{A}(\lambda_0, c_1^*)}{\lambda_0 \mathcal{A}^2(\lambda_0, c_1^*) + \alpha \lambda_0^2 \tau \partial_I F(S_0, 0) e^{-\lambda_0 c_1^* \tau} \mathcal{A}(\lambda_0, c_1^*) + \alpha \lambda_0 \partial_I F(S_0, 0) \mathcal{R}(\lambda_0, c_1^*)} < 0, \\ & \frac{dc_1^*(d_2, d_3, \tau)}{dd_2} \\ &= -\frac{\lambda_0^2 \mathcal{A}^2(\lambda_0, c_1^*)}{\lambda_0 \mathcal{A}^2(\lambda_0, c_1^*) + \alpha \lambda_0^2 \tau \partial_I F(S_0, 0) e^{-\lambda_0 c_1^* \tau} \mathcal{A}(\lambda_0, c_1^*) + \alpha \lambda_0 \partial_I F(S_0, 0) \mathcal{R}(\lambda_0, c_1^*)} > 0, \\ & \frac{dc_1^*(d_2, d_3, \tau)}{dd_3} \\ &= -\frac{\alpha \lambda_0^2 \partial_I F(S_0, 0) e^{-\lambda_0 c_1^* \tau}}{\lambda_0 \mathcal{A}^2(\lambda_0, c_1^*) + \alpha \lambda_0^2 \tau \partial_I F(S_0, 0) e^{-\lambda_0 c_1^* \tau} \mathcal{A}(\lambda_0, c_1^*) + \alpha \lambda_0 \partial_I F(S_0, 0) \mathcal{R}(\lambda_0, c_1^*)} > 0, \\ & \frac{dc_2^*(d_2, d_3, \rho)}{d\rho} = \frac{\alpha \lambda_0 c_2^* \partial_I F(S_0, 0) e^{\rho\lambda_0^2} \mathcal{A}(\lambda_0, c_2^*)}{\lambda_0 \mathcal{A}^2(\lambda_0, c_2^*) + \alpha \lambda_0 \partial_I F(S_0, 0) \mathcal{R}(\lambda_0, c_2^*)} > 0, \\ & \frac{dc_2^*(d_2, d_3, \rho)}{dd_2} = -\frac{\lambda_0^2 \mathcal{A}^2(\lambda_0, c_2^*)}{\lambda_0 \mathcal{A}^2(\lambda_0, c_2^*) + \alpha \lambda_0 \partial_I F(S_0, 0) \mathcal{R}(\lambda_0, c_2^*)} > 0, \\ & \frac{dc_2^*(d_2, d_3, \rho)}{dd_3} = -\frac{\alpha \lambda_0^2 \partial_I F(S_0, 0) e^{\rho\lambda_0^2}}{\lambda_0 \mathcal{A}^2(\lambda_0, c_2^*) + \alpha \lambda_0 \partial_I F(S_0, 0) \mathcal{R}(\lambda_0, c_2^*)} > 0, \end{aligned}$$

where $\mathcal{A}(\lambda, c) = -d_3\lambda^2 + c\lambda + \gamma$. By the above argument we can obtain that the latent period of disease can slow down the speed of the disease and the nonlocal interaction between the infective and susceptible individuals and that the diffusion rate d_2 of the exposed individuals and the diffusion rate d_3 of the infected individuals can increase the speed of the spread of the disease.

To further illustrate our conclusions, we simulate how the latent period of disease, the nonlocal interaction between the infective and susceptible individuals and the spatial movement of expositive individuals and infective individuals affect the speed of the spread of the disease.

Define $K(x, t) = \mathcal{K}(x) \delta(t - \tau)$. Then

$$\mathcal{R}(\lambda, c) = e^{-\lambda c \tau} G(\lambda) \quad \text{and} \quad G(\lambda) = \int_{-\infty}^{+\infty} \mathcal{K}(x) e^{-\lambda x} dx.$$

Figure 1 The curves of $c^*(k)$. Let $F(S, I) = \beta S * I$, $\alpha = 1$, $\beta = 3$, $S_0 = 1$, $\gamma = 1.5m$ and $\tau = 1$ be fixed. Additionally, $d_2 = 4$ and $d_3 = 7$. We can see that the minimal wave speed c^* of the first, second, and third models converges to the minimal wave speed of the local reaction case as $k = \frac{1}{\rho} \rightarrow \infty$

First Model: $\mathcal{K}$ is of exponential decay, satisfying

$$\mathcal{K}(x) = \frac{ke^{-|x|/\rho}}{2\rho},$$

and for $\lambda \in [0, 1/\rho)$,

$$G(\lambda) = \frac{1}{1 - \rho^2 \lambda^2} \quad \text{and} \quad \mathcal{R}(\lambda, c) = \frac{e^{-\lambda c \tau}}{1 - \rho^2 \lambda^2}.$$

Second Model: $\mathcal{K}$ is of compact support, with linear decay, satisfying

$$\mathcal{K}(x) = \begin{cases} (\rho - |x|)/\rho^2, & |x| \leq \rho, \\ 0, & |x| > \rho. \end{cases}$$

By simple calculation we have

$$G(\lambda) = \frac{1}{\lambda^2 \rho^2} (e^{\rho \lambda} + e^{-\rho \lambda} - 2) \quad \text{and} \quad \mathcal{R}(\lambda, c) = \frac{e^{-\lambda c \tau}}{\lambda^2 \rho^2} (e^{\rho \lambda} + e^{-\rho \lambda} - 2).$$

Third Model: $\mathcal{K}$ is Gaussian distribution $N(0, \rho)$, that is,

$$\mathcal{K}(x) = \frac{1}{\sqrt{2\pi\rho}} e^{-\frac{x^2}{2\rho^2}}.$$

By simple calculation we obtain

$$G(\lambda) = e^{\sqrt{2\lambda}\rho} \quad \text{and} \quad \mathcal{R}(\lambda, c) = e^{\sqrt{2\lambda}\rho - \lambda c \tau}.$$

Figure 2 The curves of $c^*(\tau)$. Let $F(S, I) = \beta S * I$, $\alpha = 1$, $\beta = 3$, $S_0 = 1$, $\gamma = 1.5$, and $k = 0.5$ be fixed. Additionally, $d_2 = 4$ and $d_3 = 7$. We can see that the minimal wave speed c^* of the first, second, and third models decreases as $\tau \rightarrow \infty$

Figure 3 The surface of $c^*(d_2, d_3)$ for the first model. Let $\alpha = 1$, $\beta = 3$, $S_0 = 1$, $\gamma = 1.5$, and $\tau = 1$ be fixed. We can see that the minimal wave speed c^* increases with respect to d_2 and d_3 . Moreover, three surfaces from top to bottom are generated by taking $\rho = 3, 2, 0.2$, respectively

Fourth Model: The degenerate case. $\mathcal{K}$ is the Dirac delta function, satisfying

$$\mathcal{K}(x) = \delta(x), \quad G(\lambda) = 1 \quad \text{and} \quad \mathcal{R}(\lambda, c) = e^{-\lambda c \tau}.$$

We can see that in this case, the nonlocal reaction term degenerates to the local reaction.

Remark 5.1 We notice that $G(\lambda) \geq 1$ always holds in the above four models. Moreover, for a fixed λ , $G(\lambda)$ increases with respect to ρ , and $\lim_{\rho \rightarrow 0} G(\lambda) = 1$, whereas $K(x) \rightarrow \delta(x)$ as $\rho \rightarrow 0$, and $G(\lambda)$ measures the nonlocality of the reaction term.

Figure 4 The surface of $c^*(d_2, d_3)$ for the second model. Let $\alpha = 1$, $\beta = 3$, $S_0 = 1$, $k = 0.2$, and $\gamma = 1.5$ be fixed. Three surfaces from bottom to top are generated by taking $\tau = 4, 1, 0.2$, respectively. We can see that the minimal wave speed c^* increases with respect to d_2 and d_3 . Moreover, the surface of c^* decreases as τ increases

Figure 5 The surface of $c^*(d_2, d_3)$ for the third model. Let $\alpha = 1$, $\beta = 3$, $S_0 = 1$, $\gamma = 1.5$, and $\tau = 1$ be fixed. Three surfaces from top to bottom are generated by taking $\rho = 3, 2, 0.2$, respectively. We can see that the surface of c^* increases as ρ increases

Remark 5.2 By numerical calculation we show how the nonlocal interaction between the infective and susceptible individuals affects the minimal wave speed c^* ; see Fig. 1.

Remark 5.3 We show how the latent period of disease affects the minimal wave speed c^* in Fig. 2.

Furthermore, we are interested in the relation between the minimal wave speed c^* and the diffusion rates of the exposed and the infected (see Fig. 3). For the second and third models, we have similar results given in Figs. 4 and 5.

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Availability of data and materials

The authors declare that all data and materials in this paper are available and veritable.

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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