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# On strong singular fractional version of the Sturm–Liouville equation

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## Abstract

The Sturm–Liouville equation is among the significant differential equations having many applications, and a lot of researchers have studied it. Up to now, different versions of this equation have been reviewed, but one of its most attractive versions is its strong singular version. In this work, we investigate the existence of solutions for the strong singular version of the fractional Sturm–Liouville differential equation with multi-points integral boundary conditions. Also, the continuity depending on coefficients of the initial condition of the equation is examined. An example is proposed to demonstrate our main result.

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## 1 Preliminaries

Although there are many different works in the field of fractional calculus via many applications (see, for example, [1–12]), some researchers like to focus on some famous differential equations. One of the well-known differential equations is the Sturm–Liouville, and so far many researchers have studied the equation. Up to now, distinct fractional differential equations and especially different versions of the Strum–Liouville equation have been reviewed (see, for example, [13–28]). On the other hand, some phenomena could be described by singular differential equations. For this reason, some researchers have tried to study different singular equations.

In 2015, the fractional problem  ${}^C\mathcal{D}^\alpha x(t) = f(t, x(t), \mathcal{D}^\beta x(t))$  with boundary value conditions  $x(0) + x'(0) = y(x)$ ,  $\int_0^1 x(t) dt = m$  and  $x''(0) = x^{(3)} = \dots = x^{(n-1)}(0) = 0$  was investigated, where  $0 < t < 1$ ,  $m$  is a real number,  $n \geq 2$ ,  $\alpha \in (n-1, n)$ ,  $0 < \beta < 1$ ,  $\mathcal{D}^\alpha$  and  $\mathcal{D}^\beta$  are the Caputo fractional derivatives,  $y \in C([0, 1], \mathbb{R})$  and  $f : (0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous with  $f(t, x, y)$  may be singular at  $t = 0$  [29]. In 2019, the fractional Sturm–Liouville differential equation  $\mathcal{D}^\alpha(\rho(t)\mathcal{D}^\beta y'(t)) + \theta(t)y(t) = h(t)\kappa(y(t))$  with boundary conditions  $y'(0) = 0$  and  $\sum_{k=1}^m \xi_k y(a_k) = y \sum_{i=1}^n \eta_i y(b_i)$  was considered, where  $\alpha \in (0, 1]$ ,  $\rho(t) \in C^1(J, \mathbb{R})$ , and  $\theta(t)$  and  $h(t)$  are absolute continuous functions on  $J = [0, T]$ ,  $T < \infty$  with  $\rho(t) \neq 0$  for all  $t \in J$ ;  $\kappa(y(t)) : \mathbb{R} \rightarrow \mathbb{R}$  is defined and differentiable on the interval  $J$ ,  $0 \leq a_1 < \dots < a_m < c$ ,  $d \leq b_1 < b_2 < \dots < b_n \leq T$  and  $\xi_k, \eta_j$  and  $v \in \mathbb{R}$  [14]. The hybrid version of this problem has

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been studied recently in [13]. From the background of the research, it became clear to us that there are different methods for solving weakly singular equations, but generally these methods are not able to solve the strongly singular case (see [30, 31]). Thus, it is very important to study the strong singular fractional differential equations with new techniques [32]. Therefore, considering the existing gap, we intend to introduce a new method for solving strongly singular equations in this research, which has not been presented so far. Regarding the main idea of the works, we examine the existence of solutions for the strong singular pointwisely defined fractional Sturm–Liouville differential equation

$$\mathcal{D}^\alpha (p(t)\mathcal{D}^\beta v(t)) + q(t)v(t) = h(t)f(v(t)) \quad (1)$$

with boundary conditions  $v^{(i)}(0) = \mathcal{D}^{(\beta+j)}v(0) = 0$  for  $0 \leq j \leq n-1$ ,  $0 \leq i \leq k-1$ , and  $\alpha v(\mu) = \sum_{i=1}^{n_0} \lambda_i \mathcal{I}^{p_i} v(a_i)$ , where  $\alpha \geq 1$ ,  $\alpha \in (n-1, n]$ ,  $\beta \in (k-1, k]$ ,  $\mu, a_i \in [0, 1]$ ,  $\alpha, \lambda_i \in \mathbb{R}$ ,  $p_i \geq 0$ ,  $q, h : [0, 1] \rightarrow \mathbb{R}$  are singular at some points  $[0, 1]$ ,  $p : [0, 1] \rightarrow [0, \infty)$  is  $n-1$  times differentiable and can be zero at some points in  $[0, 1]$ ,  $\mathcal{D}^\beta$  is the Caputo derivative of fractional order  $\beta$ , and  $\mathcal{I}^{p_i}$  is the Riemann–Liouville integral of fractional order  $p_i$ .

By carefully checking the used techniques in related works, we find that equation (1) is singular at  $t_0 \in [0, 1]$  whenever  $p(t_0) = 0$  or  $q$  or  $h$  is singular at the point  $t_0$ . Problem (1) is strong singular at the point  $t_0$  whenever at least one of the functions  $\frac{1}{p(t)}$  or  $q(t)$  or  $h(t)$  is singular at the point  $t_0$ , but is not integrable on the interval  $[0, 1]$ . In this article, we use  $\|\cdot\|_1$  for the norm of  $L^1[0, 1]$  and  $\|\cdot\|$  for the sup norm of  $X = C[0, 1]$ .

The Riemann–Liouville integral of fractional order  $v$  with the lower limit  $s \geq 0$  for a function  $g : (s, \infty) \rightarrow \mathbb{R}$  is defined by  $\mathcal{I}_s^\nu g(t) = \frac{1}{\Gamma(\nu)} \int_s^t (t-\zeta)^{\nu-1} g(\zeta) d\zeta$  provided that the right-hand side is pointwisely defined on  $(s, \infty)$ . We denote  $\mathcal{I}^\nu g(t)$  for  $\mathcal{I}_{0+}^\nu g(t)$  [33]. Also, the Caputo fractional derivative of order  $\alpha > 0$  of the function  $g$  is defined by  ${}^C\mathcal{D}^\alpha g(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{g^{(m)}(\zeta)}{(t-\zeta)^{\alpha+1-m}} d\zeta$ , where  $m = [q] + 1$  [33]. We need the following two statements to prove our main results.

**Lemma 1.1** ([34]) *Let  $m-1 < \sigma \leq m$  and  $v \in C(0, 1)$ . Then  $\mathcal{I}^\sigma D^\sigma v(t) = v(t) + \sum_{i=0}^{m-1} e_i t^i$  for some real constants  $e_0, \dots, e_{m-1}$ .*

**Lemma 1.2** ([35]) *Let  $\mathcal{C}$  be a closed and convex subset of a Banach space  $X$ ,  $\Omega$  be a relatively open subset of  $\mathcal{C}$  with  $0 \in \Omega$ , and  $\mathcal{F} : \Omega \rightarrow \mathcal{C}$  be a continuous and compact mapping. Then either*

- i) *the mapping  $\mathcal{F}$  has a fixed point in  $\bar{\Omega}$ , or*
- ii) *there exist  $y \in \partial\Omega$  and  $\lambda \in (0, 1)$  with  $y = \lambda \mathcal{F}y$ .*

## 2 Main results

We first provide our key lemma.

**Lemma 2.1** *Let  $\alpha, \beta \geq 1$ ,  $\alpha \in [n-1, n]$ ,  $\beta \in [k-1, k]$ ,  $\mu, a_i \in [0, 1]$ ,  $\alpha, \lambda_i \in \mathbb{R}$ ,  $p_i \geq 0$ , where  $\alpha \neq \sum_{i=1}^{n_0} \frac{\lambda_i p_i}{\Gamma(\beta+p_i)}$ ,  $q, h : [0, 1] \rightarrow \mathbb{R}$  may be singular at some points in  $[0, 1]$ ,  $p : [0, 1] \rightarrow [0, \infty)$  is  $n-1$  times differentiable and can be zero at some points in  $[0, 1]$ , and  $f \in L^1$ . Then a map  $v$  is a solution for the equation*

$$\mathcal{D}^\alpha (p(t)\mathcal{D}^\beta v(t)) + q(t)v(t) = h(t)f(v(t)),$$

with boundary conditions  $v^{(i)}(0) = \mathcal{D}^{(\beta+j)}v(0) = 0$  for  $0 \leq j \leq n-1$  and  $0 \leq i \leq k-1$  and  $a v(\mu) = \sum_{i=1}^{n_0} \lambda_i \mathcal{I}^{p_i} v(a_i)$  if and only if

$$\begin{aligned} v(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ &\quad + \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ &\quad - \frac{a}{\Delta \Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(s, v(\zeta)))}{p(\zeta)} d\zeta, \end{aligned}$$

where  $\Delta = a - \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(\beta + p_i)}$ ,  $A_\alpha(\zeta, v(\zeta)) = \frac{1}{\Gamma(\alpha)} \int_0^\zeta (t-\zeta)^{\alpha-1} q(\zeta) v(\zeta) d\zeta$  and

$$B_\alpha(\zeta, v(\zeta)) = \frac{1}{\Gamma(\alpha)} \int_0^\zeta (t-\zeta)^{\alpha-1} h(\zeta) f(v(\zeta)) d\zeta.$$

*Proof* By using the same strategy in [32], one can find Lemma 1.1 is valid on  $L^1[0, 1]$ . Let  $v(t)$  be a solution for the fractional boundary value problem (FBVP). Via Lemma 1.1, there are some real constants  $e_0, \dots, e_{n-1}$  such that

$$p(t) \mathcal{D}^\beta v(t) = -\mathcal{I}^\alpha(q(t)f(v(t))) + \mathcal{I}^\alpha(q(t)f(v(t))) + e_0 + e_1 t + \dots + e_{n-1} t^{n-1}.$$

Since  $\mathcal{D}^\beta v(0) = 0$ , we get  $e_0 = 0$ . Also since  $\frac{d}{dt}(\mathcal{I}^\alpha(q(t)f(v(t)))) = \mathcal{I}^{\alpha-1}(q(t)f(v(t)))$ , by derivation from the last equality, we have

$$(p(t) \mathcal{D}^\beta v(t))' |_{t=0} = -\mathcal{I}^{\alpha-1}(q(t)f(v(t)))|_{t=0} + \mathcal{I}^{\alpha-1}(q(t)f(v(t)))|_{t=0} + e_1.$$

Since  $\mathcal{I}^\alpha(q(t)f(v(t)))|_{t=0} = 0$ , it results that  $e_1 = (p'(t) \mathcal{D}^\beta v(t) + p(t) \mathcal{D}^{\beta+1} v(t))|_{t=0}$ . Thus,  $e_1 = 0$ . By continuing this way, one can check that  $e_2 = \dots = e_{n-1} = 0$  and so

$$\mathcal{D}^\beta v(t) = -\frac{1}{p(t)} \mathcal{I}^\alpha(q(t)f(v(t))) + \frac{1}{p(t)} \mathcal{I}^\alpha(q(t)f(v(t))).$$

If

$$A_\alpha(t, v(t)) = \mathcal{I}^\alpha(q(t)f(v(t)))$$

and

$$B_\alpha(t, v(t)) = \mathcal{I}^\alpha(q(t)f(v(t))),$$

then it is evolved that

$$\mathcal{D}^\beta v(t) = -\frac{A_\alpha(t, v(t))}{p(t)} + \frac{B_\alpha(t, v(t))}{p(t)}.$$

Once again for the above equality, by using Lemma 1.1, it is concluded that there are some real constants  $d_0, \dots, d_{k-1}$  such that

$$v(t) = -\mathcal{I}^\beta \left( \frac{A_\alpha(t, v(t))}{p(t)} \right) + \mathcal{I}^\beta \left( \frac{B_\alpha(t, v(t))}{p(t)} \right) + d_0 + d_1 t + \dots + d_{k-1} t^{k-1}.$$

Since  $v^{(i)}(0) = 0$  for  $1 \leq i \leq k - 1$ , we get  $d_1 = \dots = d_{k-1} = 0$  therefore it is concluded that

$$\begin{aligned} v(t) &= -\frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta + d_0. \end{aligned} \quad (2)$$

Hence,

$$\begin{aligned} av(\mu) &= \frac{-a}{\Gamma(\beta)} \int_0^\mu (\mu-\zeta)^{\beta-1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\ &\quad + \frac{a}{\Gamma(\beta)} \int_0^\mu (\mu-\zeta)^{\beta-1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta + d_0 a \\ &= \frac{a}{\Gamma(\beta)} \int_0^\mu (\mu-s)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta + d_0 a. \end{aligned}$$

Also, by integration of order  $p_i$  from (2), for each  $1 \leq i \leq n_0$ , we have

$$\begin{aligned} \mathcal{I}^{p_i} v(t) &= -\frac{1}{\Gamma(\beta+p_i)} \int_0^t (t-\zeta)^{\beta+p_i-1} \frac{A_\alpha(\zeta, v(\zeta))}{p(s)} ds \\ &\quad + \frac{1}{\Gamma(\beta+p_i)} \int_0^t (t-\zeta)^{\beta+p_i-1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta + \frac{d_0 t_i^p}{\Gamma(p_i+1)}, \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{i=1}^{n_0} \lambda_i \mathcal{I}^{p_i} v(t) &= -\sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\beta+p_i)} \int_0^t (t-\zeta)^{\beta+p_i-1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\ &\quad + \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\beta+p_i)} \int_0^t (t-\zeta)^{\beta+p_i-1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta + d_0 \sum_{i=1}^{n_0} \frac{\lambda_i t_i^p}{\Gamma(p_i+1)}. \end{aligned}$$

Thus it results in

$$\begin{aligned} \sum_{i=1}^{n_0} \lambda_i \mathcal{I}^{p_i} v(a_i) &= -\sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\beta+p_i)} \int_0^{a_i} (a_i-\zeta)^{\beta+p_i-1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\ &\quad + \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\beta+p_i)} \int_0^{a_i} (a_i-\zeta)^{\beta+p_i-1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta + d_0 \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(p_i+1)} \\ &= \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\beta+p_i)} \int_0^{a_i} (a_i-\zeta)^{\beta+p_i-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ &\quad + d_0 \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(p_i+1)}. \end{aligned}$$

Since  $\alpha v(\mu) = \sum_{i=1}^{n_0} \lambda_i \mathcal{I}^{p_i} v(a_i)$ , we obtain

$$\begin{aligned} & \frac{\alpha}{\Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta + d_0 \alpha \\ &= \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ &+ d_0 \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(p_i + 1)}. \end{aligned}$$

Hence

$$\begin{aligned} & d_0 \left( \alpha - \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(p_i + 1)} \right) \\ &= \sum_{i=1}^{n_0} \frac{\lambda_i}{\Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ &- \frac{\alpha}{\Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta, \end{aligned}$$

and so

$$\begin{aligned} d_0 &= \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ &- \frac{\alpha}{\Delta \Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta, \end{aligned}$$

where  $\Delta = (\alpha - \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(p_i + 1)})$ . This indicates that

$$\begin{aligned} v(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t - \zeta)^{\beta-1} \frac{(B_\alpha(s, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ &+ \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ &- \frac{\alpha}{\Delta \Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta. \end{aligned}$$

One can obtain the other part by using some calculations. This completes the proof.  $\square$

Note that the generalized boundary conditions of the Sturm–Liouville problem lead us to attaining a different integral equation to consider. Also, as we have a strong singularity in the problem, we need to investigate the equation by a novel method.

Designate the space  $X = C[0, 1]$  with the supremum norm. Define the map  $H : X \rightarrow X$  by

$$\begin{aligned} H_v(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t - \zeta)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ & + \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta+p_i-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \\ & - \frac{a}{\Delta \Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{(B_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, v(\zeta)))}{p(\zeta)} d\zeta \end{aligned}$$

for all  $t \in [0, 1]$ . Note that, if  $v_0 \in X$  is a solution for SBVP (1), then  $v_0$  is a fixed point of the map  $H$ . Vice versa,  $v_0 \in X$  is a solution for the problem when  $v_0$  is a fixed point of the mapping. In the next result, we suppose that the maps  $q, h : [0, 1] \rightarrow \mathbb{R}$  may be singular at some points in  $[0, 1]$  and the function  $p : [0, 1] \rightarrow [0, \infty)$  in equation (1) is  $n - 1$  times differentiable but can be zero at some points in  $[0, 1]$ . In the next theorem, using inequalities for controlling singular points by some functions that are called control functions, and by the fixed point method, we will investigate the existence of a solution for the singular fractional differential problem (SFDP).

**Theorem 2.2** Assume that  $\alpha, \beta \geq 1$ ,  $\alpha \in [n - 1, n]$ ,  $\beta \in [k - 1, k]$ ,  $n_0$  is a natural number,  $\mu, a_1, \dots, a_{n_0} \in [0, 1]$ ,  $a, \lambda_1, \dots, \lambda_{n_0} \in \mathbb{R}$ ,  $p_i \geq 0$  with  $a \neq \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(\beta + p_i)}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $|f(x) - f(y)| \leq \Lambda(|x - y|)$  and  $|f(z)| \leq M(z) + N(z)$  for all  $x, y, z \in \mathbb{R}$ , where  $\Lambda, M, N : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are increasing functions with  $\lim_{\omega \rightarrow 0^+} \frac{\Lambda(\omega)}{\omega} = Q \in [0, \infty)$ ,  $\lim_{\omega \rightarrow \infty} \frac{M(\omega)}{\omega} = m \in [0, \infty)$ , and  $\lim_{\omega \rightarrow \infty} N(\omega) < \infty$ . Suppose that  $\tilde{h}_p[0, 1] = \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(1, \xi) d\xi < \infty$  and  $\tilde{h}_q[0, 1] = \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(1, \xi) d\xi < \infty$ , where  $\hat{p}(t, \xi) = \int_\xi^t \frac{ds}{p(s)}$ . If

$$\begin{aligned} & \left( \frac{m}{\Gamma(\alpha)\Gamma(\beta)} + \frac{m}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\ & + \left( \frac{1}{\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\ & + \frac{|a|m\tilde{h}_p[0, \mu] + |a|\tilde{h}_q[0, \mu]}{|\Delta| \Gamma(\alpha)\Gamma(\beta)} < 1, \end{aligned}$$

then the singular boundary value problem (SBVP)  $\mathcal{D}^\alpha(p(t)\mathcal{D}^\beta v(t)) + q(t)v(t) = h(t)f(v(t))$  with  $v^{(i)}(0) = \mathcal{D}^{(\beta+j)}v(0) = 0$  for  $0 \leq j \leq n - 1$  and  $0 \leq i \leq k - 1$  and  $a v(\mu) = \sum_{i=1}^{n_0} \lambda_i \mathcal{I}^{p_i} v(a_i)$  has a solution, in which  $\Delta = a - \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(\beta + p_i)}$ .

*Proof* First, we show that  $H$  is continuous. Let  $v, v^* \in X$  and  $t \in [0, 1]$ . Then we have

$$\begin{aligned} & |H_v(t) - H_{v^*}(t)| \\ & \leq \frac{1}{\Gamma(\beta)} \int_0^t (t - \zeta)^{\beta-1} \frac{|B_\alpha(\zeta, v(\zeta)) - B_\alpha(\zeta, v^*(\zeta)) + A_\alpha(\zeta, v^*(\zeta)) - A_\alpha(\zeta, v(\zeta))|}{p(\zeta)} d\zeta \\ & + \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \end{aligned}$$

$$\begin{aligned}
& \times \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{|B_\alpha(\zeta, v(\zeta)) - B_\alpha(\zeta, v^*(\zeta)) + A_\alpha(\zeta, v^*(\zeta)) - A_\alpha(\zeta, v(\zeta))|}{p(\zeta)} d\zeta \\
& + \frac{|a|}{|\Delta| \Gamma(\beta)} \\
& \times \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{|B_\alpha(\zeta, v(\zeta)) - B_\alpha(\zeta, v^*(\zeta)) + A_\alpha(\zeta, v^*(\zeta)) - A_\alpha(\zeta, v(\zeta))|}{p(\zeta)} d\zeta \\
& \leq \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \frac{(t - \zeta)^{\beta - 1}}{p(\zeta)} \left[ \int_0^\zeta (\zeta - \xi)^{\alpha - 1} |h(\xi)| |f(v(\xi)) - f(v^*(\xi))| d\xi \right. \\
& \quad \left. + \int_0^\zeta (\zeta - \xi)^{\alpha - 1} |q(\xi)| |v(\xi) - v^*(\xi)| d\xi \right] d\zeta \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \\
& \times \int_0^{a_i} \frac{(a_i - \zeta)^{\beta + p_i - 1}}{p(\zeta)} \left[ \int_0^\zeta (\zeta - \xi)^{\alpha - 1} |h(\xi)| |f(v(\xi)) - f(v^*(\xi))| d\xi \right. \\
& \quad \left. + \int_0^\zeta (\zeta - \xi)^{\alpha - 1} |q(\xi)| |v(\xi) - v^*(\xi)| d\xi \right] d\zeta \\
& + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \frac{(\mu - \zeta)^{\beta - 1}}{p(\zeta)} \left[ \int_0^\zeta (\zeta - \xi)^{\alpha - 1} |h(\xi)| |f(v(\xi)) - f(v^*(\xi))| d\xi \right. \\
& \quad \left. + \int_0^\zeta (\zeta - \xi)^{\alpha - 1} |q(\xi)| |v(\xi) - v^*(\xi)| d\xi \right] ds \\
& = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta - 1} (\zeta - \xi)^{\alpha - 1}}{p(\zeta)} |h(\xi)| |f(v(\xi)) - f(v^*(\xi))| d\xi \\
& \quad + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta - 1} (\zeta - \xi)^{\alpha - 1}}{p(\zeta)} |q(\xi)| |v(\xi) - v^*(\xi)| d\xi d\zeta \\
& \quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \\
& \quad \times \int_0^{a_i} \int_0^s \frac{(a_i - \zeta)^{\beta + p_i - 1} (\zeta - \xi)^{\alpha - 1}}{p(\zeta)} |h(\xi)| |f(v(\xi)) - f(v^*(\xi))| d\xi \\
& \quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \\
& \quad \times \int_0^{a_i} \int_0^\zeta \frac{(a_i - \zeta)^{\beta + p_i - 1} (\zeta - \xi)^{\alpha - 1}}{p(\zeta)} |q(\xi)| |v(\xi) - v^*(\xi)| d\xi d\zeta \\
& \quad + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \zeta)^{\beta - 1} (\zeta - \xi)^{\alpha - 1}}{p(\zeta)} |h(\xi)| |f(v(\xi)) - f(v^*(\xi))| d\xi \\
& \quad + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \zeta)^{\beta - 1} (\zeta - \xi)^{\alpha - 1}}{p(\zeta)} |q(\xi)| |v(\xi) - v^*(\xi)| d\xi d\zeta \\
& \leq \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta - 1} (\zeta - \xi)^{\alpha - 1}}{p(\zeta)} |h(\xi)| \Lambda(|v(\xi) - v^*(\xi)|) d\xi \\
& \quad + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta - 1} (\zeta - \xi)^{\alpha - 1}}{p(\zeta)} |q(\xi)| \|v - v^*\| d\xi d\zeta \\
& \quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)}
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^{a_i} \int_0^\zeta \frac{(a_i - \zeta)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| \Lambda(|v(\xi) - v^*(\xi)|) d\xi \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} \int_0^\zeta \frac{(a_i - \zeta)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| \|v - v^*\| d\xi d\zeta \\
& + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| \Lambda(|v(\xi) - v^*(\xi)|) d\xi \\
& + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - s)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| \|v - v^*\| d\xi d\zeta \\
& \leq \frac{\Lambda(\|v - v^*\|)}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| d\xi \\
& + \frac{\|v - v^*\|}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta \\
& + \frac{\Lambda(\|v - v^*\|)}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} \int_0^\zeta \frac{(a_i - \zeta)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| d\xi \\
& + \frac{\|v - v^*\|}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} \int_0^\zeta \frac{(a_i - \zeta)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta \\
& + \frac{|a| \Lambda(\|v - v^*\|)}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| d\xi \\
& + \frac{|a| \|v - v^*\|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta.
\end{aligned}$$

Let  $\epsilon > 0$  be given. Since  $\lim_{z \rightarrow 0^+} \frac{\Lambda(z)}{z} = Q \in [0, \infty)$ , there exists  $\delta(\epsilon) > 0$  such that  $z \in (0, \delta(\epsilon))$  implies  $|\frac{\Lambda(z)}{z}| \leq Q + \epsilon$ . Hence,  $z \in (0, \delta(\epsilon))$  implies  $\Lambda(z) \leq (Q + \epsilon)z$ . Put  $\delta_m(\epsilon) = \min\{\epsilon, \delta(\epsilon)\}$ . Then  $\|v - v^*\| \leq \delta_m(\epsilon)$  implies  $\Lambda(\|v - v^*\|) \leq (Q + \epsilon)\|v - v^*\|$ . Also,

$$\begin{aligned}
& \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta \\
& = \int_0^t \int_\xi^t \frac{(t - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta \\
& = \int_0^t |q(\xi)| \left( \int_\xi^t \frac{(t - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} d\zeta \right) d\xi \\
& \leq \int_0^t |q(\xi)| \left( \int_\xi^t \frac{(t - \xi)^{\beta-1} (t - \xi)^{\alpha-1}}{p(\zeta)} d\zeta \right) d\xi.
\end{aligned}$$

Since  $\alpha, \beta \geq 1$  and  $\zeta \in [\xi, t]$ , we get  $(t - \zeta)^{\beta-1} \leq (t - \xi)^{\beta-1}$  and  $(\zeta - \xi)^{\alpha-1} \leq (t - \xi)^{\alpha-1}$ , so

$$\begin{aligned}
& \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta \\
& \leq \int_0^t (t - \xi)^{\alpha+\beta-2} |q(\xi)| \left( \int_\xi^t \frac{d\zeta}{p(\zeta)} \right) d\xi \\
& \leq \int_0^t (t - \xi)^{\alpha+\beta-2} |q(\xi)| \hat{p}(t, \xi) d\xi,
\end{aligned}$$

where  $\hat{p}(t, \xi) = \int_{\xi}^t \frac{d\xi}{p(\xi)}$ . Obviously,  $\hat{p}(t, \xi)$  is increasing with respect to  $t$  and is decreasing with respect to  $\xi$ . Therefore, we get

$$\begin{aligned} & \int_0^t \int_0^\xi \frac{(t-\xi)^{\beta-1}(\xi-\xi)^{\alpha-1}}{p(\xi)} |h(\xi)| d\xi d\xi \\ & \leq \int_0^t (t-\xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi, \\ & \int_0^{a_i} \int_0^\xi \frac{(a_i-\xi)^{\beta+p_i-1}(\xi-\xi)^{\alpha-1}}{p(\xi)} |h(\xi)| d\xi d\xi \\ & \leq \int_0^{a_i} (a_i-\xi)^{\alpha+\beta+p_i-2} |h(\xi)| \hat{p}(t, \xi) d\xi, \end{aligned}$$

and

$$\int_0^\mu \int_0^\xi \frac{(\mu-\xi)^{\beta-1}(\xi-\xi)^{\alpha-1}}{p(\xi)} |h(\xi)| d\xi d\xi \leq \int_0^\mu (\mu-\xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi.$$

Thus it evolved that

$$\begin{aligned} |H_v(t) - H_{v^*}(t)| & \leq \frac{\Lambda(\|v - v^*\|)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-\xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi \\ & + \frac{\|v - v^*\|}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-\xi)^{\alpha+\beta-2} |q(\xi)| \hat{p}(t, \xi) d\xi \\ & + \frac{\Lambda(\|v - v^*\|)}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \int_0^{a_i} (a_i-\xi)^{\alpha+p_i+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi \\ & + \frac{\|v - v^*\|}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \int_0^{a_i} (a_i-\xi)^{\alpha+p_i+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi \\ & + \frac{|\alpha|\Lambda(\|v - v^*\|)}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \int_0^\mu (\mu-\xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi \\ & + \frac{|\alpha|\|v - v^*\|}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \int_0^\mu (\mu-\xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi. \end{aligned}$$

Let  $\epsilon > 0$  be given and  $\|v - v^*\| \leq \delta_m(\epsilon)$ . Then we have

$$\begin{aligned} |H_v(t) - H_{v^*}(t)| & \leq \frac{(Q+\epsilon)\|v - v^*\|}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-\xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi \\ & + \frac{\|v - v^*\|}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-\xi)^{\alpha+\beta-2} |q(\xi)| \hat{p}(t, \xi) d\xi \\ & + \frac{(Q+\epsilon)\|v - v^*\|}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \\ & \times \int_0^{a_i} (a_i-\xi)^{\alpha+p_i+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi \\ & + \frac{\|v - v^*\|}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \int_0^{a_i} (a_i-\xi)^{\alpha+p_i+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi \end{aligned}$$

$$\begin{aligned}
& + \frac{|\alpha|(Q+\epsilon)\|v - v^*\|}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \int_0^\mu (\mu - \xi)^{\alpha+\beta-2} |h(\xi)| |\hat{p}(t, \xi)| d\xi \\
& + \frac{|\alpha|\|v - v^*\|}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \int_0^\mu (\mu - \xi)^{\alpha+\beta-2} |h(\xi)| |\hat{p}(t, \xi)| d\xi \\
& \leq \frac{(Q+\epsilon)\epsilon}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| |\hat{p}(1, \xi)| d\xi \\
& + \frac{\epsilon}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1 - \xi)^{\alpha+\beta-2} |q(\xi)| |\hat{p}(1, \xi)| d\xi \\
& + \frac{(Q+\epsilon)\epsilon}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| |\hat{p}(1, \xi)| d\xi \\
& + \frac{\epsilon}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| |\hat{p}(1, \xi)| d\xi \\
& + \frac{|\alpha|(Q+\epsilon)\epsilon}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| |\hat{p}(1, \xi)| d\xi \\
& + \frac{|\alpha|\epsilon}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| |\hat{p}(1, \xi)| d\xi.
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
|H_v(t) - H_{v^*}(t)| & \leq \frac{(Q+\epsilon)\epsilon}{\Gamma(\alpha)\Gamma(\beta)} \tilde{h}_p[0, 1] + \frac{\epsilon}{\Gamma(\alpha)\Gamma(\beta)} \tilde{q}_p[0, 1] \\
& + \frac{\tilde{h}_p[0, 1](Q+\epsilon)\epsilon}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} + \frac{\tilde{q}_p[0, 1]\epsilon}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \\
& + \frac{|\alpha|\tilde{h}_p[0, 1](Q+\epsilon)\epsilon}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} + \frac{|\alpha|\tilde{q}_p[0, 1]\epsilon}{|\Delta|\Gamma(\alpha)\Gamma(\beta)}
\end{aligned}$$

for all  $t \in [0, 1]$ , where

$$\tilde{h}_p[0, t] = \int_0^t (1 - \xi)^{\alpha+\beta-2} |h(\xi)| |\hat{p}(1, \xi)| d\xi$$

and

$$\tilde{q}_p[0, t] = \int_0^t (1 - \xi)^{\alpha+\beta-2} |q(\xi)| |\hat{p}(1, \xi)| d\xi.$$

By using the supremum norm on  $[0, 1]$ , it is deduced that

$$\begin{aligned}
\|H_v - H_{v^*}\| & \leq \left( \frac{(Q+\epsilon)\tilde{h}_p[0, 1] + \tilde{q}_p[0, 1]}{\Gamma(\alpha)\Gamma(\beta)} \right. \\
& + \frac{\tilde{h}_p[0, 1](Q+\epsilon) + \tilde{q}_p[0, 1]}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \\
& \left. + \frac{|\alpha|\tilde{h}_p[0, 1](Q+\epsilon) + |\alpha|\tilde{q}_p[0, 1]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \right) \epsilon.
\end{aligned}$$

This implies that  $\|H_v - H_{v^*}\|$  tends to zero as  $\|v - v^*\| \rightarrow 0$ . Thus,  $H$  is continuous. Since  $|f(\omega)| \leq M(\omega) + N(\omega)$  for all  $\omega \in [0, \infty)$  and  $\lim_{\omega \rightarrow \infty} \frac{M(\omega)}{\omega} = m \in [0, \infty)$  for each  $\epsilon > 0$ , there exists  $R(\epsilon) > 0$  such that  $\omega \in [R(\epsilon), \infty)$  implies

$$M(\omega) \leq (m + \epsilon)\omega. \quad (3)$$

Likewise, by the assumptions  $\lim_{\omega \rightarrow \infty} N(\omega) < \infty$ , it results that  $\lim_{\omega \rightarrow \infty} \frac{N(\omega)}{\omega} = 0$ , so there exists  $R'(\epsilon) > 0$  such that  $\frac{N(\omega)}{\omega} < \epsilon$  for all  $\omega \in [R'(\epsilon), \infty)$ . Therefore,  $\omega \in [R'(\epsilon), \infty)$  implies

$$N(\omega) \leq \epsilon\omega. \quad (4)$$

On the other side, we have

$$\begin{aligned} & \left( \frac{m}{\Gamma(\alpha)\Gamma(\beta)} + \frac{m}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\ & + \left( \frac{1}{\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\ & + \frac{|\alpha|m\tilde{h}_p[0, \mu] + |\alpha|\tilde{h}_q[0, \mu]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} < 1. \end{aligned}$$

Choose  $\epsilon_0 > 0$  such that

$$\begin{aligned} & \left( \frac{m+2\epsilon_0}{\Gamma(\alpha)\Gamma(\beta)} + \frac{m+2\epsilon_0}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\ & + \left( \frac{1}{\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\ & + \frac{|\alpha|(m+2\epsilon_0)\tilde{h}_p[0, \mu] + |\alpha|\tilde{h}_q[0, \mu]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} < 1. \end{aligned}$$

Put  $R_0 := \max\{R(\epsilon_0), R'(\epsilon_0)\}$ . By using (3) and (4), for  $\omega = R_0$ , we get  $M(R_0) \leq (m + \epsilon_0)R_0$  and  $N(R_0) \leq \epsilon_0 R_0$ . Define  $\Omega = \{u \in X : \|u\| < R_0\}$ . Let  $u_0 \in \partial\Omega$  and  $\lambda \in (0, 1)$  be such that  $u_0 = \lambda H_{u_0}$ . Then  $\|u_0\| = R_0$ . Then we have

$$\begin{aligned} |u_0(t)| &= |\lambda H_{u_0}(t)| \\ &\leq \lambda \left[ \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{|B_\alpha(\zeta, u_0(\zeta)) - A_\alpha(\zeta, u_0(\zeta))|}{p(\zeta)} d\zeta \right. \\ &+ \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta+p_i-1} \frac{|B_\alpha(\zeta, u_0(\zeta)) - A_\alpha(\zeta, u_0(\zeta))|}{p(\zeta)} d\zeta \\ &+ \left. \frac{|\alpha|}{|\Delta|\Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{|B_\alpha(\zeta, u_0(\zeta)) - A_\alpha(\zeta, u_0(\zeta))|}{p(\zeta)} d\zeta \right] \\ &\leq \lambda \left[ \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{(t-\zeta)^{\beta-1}}{p(\zeta)} \left( \int_0^\zeta (\zeta - \xi)^{\alpha-1} |h(\xi)| |f(u_0(\xi))| d\xi \right) d\zeta \right. \\ &+ \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \int_0^{a_i} \frac{(a_i - \zeta)^{\beta+p_i-1}}{p(\zeta)} \left( \int_0^\zeta (\zeta - \xi)^{\alpha-1} |h(\xi)| |f(u_0(\xi))| d\xi \right) d\zeta \\ &+ \left. \frac{|\alpha|}{|\Delta|\Gamma(\beta)} \int_0^\mu \frac{(\mu - \zeta)^{\beta-1}}{p(\zeta)} \left( \int_0^\zeta (\zeta - \xi)^{\alpha-1} |h(\xi)| |f(u_0(\xi))| d\xi \right) d\zeta \right]. \end{aligned}$$

$$\begin{aligned}
& + \int_0^\zeta (\zeta - \xi)^{\alpha-1} |q(\xi)| |u_0(\xi)| d\xi \Big) d\xi + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \\
& \times \int_0^{a_i} \frac{(a_i - \xi)^{\beta+p_i-1}}{p(\xi)} \left( \int_0^\zeta (\zeta - \xi)^{\alpha-1} |h(\xi)| |f(u_0(\xi))| d\xi \right. \\
& + \int_0^\zeta (\zeta - \xi)^{\alpha-1} |q(\xi)| |u_0(\xi)| d\xi \Big) d\xi \\
& + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \frac{(\mu - \xi)^{\beta-1}}{p(\xi)} \left( \int_0^\zeta (\zeta - \xi)^{\alpha-1} |h(\xi)| |f(u_0(\xi))| d\xi \right. \\
& \left. + \int_0^\zeta (\zeta - \xi)^{\alpha-1} |q(\xi)| |u_0(\xi)| d\xi \right) d\xi \Big] \\
= & \lambda \left[ \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \xi)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |h(\xi)| |f(u_0(\xi))| d\xi d\zeta \right. \\
& + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \xi)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |q(\xi)| |u_0(\xi)| d\xi d\zeta \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \\
& \times \int_0^{a_i} \int_0^\zeta \frac{(a_i - \xi)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |h(\xi)| |f(u_0(\xi))| d\xi d\zeta \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} \int_0^\zeta \frac{(a_i - \xi)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |q(\xi)| |u_0(\xi)| d\xi d\zeta \\
& + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \xi)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |h(\xi)| |f(u_0(\xi))| d\xi d\zeta \\
& \left. + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \xi)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |q(\xi)| |u_0(\xi)| d\xi d\zeta \right] \\
\leq & \lambda \left[ \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \xi)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |h(\xi)| (M(u_0(\xi)) + N(u_0(\xi))) d\xi d\zeta \right. \\
& + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \xi)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |q(\xi)| |u_0(\xi)| d\xi d\zeta \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \\
& \times \int_0^{a_i} \int_0^\zeta \frac{(a_i - \xi)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(s)} |h(\xi)| (M(u_0(\xi)) + N(u_0(\xi))) d\xi d\zeta \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} \int_0^\zeta \frac{(a_i - \xi)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |q(\xi)| |u_0(\xi)| d\xi d\zeta \\
& + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \\
& \times \int_0^\mu \int_0^\zeta \frac{(\mu - \xi)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\xi)} |h(\xi)| (M(u_0(\xi)) + N(u_0(\xi))) d\xi d\zeta
\end{aligned}$$

$$\begin{aligned}
& + \frac{|a|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - s)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| |u_0(\xi)| d\xi d\zeta \Big] \\
& \leq \lambda \left[ \frac{M(\|u_0\|) + N(\|u_0\|)}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| d\xi d\zeta \right. \\
& \quad + \frac{\|u_0\|}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t \int_0^\zeta \frac{(t - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta \\
& \quad + \frac{M(\|u_0\|) + N(\|u_0\|)}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \\
& \quad \times \int_0^{a_i} \int_0^\zeta \frac{(a_i - \zeta)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| d\xi d\zeta \\
& \quad + \frac{\|u_0\|}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} \int_0^\zeta \frac{(a_i - \zeta)^{\beta+p_i-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta \\
& \quad + \frac{|a|(M(\|u_0\|) + N(\|u_0\|))}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| d\xi d\zeta \\
& \quad \left. + \frac{|a| \|u_0\|}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu \int_0^\zeta \frac{(\mu - \zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |q(\xi)| d\xi d\zeta \right] \\
& \leq \lambda \left[ \frac{M(R_0) + N(R_0)}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (t - \xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi \right. \\
& \quad + \frac{R_0}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (t - \xi)^{\alpha+\beta-2} |q(\xi)| \hat{p}(t, \xi) d\xi \\
& \quad + \frac{M(R_0) + N(R_0)}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \xi)^{\alpha+\beta+p_i-2} |h(\xi)| \hat{p}(a_i, \xi) d\xi \\
& \quad + \frac{R_0}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \xi)^{\alpha+\beta+p_i-2} |q(\xi)| \hat{p}(a_i, \xi) d\xi \\
& \quad + \frac{|a|(M(R_0) + N(R_0))}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu (\mu - \xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(\mu, \xi) d\xi \\
& \quad \left. + \frac{|a|R_0}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \int_0^\mu (\mu - \xi)^{\alpha+\beta-2} |q(\xi)| \hat{p}(\mu, \xi) d\xi \right]
\end{aligned}$$

for all  $t \in [0, 1]$ . Hence,

$$\begin{aligned}
|u_0(t)| & \leq \lambda \left[ \frac{(m + \epsilon_0)R_0 + \epsilon_0 R_0}{\Gamma(\alpha) \Gamma(\beta)} \int_0^t (1 - \xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(1, \xi) d\xi \right. \\
& \quad + \frac{R_0}{\Gamma(\alpha) \Gamma(\beta)} \int_0^1 (1 - \xi)^{\alpha+\beta-2} |q(\xi)| \hat{p}(1, \xi) d\xi \\
& \quad + \frac{(m + \epsilon_0)R_0 + \epsilon_0 R_0}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^1 (1 - \xi)^{\alpha+\beta+p_i-2} |h(\xi)| \hat{p}(1, \xi) d\xi \\
& \quad \left. + \frac{R_0}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^1 (1 - \xi)^{\alpha+\beta+p_i-2} |q(\xi)| \hat{p}(1, \xi) d\xi \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{|\alpha|((m+\epsilon_0)R_0 + \epsilon_0 R_0)}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \int_0^\mu (\mu - \xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(1, \xi) d\xi \\
& + \frac{|\alpha|R_0}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \int_0^\mu (\mu - \xi)^{\alpha+\beta-2} |q(\xi)| \hat{p}(1, \xi) d\xi
\end{aligned}$$

for all  $t \in [0, 1]$ . Thus we infer that

$$\begin{aligned}
|u_0(t)| & \leq \lambda \left[ \frac{(m+\epsilon_0)R_0 + \epsilon_0 R_0}{\Gamma(\alpha)\Gamma(\beta)} \tilde{h}_p[0, 1] + \frac{R_0}{\Gamma(\alpha)\Gamma(\beta)} \tilde{q}_p[0, 1] \right. \\
& + \frac{(m+\epsilon_0)R_0 + \epsilon_0 R_0}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \tilde{h}_p[0, 1] \\
& + \frac{R_0}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \tilde{h}_q[0, 1] \\
& \left. + \frac{|\alpha|((m+\epsilon_0)R_0 + \epsilon_0 R_0)}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \tilde{h}_p[0, \mu] + \frac{|\alpha|R_0}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \tilde{h}_q[0, \mu] \right],
\end{aligned}$$

and so by using the supremum on  $[0, 1]$ , we have

$$\begin{aligned}
\|u_0\| & \leq \lambda \left[ \left( \frac{m+2\epsilon_0}{\Gamma(\alpha)\Gamma(\beta)} + \frac{m+2\epsilon_0}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \right) \tilde{h}_p[0, 1] \right. \\
& + \left( \frac{1}{\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \right) \tilde{q}_p[0, 1] \\
& \left. + \frac{|\alpha|(m+2\epsilon_0)\tilde{h}_p[0, \mu] + |\alpha|\tilde{h}_q[0, \mu]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \right] R_0 < R_0.
\end{aligned}$$

This implies that  $u_0 \notin \partial\Omega$ . By the same way, via Lemma 1.2,  $H$  has a fixed point in  $\bar{\Omega}$  which is a solution for FBVP (1).  $\square$

*Example 2.3* Consider the strong singular Sturm–Liouville equation

$$(t^2 u'(t))' + \frac{\sqrt{t}u(t)}{5c(t)} = \frac{\sqrt{t}}{10}(u(t) + 1) \quad (5)$$

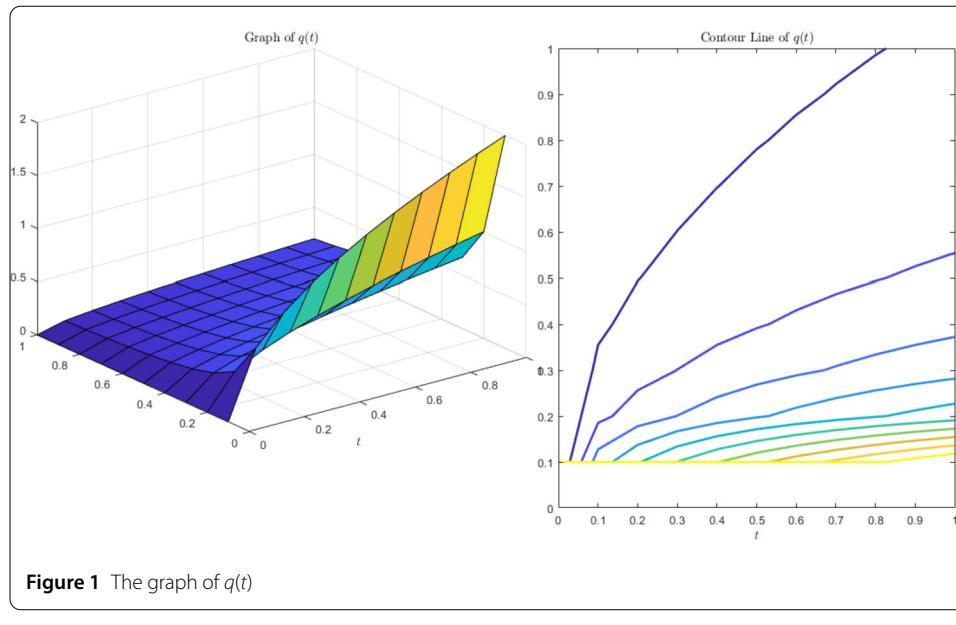
with boundary conditions  $u'(0) = 0$  and  $u(\frac{1}{2}) = u(1) = 0$ . Put  $\alpha = \beta = 1$ ,  $n_0 = 2$ ,  $p_1 = p_2 = 0$ ,  $a = 0$ ,  $\mu \in [0, 1]$ ,  $\alpha_1 = \frac{1}{2}$ ,  $\alpha_2 = 1$ ,  $\lambda_1 = \lambda_2 = 1$ ,  $p(t) = t^2$ ,  $h(t) = \frac{\sqrt{t}}{10}$ ,  $q(t) = \frac{\sqrt{t}}{5c(t)}$ , where  $c(t) = 0$  when  $t \in Q \cap [0, 1]$ , and  $c(t) = 1$  for  $t \in Q^c \cap [0, 1]$ . If  $f(u) = u + 1$ , then we have

$$\hat{p}(1, \xi) = \int_\xi^1 \frac{d\xi}{\xi^2} = \frac{1}{\xi} - 1, \quad \tilde{q}_p[0, 1] = \frac{1}{5} \int_0^1 \frac{\sqrt{\xi}}{c(\xi)} \left( \frac{1}{\xi} - 1 \right) d\xi = \frac{4}{15}$$

and  $\tilde{h}_p[0, 1] = \frac{1}{10} \int_0^1 \sqrt{\xi} \left( \frac{1}{\xi} - 1 \right) d\xi = \frac{4}{30}$ . Note that  $\Delta = a - \sum_{i=1}^{n_0} \frac{\lambda_i \alpha_i^{p_i}}{\Gamma(\beta+p_i)} = -(\frac{1}{2} + 1) = -\frac{3}{2}$ ,

$$|f(u) - f(v)| = |u - v| := \Lambda(|u - v|),$$

$$\lim_{\omega \rightarrow 0^+} \frac{\Lambda(\omega)}{\omega} = \lim_{\omega \rightarrow 0^+} \frac{\omega}{\omega} = 1 \in [0, \infty) \quad \text{and} \quad |f(u)| \leq |u| + 1 := M(u) + N(u),$$



where  $M(u) = |u|$ ,  $N(u) = 1$ ,  $m = \lim_{\omega \rightarrow \infty} \frac{M(\omega)}{\omega} = 1$ , and  $\lim_{\omega \rightarrow \infty} N(\omega) < \infty$ . Note that

$$\begin{aligned} & \left( \frac{m}{\Gamma(\alpha)\Gamma(\beta)} + \frac{m}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \right) \tilde{h}_p[0,1] \\ & + \left( \frac{1}{\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \right) \tilde{q}_p[0,1] \\ & + \frac{|a|m\tilde{h}_p[0,\mu] + |a|\tilde{h}_q[0,\mu]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} = \left( 1 + \frac{4}{3} \right) \times \frac{4}{30} + \left( 1 + \frac{4}{3} \right) \times \frac{4}{15} < 1. \end{aligned}$$

Now, by using Theorem 2.2, the Sturm–Liouville problem (5) has a solution. Also for a better graphical understanding of the problem, the graph of  $q(t)$  is shown in Fig. 1.

### 3 Continuous dependence

In this part, according to the topics raised in [14], we verify continuous dependence of the solution for the fractional Sturm–Liouville differential equation (1).

**Definition 3.1** We say that the solution of the fractional Sturm–Liouville differential equation

$$\mathcal{D}^\alpha(p(t)\mathcal{D}^\beta v(t)) + q(t)v(t) = h(t)f(v(t)) \quad (6)$$

is continuously dependent on  $\lambda_i$  whenever, for each  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$  such that, for any two solutions  $v$  and  $\tilde{v}$  of (6), that  $v$  satisfies conditions (1) and  $\tilde{v}$  satisfies the following initial conditions:

$$\begin{cases} \tilde{v}^{(i)}(0) = \mathcal{D}^{(\beta+j)}\tilde{v}(0) = 0 & (\text{for } 0 \leq j \leq n-1 \text{ and } 0 \leq i \leq k-1), \\ a\tilde{v}(\mu) = \sum_{i=1}^{n_0} \tilde{\lambda}_i \mathcal{I}^{p_i} v(a_i), \end{cases} \quad (7)$$

$$\sum_{i=1}^{n_0} |\lambda_i - \tilde{\lambda}_i| < \delta \text{ implies } \|v - \tilde{v}\| < \epsilon.$$

In the next result, we again suppose that the maps  $q, h : [0, 1] \rightarrow \mathbb{R}$  may be singular at some points in  $[0, 1]$  and the function  $p : [0, 1] \rightarrow [0, \infty)$  is  $n - 1$  times differentiable, but it can be zero at some points in  $[0, 1]$ .

**Theorem 3.2** Assume that  $\alpha, \beta \geq 1$ ,  $\alpha \in [n - 1, n]$ ,  $\beta \in [k - 1, k]$ ,  $n_0$  is a natural number,  $\mu, a_1, \dots, a_{n_0} \in [0, 1]$ ,  $a, \lambda_1, \dots, \lambda_{n_0} \in \mathbb{R}$ ,  $p_i \geq 0$  with  $a \neq \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(\beta + p_i)}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function such that  $|f(x) - f(y)| \leq \Lambda(|x - y|)$  and  $|f(z)| \leq M(z) + N(z)$  for all  $x, y, z \in \mathbb{R}$ , where  $\Lambda, M, N : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are nondecreasing functions with  $\sup_{z \in (0, \infty)} \frac{\Lambda(z)}{z} = Q' \in [0, \infty)$ ,  $\lim_{\omega \rightarrow \infty} \frac{M(\omega)}{\omega} = m \in [0, \infty)$ , and  $\lim_{\omega \rightarrow \infty} N(\omega) < \infty$ . Suppose that

$$\tilde{h}_p[0, 1] = \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(1, \xi) d\xi < \infty$$

and

$$\tilde{h}_q[0, 1] = \int_0^1 (1 - \xi)^{\alpha+\beta-2} |h(\xi)| \hat{p}(1, \xi) d\xi < \infty,$$

where  $\hat{p}(t, \xi) = \int_\xi^t \frac{ds}{p(s)}$ . If

$$\begin{aligned} & \left( \frac{\Xi}{\Gamma(\alpha)\Gamma(\beta)} + \frac{\Xi}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\ & + \left( \frac{1}{\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \right) \tilde{h}_q[0, 1] \\ & + \frac{|\alpha|\Xi \tilde{h}_p[0, \mu] + |\alpha|\tilde{h}_q[0, \mu]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} < 1, \end{aligned}$$

then the solutions of the equation  $D^\alpha(p(t)D^\beta v(t)) + q(t)v(t) = h(t)f(v(t))$  with the initial conditions  $v^{(i)}(0) = D^{(\beta+i)}v(0) = 0$  (for  $0 \leq j \leq n - 1$  and  $0 \leq i \leq k - 1$ ) and  $av(\mu) = \sum_{i=1}^{n_0} \lambda_i \mathcal{I}^{p_i} v(a_i)$  are continuously dependent on the coefficients  $\lambda_i$ , where  $\Delta = a - \sum_{i=1}^{n_0} \frac{\lambda_i a_i^{p_i}}{\Gamma(\beta + p_i)}$  and  $\Xi = \max\{Q', m\}$ .

*Proof* Since  $\lim_{\omega \rightarrow 0^+} \frac{\Lambda(\omega)}{\omega} := Q \leq Q'$ , we get

$$\begin{aligned} & \left( \frac{m}{\Gamma(\alpha)\Gamma(\beta)} + \frac{m}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\ & + \left( \frac{1}{\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta + p_i)} \right) \tilde{h}_q[0, 1] \\ & + \frac{|\alpha|m \tilde{h}_p[0, \mu] + |\alpha|\tilde{h}_q[0, \mu]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

$$\begin{aligned} &\leq \left( \frac{\Xi}{\Gamma(\alpha)\Gamma(\beta)} + \frac{\Xi}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \right) \tilde{h}_p[0,1] \\ &+ \left( \frac{1}{\Gamma(\alpha)\Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta|\Gamma(\beta+p_i)} \right) \tilde{q}_p[0,1] \\ &+ \frac{|a|\Xi\tilde{h}_p[0,\mu] + |a|\tilde{h}_q[0,\mu]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} < 1. \end{aligned}$$

Using Theorem 2.2, it is obtained that the equation has a solution. Let  $v(t)$  and  $\tilde{v}(t)$  be two solutions for the problem with initial conditions (7). Then we have

$$\begin{aligned} \tilde{v}(t) = &\frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{(B_\alpha(\zeta, \tilde{v}(\zeta)) - A_\alpha(\zeta, \tilde{v}(\zeta)))}{p(\zeta)} d\zeta \\ &+ \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta\Gamma(\beta+p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta+p_i-1} \frac{(B_\alpha(\zeta, \tilde{v}(\zeta)) - A_\alpha(\zeta, \tilde{v}(\zeta)))}{p(\zeta)} d\zeta \\ &- \frac{a}{\Delta\Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{(B_\alpha(\zeta, \tilde{v}(\zeta)) - A_\alpha(s, \tilde{v}(\zeta)))}{p(\zeta)} d\zeta, \end{aligned}$$

where  $\tilde{\Delta} = a - \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i a_i^{p_i}}{\Gamma(\beta+p_i)} \neq 0$ . Thus, it results in

$$\begin{aligned} |v(t) - \tilde{v}(t)| = &\left| \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{B_\alpha(\zeta, v(\zeta)) - B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right. \\ &+ \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta)) - A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\ &+ \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta\Gamma(\beta+p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta+p_i-1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\ &- \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i}{\tilde{\Delta}\Gamma(\beta+p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta+p_i-1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \\ &+ \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i}{\tilde{\Delta}\Gamma(\beta+p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta+p_i-1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \\ &- \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta\Gamma(\beta+p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta+p_i-1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\ &+ \frac{a}{\Delta\Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\ &- \frac{a}{\tilde{\Delta}\Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \\ &+ \frac{a}{\tilde{\Delta}\Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \\ &\left. - \frac{a}{\Delta\Gamma(\beta)} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right| \end{aligned}$$

for all  $t \in [0, 1]$ . Hence, it implies that

$$\begin{aligned}
|v(t) - \tilde{v}(t)| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{|B_\alpha(\zeta, v(\zeta)) - B_\alpha(\zeta, \tilde{v}(\zeta))|}{p(\zeta)} d\zeta \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-\zeta)^{\beta-1} \frac{|A_\alpha(\zeta, \tilde{v}(\zeta)) - A_\alpha(\zeta, v(\zeta))|}{p(\zeta)} d\zeta \\
&\quad + \left| \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right. \\
&\quad \left. - \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i}{\tilde{\Delta} \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right| \\
&\quad + \left| \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i}{\tilde{\Delta} \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right. \\
&\quad \left. - \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right| \\
&\quad + \frac{|a|}{\Gamma(\beta)} \left| \frac{1}{\Delta} \int_0^\mu (\mu - s)^{\beta-1} \frac{A_\alpha(s, v(s))}{p(s)} ds \right. \\
&\quad \left. - \frac{1}{\tilde{\Delta}} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{A_\alpha(s, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right| \\
&\quad + \frac{|a|}{\Gamma(\beta)} \left| \frac{1}{\tilde{\Delta}} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right. \\
&\quad \left. - \frac{1}{\Delta} \int_0^\mu (\mu - \zeta)^{\beta-1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right|. \tag{8}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_0^t (t-\zeta)^{\beta-1} \frac{|B_\alpha(\zeta, v(\zeta)) - B_\alpha(\zeta, \tilde{v}(\zeta))|}{p(\zeta)} d\zeta \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\zeta \frac{(t-\zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| |f(v(\xi)) - f(\tilde{v}(\xi))| d\xi d\zeta \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\zeta \frac{(t-\zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| \Lambda(|v(\xi) - \tilde{v}(\xi)|) d\xi d\zeta \\
&\leq \frac{\Lambda(\|v - \tilde{v}\|)}{\Gamma(\alpha)} \int_0^t \int_\xi^t \frac{(t-\zeta)^{\beta-1} (\zeta - \xi)^{\alpha-1}}{p(\zeta)} |h(\xi)| d\zeta d\xi \\
&\leq \frac{\Lambda(\|v - \tilde{v}\|)}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha+\beta-2} |h(\xi)| \hat{p}(t, \xi) d\xi,
\end{aligned}$$

and so

$$\int_0^t (t-\zeta)^{\beta-1} \frac{|B_\alpha(\zeta, v(\zeta)) - B_\alpha(\zeta, \tilde{v}(\zeta))|}{p(\zeta)} d\zeta \leq \frac{\Lambda(\|v - \tilde{v}\|)}{\Gamma(\alpha)} \tilde{h}_p[0, 1].$$

Similarly, it is obtained that

$$\int_0^t (t-\zeta)^{\beta-1} \frac{|A_\alpha(\zeta, \tilde{v}(\zeta)) - A_\alpha(\zeta, v(\zeta))|}{p(\zeta)} d\zeta \leq \frac{\|v - \tilde{v}\|}{\Gamma(\alpha)} \tilde{q}_p[0, 1].$$

Also, we have

$$\begin{aligned}
& \left| \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right. \\
& \quad \left. - \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i}{\tilde{\Delta} \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right| \\
& = \left| \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right. \\
& \quad \left. - \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i}{\tilde{\Delta} \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right. \\
& \quad \left. + \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right. \\
& \quad \left. - \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right| \\
& \leq \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{|B_\alpha(\zeta, v(\zeta)) - B_\alpha(\zeta, \tilde{v}(\zeta))|}{p(\zeta)} d\zeta \\
& \quad + \sum_{i=1}^{n_0} \frac{|\tilde{\lambda}_i - \Delta \tilde{\lambda}_i + \tilde{\Delta} \tilde{\lambda}_i - \tilde{\Delta} \tilde{\lambda}_i|}{|\Delta \tilde{\Delta}| \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \\
& \leq \frac{\Lambda(\|v - \tilde{v}\|)}{\Gamma(\alpha)} \tilde{h}_p[0, 1] \\
& \quad + \frac{|\tilde{\Delta}|}{\Gamma(\alpha) |\tilde{\Delta} \Delta|} \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} (M(\|\tilde{v}\|) + N(\|\tilde{v}\|)) \tilde{h}_p[0, 1] \\
& \quad + \frac{|\tilde{\Delta} - \Delta|}{\Gamma(\alpha) |\tilde{\Delta} \Delta|} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} (M(\|\tilde{v}\|) + N(\|\tilde{v}\|)) \tilde{h}_p[0, 1].
\end{aligned}$$

Note that  $|\Delta - \tilde{\Delta}| \leq \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)}$ , and so

$$\begin{aligned}
& \left| \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right. \\
& \quad \left. - \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i}{\tilde{\Delta} \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right| \\
& \leq \frac{\Lambda(\|v - \tilde{v}\|)}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& \quad + \frac{M(\|\tilde{v}\|) + N(\|\tilde{v}\|)}{\Gamma(\alpha) |\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& \quad + \frac{M(\|\tilde{v}\|) + N(\|\tilde{v}\|)}{|\tilde{\Delta} \Delta| \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1].
\end{aligned}$$

By using a similar method, we can show that

$$\begin{aligned} & \left| \sum_{i=1}^{n_0} \frac{\tilde{\lambda}_i}{\tilde{\Delta} \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right. \\ & \quad \left. - \sum_{i=1}^{n_0} \frac{\lambda_i}{\Delta \Gamma(\beta + p_i)} \int_0^{a_i} (a_i - \zeta)^{\beta + p_i - 1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta \right| \\ & \leq \frac{\|v - \tilde{v}\|}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] + \frac{\|\tilde{v}\|}{\Gamma(\alpha)|\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\ & \quad + \frac{\|\tilde{v}\|}{|\tilde{\Delta}| \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1}{\Delta} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta - \frac{1}{\tilde{\Delta}} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right| \\ & = \left| \frac{1}{\Delta} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{A_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta - \frac{1}{\tilde{\Delta}} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right. \\ & \quad \left. - \frac{1}{\Delta} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta + \frac{1}{\Delta} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{A_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right| \\ & \leq \frac{1}{|\Delta|} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{|A_\alpha(\zeta, v(\zeta)) - A_\alpha(\zeta, \tilde{v}(\zeta))|}{p(\zeta)} d\zeta \\ & \quad + \left| \frac{\Delta - \tilde{\Delta}}{\Delta \tilde{\Delta}} \right| \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{|A_\alpha(\zeta, \tilde{v}(\zeta))|}{p(\zeta)} d\zeta \\ & \leq \frac{\|v - \tilde{v}\|}{|\Delta| \Gamma(\alpha)} \tilde{q}_p[0, \mu] + \frac{\|\tilde{v}\|}{|\Delta| \tilde{\Delta} \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{q}_p[0, \mu], \end{aligned}$$

which implies

$$\begin{aligned} & \left| \frac{1}{\Delta} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{B_\alpha(\zeta, v(\zeta))}{p(\zeta)} d\zeta - \frac{1}{\tilde{\Delta}} \int_0^\mu (\mu - \zeta)^{\beta - 1} \frac{B_\alpha(\zeta, \tilde{v}(\zeta))}{p(\zeta)} d\zeta \right| \\ & \leq \frac{\Lambda(\|v - \tilde{v}\|)}{|\Delta| \Gamma(\alpha)} \tilde{h}_p[0, \mu] + \frac{M(\|\tilde{v}\|) + N(\|\tilde{v}\|)}{|\Delta| \tilde{\Delta} \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{h}_p[0, \mu]. \end{aligned}$$

Now, by using the above inequalities and (8), it is acquired

$$\begin{aligned} |\nu(t) - \tilde{v}(t)| & \leq \frac{\Lambda(\|v - \tilde{v}\|)}{\Gamma(\alpha) \Gamma(\beta)} \tilde{h}_p[0, 1] + \frac{\|v - \tilde{v}\|}{\Gamma(\alpha) \Gamma(\beta)} \tilde{q}_p[0, 1] \\ & \quad + \frac{\Lambda(\|v - \tilde{v}\|)}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\ & \quad + \frac{M(\|\tilde{v}\|) + N(\|\tilde{v}\|)}{\Gamma(\alpha) |\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\ & \quad + \frac{M(\|\tilde{v}\|) + N(\|\tilde{v}\|)}{|\tilde{\Delta}| \Delta \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \end{aligned}$$

$$\begin{aligned}
& + \frac{\|\nu - \tilde{\nu}\|}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] + \frac{\|\tilde{\nu}\|}{\Gamma(\alpha)|\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{\|\tilde{\nu}\|}{|\tilde{\Delta}\Delta|\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|a_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{|a|\|\nu - \tilde{\nu}\|}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \tilde{q}_p[0, \mu] + \frac{|a|\|\tilde{\nu}\|}{|\Delta\tilde{\Delta}|\Gamma(\alpha)\Gamma(\beta)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|a_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{q}_p[0, \mu] \\
& + \frac{|a|\Lambda(\|\nu - \tilde{\nu}\|)}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \tilde{h}_p[0, \mu] \\
& + \frac{|a|(M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|))}{|\Delta\tilde{\Delta}|\Gamma(\alpha)\Gamma(\beta)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|a_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{h}_p[0, \mu]
\end{aligned}$$

for all  $t \in [0, 1]$ . By the above inequality and taking the supremum norm, we have

$$\begin{aligned}
\|\nu - \tilde{\nu}\| & \leq \frac{\Lambda(\|\nu - \tilde{\nu}\|)}{\Gamma(\alpha)\Gamma(\beta)} \tilde{h}_p[0, 1] + \frac{\|\nu - \tilde{\nu}\|}{\Gamma(\alpha)\Gamma(\beta)} \tilde{q}_p[0, 1] \\
& + \frac{\Lambda(\|\nu - \tilde{\nu}\|)}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& + \frac{M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|)}{\Gamma(\alpha)|\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& + \frac{M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|)}{|\tilde{\Delta}\Delta|\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|a_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& + \frac{\|\nu - \tilde{\nu}\|}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] + \frac{\|\tilde{\nu}\|}{\Gamma(\alpha)|\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{\|\tilde{\nu}\|}{|\tilde{\Delta}\Delta|\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|a_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{|a|\|\nu - \tilde{\nu}\|}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \tilde{q}_p[0, \mu] + \frac{|a|\|\tilde{\nu}\|}{|\Delta\tilde{\Delta}|\Gamma(\alpha)\Gamma(\beta)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|a_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{q}_p[0, \mu] \\
& + \frac{|a|\Lambda(\|\nu - \tilde{\nu}\|)}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \tilde{h}_p[0, \mu] \\
& + \frac{|a|(M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|))}{|\Delta\tilde{\Delta}|\Gamma(\alpha)\Gamma(\beta)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|a_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{h}_p[0, \mu].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\nu - \tilde{\nu}\| & \left( 1 - \left[ \frac{\Lambda(\|\nu - \tilde{\nu}\|)}{\|\nu - \tilde{\nu}\|\Gamma(\alpha)\Gamma(\beta)} \tilde{h}_p[0, 1] + \frac{\tilde{q}_p[0, 1]}{\Gamma(\alpha)\Gamma(\beta)} \right. \right. \\
& \left. \left. + \frac{\Lambda(\|\nu - \tilde{\nu}\|)}{\|\nu - \tilde{\nu}\|\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \right. \\
& \left. + \frac{\tilde{q}_p[0, 1]}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) + \frac{|a|\tilde{q}_p[0, \mu]}{|\Delta|\Gamma(\alpha)\Gamma(\beta)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{|\alpha| \Lambda(\|\nu - \tilde{\nu}\|)}{\|\nu - \tilde{\nu}\| |\Delta| \Gamma(\alpha) \Gamma(\beta)} \tilde{h}_p[0, \mu] \Big] \Big) \\
& \leq \frac{M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|)}{\Gamma(\alpha) |\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& + \frac{M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|)}{|\tilde{\Delta} \Delta| \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| \alpha_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& + \frac{\|\tilde{\nu}\|}{\Gamma(\alpha) |\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{\|\tilde{\nu}\|}{|\tilde{\Delta} \Delta| \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| \alpha_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{|\alpha| \|\tilde{\nu}\|}{|\Delta \tilde{\Delta}| \Gamma(\alpha) \Gamma(\beta)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| \alpha_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{q}_p[0, \mu] \\
& + \frac{|\alpha| (M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|))}{|\Delta \tilde{\Delta}| \Gamma(\alpha) \Gamma(\beta)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| \alpha_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{h}_p[0, \mu].
\end{aligned}$$

If we put

$$\begin{aligned}
C = Q' & \left[ \frac{\tilde{h}_p[0, 1]}{\Gamma(\alpha) \Gamma(\beta)} + \frac{\tilde{h}_p[0, 1]}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) + \frac{|\alpha| \tilde{h}_p[0, \mu]}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} \right] \\
& + \frac{\tilde{q}_p[0, 1]}{\Gamma(\alpha) \Gamma(\beta)} + \frac{\tilde{q}_p[0, 1]}{\Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) + \frac{|\alpha| \tilde{q}_p[0, \mu]}{|\Delta| \Gamma(\alpha) \Gamma(\beta)},
\end{aligned}$$

then it is inferred that

$$\begin{aligned}
C \leq & \left( \frac{\Xi}{\Gamma(\alpha) \Gamma(\beta)} + \frac{\Xi}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& + \left( \frac{1}{\Gamma(\alpha) \Gamma(\beta)} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n_0} \frac{|\lambda_i|}{|\Delta| \Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{|\alpha| \Xi \tilde{h}_p[0, \mu] + |\alpha| \tilde{h}_q[0, \mu]}{|\Delta| \Gamma(\alpha) \Gamma(\beta)} < 1.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\nu - \tilde{\nu}\| \leq & \frac{1}{1-C} \left( \frac{M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|)}{\Gamma(\alpha) |\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \right. \\
& \left. + \frac{M(\|\tilde{\nu}\|) + N(\|\tilde{\nu}\|)}{|\tilde{\Delta} \Delta| \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| \alpha_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \right. \\
& \left. + \frac{\|\tilde{\nu}\|}{\Gamma(\alpha) |\Delta|} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\|\tilde{v}\|}{|\tilde{\Delta} \Delta| \Gamma(\alpha)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)} \right) \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{|a| \|\tilde{v}\|}{|\Delta \tilde{\Delta}| \Gamma(\alpha) \Gamma(\beta)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{q}_p[0, \mu] \\
& + \frac{|a|(M(\|\tilde{v}\|) + N(\|\tilde{v}\|))}{|\Delta \tilde{\Delta}| \Gamma(\alpha) \Gamma(\beta)} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i - \tilde{\lambda}_i| a_i^{p_i}}{\Gamma(p_i + 1)} \right) \tilde{h}_p[0, \mu].
\end{aligned}$$

Now, choose  $\delta > 0$  such that

$$\begin{aligned}
& \frac{1}{1-C} \left[ \frac{M(\|\tilde{v}\|) + N(\|\tilde{v}\|)}{\Gamma(\alpha)|\Delta|} \max_{1 \leq i \leq n_0} \left\{ \frac{1}{\Gamma(\beta + p_i)} \right\} \tilde{h}_p[0, 1] \right. \\
& + \frac{M(\|\tilde{v}\|) + N(\|\tilde{v}\|)}{|\tilde{\Delta} \Delta| \Gamma(\alpha)} \max_{1 \leq i \leq n_0} \left\{ \frac{a_i^{p_i}}{\Gamma(p_i + 1)} \right\} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{h}_p[0, 1] \\
& + \frac{\|\tilde{v}\|}{\Gamma(\alpha)|\Delta|} \max_{1 \leq i \leq n_0} \left\{ \frac{1}{\Gamma(\beta + p_i)} \right\} \tilde{q}_p[0, 1] \\
& + \frac{\|\tilde{v}\|}{|\tilde{\Delta} \Delta| \Gamma(\alpha)} \max_{1 \leq i \leq n_0} \left\{ \frac{a_i^{p_i}}{\Gamma(p_i + 1)} \right\} \left( \sum_{i=1}^{n_0} \frac{|\lambda_i|}{\Gamma(\beta + p_i)} \right) \tilde{q}_p[0, 1] \\
& + \frac{|a| \|\tilde{v}\|}{|\Delta \tilde{\Delta}| \Gamma(\alpha) \Gamma(\beta)} \max_{1 \leq i \leq n_0} \left\{ \frac{a_i^{p_i}}{\Gamma(p_i + 1)} \right\} \tilde{q}_p[0, \mu] \\
& \left. + \frac{|a|(M(\|\tilde{v}\|) + N(\|\tilde{v}\|))}{|\Delta \tilde{\Delta}| \Gamma(\alpha) \Gamma(\beta)} \max_{1 \leq i \leq n_0} \left\{ \frac{a_i^{p_i}}{\Gamma(p_i + 1)} \right\} \tilde{h}_p[0, \mu] \right] \delta < \epsilon.
\end{aligned}$$

Thus,  $\sum_{i=1}^{n_0} |\lambda_i - \tilde{\lambda}_i| < \delta$ , which implies that  $\|v - \tilde{v}\| < \epsilon$ . This completes the proof.  $\square$

#### 4 Conclusion

Different versions of the Sturm–Liouville have been studied by researchers during the last decades. In this work, we review a strong singular version of this important and well-known equation. The existence of a solution for a fractional order version of the Sturm–Liouville differential equation with generalized boundary conditions is investigated. Using inequalities and controlling functions lets us control singular points, especially strong singularity in fractional differential equations to be considered, so by the controlling functions and the fixed point theory, we control the strong singular points and prove the existence of a solution. The methods are novel, and a lot of differential equations could be examined in this way. In the following, by introducing the concept of continuous dependence for the generalized equation of Sturm–Liouville, we indicate that the solutions of the fractional strong singular version of Sturm–Liouville equation are dependent on the existent coefficients in the initial conditions, and any change can impact the solution of the equation. The existence of the strong singular points in this version of the Sturm–Liouville differential equation as well as the applied techniques are the most prominent novelty in this article. Likewise, these techniques can be used for investigating the singular version of other differential equations. An example is presented to demonstrate our main result.

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**Authors' contributions**

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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