Su Boundary Value Problems (2021) 2021:92
https://doi.org/10.1186/s13661-021-01570-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On global classical solutions to one-dimensional compressible Navier–Stokes/Allen–Cahn system with density-dependent viscosity and vacuum

Menglong Su[1*]

*Correspondence:
mlsulynu@163.com
[1]School of Mathematical Sciences, Luoyang Normal University, Luoyang, 471934, P.R. China

Abstract

In this paper, by using the energy estimates, the structure of the equations, and the properties of one dimension, we establish the global existence and uniqueness of strong and classical solutions to the initial boundary value problem of compressible Navier–Stokes/Allen–Cahn system in one-dimensional bounded domain with the viscosity depending on density. Here, we emphasize that the time does not need to be bounded and the initial vacuum is still permitted. Furthermore, we also show the large time behavior of the velocity.

MSC: 35D35; 76D03

1 Introduction

The Navier–Stokes/Allen–Cahn system, which is a combination of the compressible Navier–Stokes equations with an Allen–Cahn phase field description, is considered in this paper. Mathematically, in one dimension, this model reads as follows [5] (cf. [1]):

$$\rho_t + (\rho u)_x = 0, \tag{1}$$

$$\rho u_t + \rho u u_x + (\rho^\gamma)_x = (\nu(\rho)u_x)_x - \frac{\delta}{2}(\chi_x^2)_x, \tag{2}$$

$$\rho\chi_t + \rho u\chi_x = -\mu, \tag{3}$$

$$\rho\mu = -\delta\chi_{xx} + \frac{\rho}{\delta}(\chi^3 - \chi) \tag{4}$$

for $(t,x) \in (0,+\infty) \times [0,1]$. Here, ρ, u, and χ represent the density of the fluid, the mean velocity of the fluid mixture, and the concentration of one selected constituent, respectively; μ is the chemical potential, $\sqrt{\delta}$ represents the thickness of the interfacial region. The viscous coefficient $\nu(\rho) > 0$ satisfies

$$0 < \bar{\nu} \le \nu(\rho). \tag{5}$$

We supplement (1)–(4) with the initial value conditions

$$(\rho, u, \chi)(0, x) = (\rho_0, u_0, \chi_0)(x), \quad x \in [0, 1], \tag{6}$$

and the no-slip boundary conditions for viscous fluids and the concentration difference

$$(u, \chi)(t, 0) = (u, \chi)(t, 1) = (0, 0), \quad t \geq 0. \tag{7}$$

Before stating our main results, we review some previous works on this topic. For 1-dimensional compressible Navier–Stokes/Allen–Cahn system, Ding et al. [5] established the existence and uniqueness of local and global classical solutions for initial data ρ_0 without vacuum states. Besides, Ding et al. [6] proved the existence and uniqueness of global strong solutions to (1)–(4) with free boundary conditions and with the lower bound of the initial density. Yin et al. [18] investigated the large time behavior of the solutions to the inflow problem in the half space, and they obtained that the nonlinear wave is asymptotically stable if the initial data has a small perturbation. Recently, Luo et al. [15] (see also [14]) proved that the system tends to the rarefaction wave time-asymptotically, where the strength of the rarefaction wave is not required to be small. Chen et al. [2] established the global strong and classical solutions with initial vacuum in bounded domains. After that, Chen et al. [4] established the blowup criterion of the strong solutions with the viscosity depending on the density and the concentration of one selected constituent. Very recently, Yan et al. [17] considered the global existence of strong solutions with the phase variable dependent viscosity and the temperature dependent heat-conductivity without vacuum.

For the multi-dimensional compressible Navier–Stokes/Allen–Cahn system, Kotschote [11] established the local existence of a unique strong solution without initial vacuum. Later on, Feireisl et al. [8] proved the existence of weak solutions in 3D, where the density ρ is a measurable function, and they [9] obtained the global weak solutions in the bounded domain of $\mathbf{R}^3$ without any restriction on the initial data for $\gamma > 6$, which was extended to $\gamma > 2$ by Chen et al. [3]. Hošek et al. [10] considered the weak-strong uniqueness result in a bounded domain of $\mathbf{R}^3$ under the incompressibility assumption, which is relying on the relative entropy method. Very recently, Feireisl et al. [7] proved that the model is thermodynamically consistent, particularly, a variant of the relative energy inequality holds. At the same time, they obtained the weak-strong uniqueness principle and showed the low Mach number limit to the standard incompressible model. For more related results, we refer the readers to Zheng et al. [19], Liu et al. [12], and Ma et al. [16].

Although considerable progress has been made to the compressible Navier–Stokes/Allen–Cahn system, one of the natural questions is whether one could obtain the global classical solutions without any small assumption on the initial data or perturbations, where the time t could tend to $+\infty$? Motivated by [13], we give a partial answer to this question.

Our first main result in the paper is the following.

Theorem 1 *Assume that* $0 \leq \rho_0 \in H^1$, $\mu_0 \in L^2$, $u_0 \in H_0^1$, *and* $\chi_0 \in H_0^1 \cap H^2$. *Then there exists a global strong solution* (ρ, u, χ) *to the initial boundary value problem* (1)–(7) *such* *that, for all* $T \in (0, +\infty)$,

$$\begin{cases} \rho \in C([0,T];H^1), \\ \rho_t \in L^\infty(0,T;L^2), \\ u \in L^\infty(0,T;H_0^1) \cap L^2(0,T;H^2), \\ \sqrt{\rho}u_t \in L^2(0,T;L^2), \\ \chi \in L^\infty(0,T;H_0^1 \cap H^2), \\ (\sqrt{t}u,\chi) \in L^\infty(0,T;H^2), \\ (\sqrt{t}u_t,\chi_t) \in L^2(0,T;H_0^1). \end{cases} \tag{8}$$

Especially, the density can remain uniformly bounded for all time, that is,

$$\sup_{0\le t<+\infty} \|\rho(t,\cdot)\|_{L^\infty} < +\infty, \tag{9}$$

$$\sup_{0\le t<+\infty} \|\chi(t,\cdot)\|_{H_0^1\cap H^2} < +\infty, \tag{10}$$

and

$$\lim_{t\to+\infty} \|u(t,\cdot)\|_{W^{1,p}} = 0, \quad \forall p \in [1,\infty). \tag{11}$$

The following result means that the strong solution obtained by Theorem 1 is a classical solution provided that the initial data (ρ_0, u_0, χ_0) satisfies some additional conditions.

Theorem 2 *Assume* $\nu(\rho) \in C^2[0,\infty)$, *and the initial data* (ρ_0, u_0, χ_0) *satisfies*

$$(\rho_0, \rho_0^\gamma) \in H^2, \qquad (u_0, \chi_0) \in H_0^1 \cap H^2$$

and the following compatibility condition

$$\begin{cases} [\nu(\rho_0)u_{0x}]_x - a(\rho_0^\gamma)_x = \sqrt{\rho_0}g, \\ \mu_0 = \sqrt{\rho_0}h, \end{cases} \tag{12}$$

where $g \in L^2$ *and* $h \in H^1$. *Then the strong solution* (ρ, u, χ) *obtained in Theorem 1 becomes a classical solution and satisfies, for any* $0 < T < +\infty$,

$$\begin{cases} \rho, \rho^\gamma \in C([0,T];H^2), \\ \rho_t, \rho_t^\gamma \in C([0,T];H^1), \\ \rho_{tt}, \rho_{tt}^\gamma \in L^2(0,T;L^2), \\ u \in C([0,T];H^2) \cap L^2(0,T;H^3), \\ \chi \in C([0,T];H^3), \\ (u_t, \chi_t) \in L^2(0,T;H_0^1), \\ (\sqrt{t}u, \chi) \in L^\infty(0,T;H^3), \\ (\sqrt{t}u_t, \sqrt{t}\chi_t) \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2), \\ (\sqrt{t}\sqrt{\rho}u_{tt}, \sqrt{\rho}\chi_{tt}) \in L^2(0,T;L^2). \end{cases} \tag{13}$$

A few remarks are listed in order:

Remark 1 Compared with the previous results [2], ours are more general. First, the viscosity $\nu(\rho)$ depends on the density; Second, we remove the compatibility condition in obtaining the strong solution in Theorem 1, and Theorem 2 is established under (12), which has improved Theorem 2 in [2]; Third, the density ρ is uniformly bounded for all time and the large time behavior of u is also obtained, see (11) for details.

Remark 2 Similar to [2], we have to use the no-slip boundary condition on χ to deal with the term $\int \rho^2 u_x \chi_t^2\,dx$ in (29) because $\|u_x\|_{L^\infty}$ is not time-integrable when we establish the time-independent lower order estimates, see (26) below.

Remark 3 The concentration χ is uniformly (in time) bounded with higher order estimates in (10), without any decay as $t \to +\infty$, perhaps because it appears in the hyperbolicity in (3) rather than in the parabolicity in (24).

We now make some comments on the analysis of this paper. To obtain the results stated in Theorems 1 and 2, which mainly establish the time-independent lower order estimates and the time-dependent higher order ones, the method used in [2] is not suitable here, due to the all time-dependent a priori estimates. Moreover, it is difficult to obtain the large time behavior of solutions (11). Here, it is noted that we borrow some ideas from [13], where they discussed the global large classical solutions to the compressible Navier–Stokes equations. The key uniform upper bound of the density is obtained by Zlotnik's inequality, which is also successfully used to system (1)–(7) (see Lemma 4). Furthermore, the key time-independent L^2-norm of u_x is bounded by the material derivative $u_t + uu_x$ (see Lemma 5). With the lower order estimates obtained in Lemmas 3–6, the time-dependent higher order estimates on (ρ, u, χ) are obtained by standard energy estimates and the properties of one dimension.

The paper is organized as follows. In the next section, we deduce the desired estimates globally in time. By using the a priori estimates obtained in Sect. 2, we complete the proofs of Theorems 1 and 2 in Sect. 3.

2 A priori estimates

In this section, we establish some necessary a priori estimates of the solutions to (1)–(7) to extend the local solution to a global one, which is guaranteed by the following Lemma 1, whose proof can be obtained by similar arguments as those in [5].

Lemma 1 *Assume that* $\rho_0 \in C^{1,\alpha}$ *satisfies* $0 < C_0^{-1} \le \rho_0 \le C_0$ *for some constant* $\alpha \in (0,1)$ *and* $C_0 > 0$, $u_0, \chi_0 \in C^{2,\alpha}$. *Then there exists a small time* $T_0 > 0$ *depending only on* (ρ_0, u_0, χ_0) *such that the initial boundary value problem* (1)–(7) *admits a unique classical solution* (ρ, u, χ) *satisfying that*

$$(\rho_x, \rho_t) \in C^{\alpha,\frac{\alpha}{2}}, \qquad 0 < C^{-1} \le \rho \le C, \qquad (u,\chi) \in C^{2+\alpha,\frac{2+\alpha}{2}},$$

where the $C^{a,b}$ *is the usual Hölder space.*

Before starting the a priori estimates, we list Zlotnik's inequality which could be found in [20] and will be used to establish the uniform upper bound of the density.

Lemma 2 *Let the function y satisfy*

$$y'(t) = g(y) + b'(t), \quad \text{on } [0,T], \qquad y(0) = y^0,$$

with $g \in C(\mathbf{R})$ and $y, b \in W^{1,1}(0,T)$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1),$$

for all $0 \le t_1 < t_2 \le T$ with some nonnegative constants N_0 and N_1, then

$$y(t) \le \max\{y^0, \bar{\zeta}\} + N_0 < \infty \quad \text{on } [0,T],$$

where $\bar{\zeta}$ is a constant such that

$$g(\zeta) \le -N_1 \quad \text{for } \zeta \ge \bar{\zeta}.$$

2.1 A priori estimates (I): Lower order estimates

We emphasize that, in this subsection, C denotes some positive constant, which may be changed line by line and depends only on $\bar{\nu}, \delta, \gamma$ and the initial data (ρ_0, u_0, χ_0), but without the lower bound of the initial density ρ_0 and the length of T. First of all, we have the following basic energy estimates.

Lemma 3 *Let (ρ, u, χ) be a smooth solution of (1)–(7) on $(0,T) \times [0,1]$. Then one has*

$$\sup_{t\in[0,T]} \int \left(\frac{\rho u^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + \frac{\rho(\chi^2 - 1)^2}{4} + \frac{\chi_x^2}{2} \right) dx + \int_0^T \int \left(u_x^2 + \mu^2 \right) dx\, dt \le C. \tag{14}$$

Proof This lemma can be obtained by standard energy estimates. Multiplying (2), (3) by u and μ, respectively, by integrating by parts and by using (1) and (4), we obtain (14), the details can be found in [2]. □

Due to the basic energy inequality (14), we first consider the uniform upper bound of the density ρ, which does not depend on the length of time T.

Lemma 4 *Let (ρ, u, χ) be a smooth solution of (1)–(7) on $(0,T) \times [0,1]$. Then one has*

$$0 \le \rho(t,x) \le C. \tag{15}$$

Proof To prove this lemma, we borrow some some ideas of [13]. First of all, integrating (2) over $(0,x)$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} \int_0^x \rho u\, dy + \rho u^2 + \rho^\gamma - \nu(\rho) u_x + \frac{\delta}{2}\chi_x^2 \\ &\quad = \left[\rho u^2 + \rho^\gamma - \nu(\rho) u_x + \frac{\delta}{2}\chi_x^2 \right](t,0), \end{aligned} \tag{16}$$

which implies that

$$\begin{aligned}
&\left[\rho u^2+\rho^\gamma-\nu(\rho)u_x+\frac{\delta}{2}\chi_x^2\right](t,0) \\
&\quad=\frac{\partial}{\partial t}\int_0^1\int_0^x\rho u\,dy\,dx+\int_0^1\rho u^2\,dx+\int_0^1\rho^\gamma\,dx-\int_0^1\nu(\rho)u_x\,dx+\frac{\delta}{2}\int_0^1\chi_x^2\,dx.
\end{aligned}\tag{17}$$

Combining (16) with (17), it follows from (14) that

$$\begin{aligned}
&\rho^\gamma-\nu(\rho)u_x+\frac{\delta}{2}\chi_x^2 \\
&\quad=\int_0^1\rho u^2\,dx+\int_0^1\rho^\gamma\,dx-\int_0^1\nu(\rho)u_x\,dx+\frac{\delta}{2}\int_0^1\chi_x^2\,dx \\
&\qquad+\frac{\partial}{\partial t}\int_0^1\int_0^x\rho u\,dy\,dx-\frac{\partial}{\partial t}\int_0^x\rho u\,dy-\rho u^2 \\
&\quad=\int_0^1\rho u^2\,dx+\int_0^1\rho^\gamma\,dx-\int_0^1\nu(\rho)u_x\,dx+\frac{\delta}{2}\int_0^1\chi_x^2\,dx \\
&\qquad+D_t\left(\int_0^1\rho\int_2^\rho\nu(s)s^{-2}\,ds\,dx+\int_0^1\int_0^x\rho u\,dy\,dx-\int_0^1\rho u\,dy\right) \\
&\quad\le C+D_t\big(B_1(t)+B_2(t)+B_3(t)\big),
\end{aligned}\tag{18}$$

where we have used the following fact (due to (1))

$$-\int_0^1\nu(\rho)u_x\,dx=D_t\int_0^1\rho\int_2^\rho\nu(s)s^{-2}\,ds\,dx.$$

The notion $D_tf(t,x)$ denotes the derivation operator $D_tf(t,x)=\partial_tf(t,x)+u\partial_xf(t,x)$.

Next, direct calculations show that

$$-\nu(\rho)u_x=D_t\int_1^\rho\nu(s)s^{-1}\,ds,$$

which together with (18) yields

$$\begin{aligned}
D_t\int_1^\rho\nu(s)s^{-1}\,ds&\le-\rho^\gamma-\frac{\delta}{2}\chi_x^2+C+D_t\big(B_1(t)+B_2(t)+B_3(t)\big) \\
&\le-\rho^\gamma+C+D_t\big(B_1(t)+B_2(t)+B_3(t)\big).
\end{aligned}\tag{19}$$

Now, we focus on the estimates of the last term on the right-hand side of (19). First, by (14) and Hölder's inequality, we easily obtain

$$\big|B_2(t)\big|+\big|B_3(t)\big|\le C\int_0^1\int_0^x\rho\,dy\int_0^x\rho u^2\,dx+\int_0^1\rho\,dx\int_0^1\rho u^2\,dx\le C.\tag{20}$$

Next, we also have

$$B_1(t)\le\frac{1}{2}\int_0^{\max\{\sup_{[0,T]\times[0,1]}\rho,2\}}\nu(s)s^{-1}\,ds.\tag{21}$$

Finally, it follows from Lemma 2, Zlotnik's inequality, and (19) that

$$\int_1^\rho \nu(s)s^{-1}\,ds \le C + \frac{1}{2}\int_1^\rho \nu(s)s^{-1}\,ds,$$

which together with (5) shows (15). This completes the proof. □

Next, we focus on L^2-estimates about $\rho\chi_t$, u_x, and χ_{xx}, which are the key estimates for the proofs of the main theorems.

Lemma 5 *Let (ρ,u,χ) be a smooth solution of (1)–(7) on $(0,T)\times[0,1]$. Then one has*

$$\begin{aligned}&\sup_{t\in[0,T]}\big(\|\rho\chi_t\|_{L^2}^2+\|u_x\|_{L^2}^2+\|\chi_{xx}\|_{L^2}^2\big)\\&\quad+\int_0^T\big(\|\chi_{xt}\|_{L^2}^2+\big\|\sqrt{\rho}|u_t+uu_x|\big\|_{L^2}^2\big)\,dt\le C.\end{aligned}\tag{22}$$

Proof First, multiplying (2) by u_t+uu_x and integrating the resultant equality by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 \nu(\rho)u_x^2\,dx+\int_0^1\rho(u_t+uu_x)^2\,dx\\&\quad=\frac{d}{dt}\bigg(\int_0^1\rho^\gamma u_x\,dx+\frac{\delta}{2}\int_0^1\chi_x^2u_x\,dx\bigg)-\frac{1}{2}\int_0^1\big[\nu(\rho)-\rho\nu'(\rho)\big]u_x^3\,dx\\&\qquad+\gamma\int_0^1\rho^\gamma u_x^2\,dx-\delta\int_0^1\chi_x\chi_{xt}u_x\,dx-\delta\int_0^1\chi_x\chi_{xx}uu_x\,dx\\&\quad\le\frac{d}{dt}\bigg(\int_0^1\rho^\gamma u_x\,dx+\frac{\delta}{2}\int_0^1\chi_x^2u_x\,dx\bigg)+C\|u_x\|_{L^3}^3+C\|u_x\|_{L^2}^2\\&\qquad+C\|\chi_x\|_{L^\infty}\|\chi_{xt}\|_{L^2}\|u_x\|_{L^2}+C\|\chi_x\|_{L^\infty}\|\chi_{xx}\|_{L^2}\|u\|_{L^\infty}\|u_x\|_{L^2}\\&\quad\le\frac{d}{dt}\bigg(\int_0^1\rho^\gamma u_x\,dx+\frac{\delta}{2}\int_0^1\chi_x^2u_x\,dx\bigg)+C\|u_x\|_{L^\infty}\|u_x\|_{L^2}^2+C\|u_x\|_{L^2}^2\\&\qquad+C\|\chi_x\|_{L^2}^{1/2}\|\chi_{xx}\|_{L^2}^{1/2}\|\chi_{xt}\|_{L^2}\|u_x\|_{L^2}+C\|\chi_x\|_{L^2}^{1/2}\|\chi_{xx}\|_{L^2}^{3/2}\|u_x\|_{L^2}^2.\end{aligned}\tag{23}$$

Next, let us rewrite (3) and (4) as

$$\rho^2\chi_t+\rho^2u\chi_x=\chi_{xx}-\frac{\rho}{\delta}\big(\chi^3-\chi\big),\tag{24}$$

from which, due to (14), (15), and Young's inequality, we obtain

$$\begin{aligned}\|\chi_{xx}\|_{L^2}&\le C\|\rho\chi_t\|_{L^2}+C\|\sqrt{\rho}u\|_{L^2}\|\chi_x\|_{L^\infty}+C\\&\le\|\rho\chi_t\|_{L^2}+\frac{1}{2}\|\chi_{xx}\|_{L^2}+C.\end{aligned}\tag{25}$$

Next, due to (2), (14), (15), and (25), we obtain

$$\begin{aligned}\|u_x\|_{L^\infty}&\le C\big\|\nu(\rho)u_x-\rho^\gamma\big\|_{L^\infty}+C\big\|\rho^\gamma\big\|_{L^\infty}\\&\le C\big\|\nu(\rho)u_x-\rho^\gamma\big\|_{L^1}+C\big\|\big[\nu(\rho)u_x-\rho^\gamma\big]_x\big\|_{L^1}+C\end{aligned}\tag{26}$$

$$
\begin{aligned}
&\le C\big\|\nu(\rho)u_x\big\|_{L^1}+C\big\|\rho^\gamma\big\|_{L^1}+C\big\|\rho(u_t+uu_x)\big\|_{L^1}+C\|\chi_x\|_{L^2}\|\chi_{xx}\|_{L^2}+C\\
&\le C\int_0^1\nu(\rho)u_x^2\,dx+C\big\|\sqrt{\rho}(u_t+uu_x)\big\|_{L^2}+C\|\chi_{xx}\|_{L^2}+C.
\end{aligned}
$$

Then, substituting (25) and (26) into (23), and integrating the resultant inequality over $(0,t)$, one has

$$
\begin{aligned}
&\int_0^1\nu(\rho)u_x^2\,dx+\int_0^t\int_0^1\rho(u_t+uu_x)^2\,dx\,dt\\
&\quad\le C+C\int_0^t\left(\int_0^1\nu(\rho)u_x^2\,dx\right)^2ds+C_1\|\rho\chi_t\|_{L^2}^2+\varepsilon\int_0^t\|\chi_{xt}\|_{L^2}^2\,ds\\
&\qquad+C\int_0^t\left(\int_0^1\nu(\rho)u_x^2\,dx\right)\|\rho\chi_t\|_{L^2}^2\,ds,
\end{aligned}
\tag{27}
$$

where we have used the following fact:

$$
\int_0^1\rho^\gamma u_x\,dx+\int_0^1\chi_x^2u_x\,dx\le\frac14\big\|\sqrt{\nu(\rho)}u_x\big\|_{L^2}^2+C_1\|\rho\chi_t\|_{L^2}^2+C,
$$

due to (15) and (25).

Next, differentiating (24) with respect to t, we deduce that

$$
\begin{aligned}
&\rho^2\chi_{tt}+\left(\rho^2\right)_t\chi_t+\rho^2u\chi_{xt}\\
&\quad=\chi_{xxt}-\left(\rho^2\right)_tu\chi_x-\rho^2u_t\chi_x-\rho_t\left(\chi^3-\chi\right)-\rho\left(3\chi^2-1\right)\chi_t.
\end{aligned}
\tag{28}
$$

Then, multiplying (28) by χ_t and integrating the resultant equality by parts, we obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int\rho^2\chi_t^2\,dx+\int\chi_{xt}^2\,dx\\
&\quad=\frac12\int\rho^2u_x\chi_t^2\,dx-2\int\rho^2u\chi_t\chi_{xt}\,dx-\int\rho^2uu_x\chi_x\chi_t\,dx-\int\rho^2u^2\chi_{xx}\chi_t\,dx\\
&\qquad-\int\rho^2u^2\chi_x\chi_{xt}\,dx-\int\rho^2u_t\chi_x\chi_t\,dx-\int\rho u\left(3\chi^2-1\right)\chi_x\chi_t\,dx\\
&\qquad-\int\rho u\left(\chi^3-\chi\right)\chi_{xt}\,dx-\int\rho\left(3\chi^2-1\right)\chi_t^2\,dx=\sum_{i=1}^9I_i.
\end{aligned}
\tag{29}
$$

Now, we estimate each term on the right-hand side of (29). First, due to Sobolev's inequality, Hölder's inequality, and Young's inequality, one has

$$
\begin{aligned}
I_1&\le C\|\rho\|_{L^\infty}\|\chi_t\|_{L^\infty}\|u_x\|_{L^2}\|\rho\chi_t\|_{L^2}\\
&\le\varepsilon\|\chi_{xt}\|_{L^2}^2+C\left(\int_0^1\nu(\rho)u_x^2\,dx\right)\|\rho\chi_t\|_{L^2}^2.
\end{aligned}
$$

Similarly, we obtain

$$I_2 \le \varepsilon \|\chi_{xt}\|_{L^2}^2 + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)\|\rho\chi_t\|_{L^2}^2,$$

$$I_3 + I_5 + I_8 \le \varepsilon \|\chi_{xt}\|_{L^2}^2 + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)^2.$$

Due to (25), we obtain

$$\begin{aligned}
I_4 &\le C\|u\|_{L^\infty}^2\|\rho\chi_t\|_{L^2}^2\|\chi_{xx}\|_{L^2} \\
&\le C\|u_x\|_{L^2}^2\|\rho\chi_t\|_{L^2}^2 + C\|u_x\|_{L^2}^3 + C\|u_x\|_{L^2}^2 \\
&\le C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)\|\rho\chi_t\|_{L^2}^2 + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)^2 + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right).
\end{aligned}$$

From (3), (14), and (25), we have

$$\begin{aligned}
I_6 &= -\int_0^1 \rho^2(u_t + uu_x)\chi_x\chi_t\,dx + \int_0^1 \rho^2 uu_x\chi_x\chi_t\,dx \\
&= \int_0^1 \rho^2(u_t + uu_x)\chi_x^2 u\,dx + \int_0^1 \rho(u_t + uu_x)\chi_x\mu\,dx + \int_0^1 \rho^2 uu_x\chi_x\chi_t\,dx \\
&\le C\left\|\sqrt{\rho}(u_t + uu_x)\right\|_{L^2}\|u_x\|_{L^2}\|\chi_x\|_{L^4}^2 + C\|\chi_x\|_{L^\infty}\left\|\sqrt{\rho}(u_t + uu_x)\right\|_{L^2}\|\mu\|_{L^2} \\
&\quad + C\|\chi_x\|_{L^\infty}\|u_x\|_{L^2}^2\|\rho\chi_t\|_{L^2} \\
&\le C\left\|\sqrt{\rho}(u_t + uu_x)\right\|_{L^2}\|u_x\|_{L^2}\|\chi_x\|_{L^2}\|\chi_{xx}\|_{L^2} \\
&\quad + C\|\chi_x\|_{L^2}^{1/2}\|\chi_{xx}\|_{L^2}^{1/2}\left\|\sqrt{\rho}(u_t + uu_x)\right\|_{L^2}\|\mu\|_{L^2} \\
&\quad + C\|\chi_x\|_{L^2}^{1/2}\|\chi_{xx}\|_{L^2}^{1/2}\|u_x\|_{L^2}^2\|\rho\chi_t\|_{L^2} \\
&\le \varepsilon\left\|\sqrt{\rho}(u_t + uu_x)\right\|_{L^2}^2 + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)\|\rho\chi_t\|_{L^2}^2 \\
&\quad + C\|\mu\|_{L^2}^2\left(\|\rho\chi_t\|_{L^2}^2 + \int_0^1 \nu(\rho)u_x^2\,dx\right) \\
&\quad + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)^2 + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right) + C\|\mu\|_{L^2}^2.
\end{aligned}$$

By (3), (14), (15), and (25), I_7 could be rewritten and estimated as

$$\begin{aligned}
I_7 &= \int_0^1 \rho u^2\big(3\chi^2 - 1\big)\chi_x^2\,dx + \int_0^1 u\big(3\chi^2 - 1\big)\chi_x\mu\,dx \\
&\le C\|u\|_{L^\infty}^2\|\chi_x\|_{L^4}^2 + C\|u\|_{L^\infty}\|\chi_x\|_{L^2}\|\mu\|_{L^2} \\
&\le C\|u_x\|_{L^2}^2\|\chi_x\|_{L^2}\|\chi_{xx}\|_{L^2} + C\|u_x\|_{L^2}\|\chi_x\|_{L^2}\|\mu\|_{L^2} \\
&\le C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)\|\rho\chi_t\|_{L^2}^2 + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)^2 \\
&\quad + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right) + C\|\mu\|_{L^2}^2.
\end{aligned}$$

Similarly, the last term I_9 is

$$\begin{aligned} I_9 &= \int_0^1 \rho u(3\chi^2-1)\chi_x\chi_t\,dx + \int_0^1 (3\chi^2-1)\mu\chi_t\,dx \\ &\le C\|u\|_{L^\infty}\|\chi_t\|_{L^\infty}\|\chi_x\|_{L^2} + C\|\chi_t\|_{L^\infty}\|\mu\|_{L^2} \\ &\le C\|u_x\|_{L^2}\|\chi_{xt}\|_{L^2} + C\|\chi_{xt}\|_{L^2}\|\mu\|_{L^2} \\ &\le \varepsilon\|\chi_{xt}\|_{L^2}^2 + C\left(\int_0^1 \nu(\rho)u_x^2\,dx\right) + C\|\mu\|_{L^2}^2. \end{aligned}$$

Then, substituting all the above estimates into (29), and then integrating it over (τ,t), where $\tau\in(0,t)$, we obtain

$$\begin{aligned} &\int_0^1 \rho^2\chi_t^2(t)\,dx + \int_\tau^t\int_0^1 \chi_{xt}^2\,dx\,ds \\ &\quad\le C\int_\tau^t\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)\|\rho\chi_t\|_{L^2}^2\,ds + C\int_\tau^t\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)ds \\ &\qquad+ \varepsilon\int_\tau^t \left\|\sqrt{\rho}(u_t+uu_x)\right\|_{L^2}^2\,ds + \left\|\rho\chi_t(\tau)\right\|_{L^2}^2 + C. \end{aligned} \tag{30}$$

On the other hand, multiplying (3) by $\rho\chi_t$, we have

$$\begin{aligned} \int_0^1 \rho^2\chi_t^2(\tau)\,dx &= -\int_0^1 \rho^2 u\chi_x\chi_t(\tau)\,dx - \int_0^1 \rho\chi_t\mu(\tau)\,dx \\ &\le C\left\|\rho\chi_t(\tau)\right\|_{L^2}\left\|u(\tau)\right\|_{L^\infty}\left\|\chi_x(\tau)\right\|_{L^2} + C\left\|\rho\chi_t(\tau)\right\|_{L^2}\left\|\mu(\tau)\right\|_{L^2} \\ &\le \frac{1}{2}\left\|\rho\chi_t(\tau)\right\|_{L^2}^2 + C\|u_x\|_{L^2}^2\|\chi_x\|_{L^2}^2 + C\left\|\mu(\tau)\right\|_{L^2}^2. \end{aligned}$$

Substituting the above inequality into (30), then letting $\tau\to0^+$, we conclude that

$$\begin{aligned} &\int_0^1 \rho^2\chi_t^2\,dx + \int_0^t\int_0^1 \chi_{xt}^2\,dx\,ds \\ &\quad\le C\int_0^t\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)\|\rho\chi_t\|_{L^2}^2\,ds + C\int_0^t\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)^2 ds \\ &\qquad+ \varepsilon\int_0^t \left\|\sqrt{\rho}(u_t+uu_x)\right\|_{L^2}^2\,ds + C. \end{aligned} \tag{31}$$

At last, multiplying (31) by $1+C_1$, then adding the resultant inequality into (27), and choosing ε sufficiently small, we have

$$\begin{aligned} &\|\rho\chi_t\|_{L^2}^2 + \int_0^1 \nu(\rho)u_x^2\,dx + \int_0^t \left(\|\chi_{xt}\|_{L^2}^2 + \left\|\sqrt{\rho}(u_t+uu_x)\right\|_{L^2}^2\right)ds \\ &\quad\le C\int_0^t\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)^2 ds + C\int_0^t\left(\int_0^1 \nu(\rho)u_x^2\,dx\right)\|\sqrt{\rho}\chi_t\|_{L^2}^2\,ds + C, \end{aligned}$$

which together with Gronwall's inequality, (4), (14), and (25) shows (22). This completes the proof. □

Lemma 6 *Let* (ρ, u, χ) *be a smooth solution of* (1)–(7) *on* $(0,T)\times[0,1]$. *Then one has*

$$\sup_{t\in[0,T]} \sigma(t)\bigl(\|u\|_{W^{1,\infty}}^2 + \bigl\|\sqrt{\rho}(u_t+uu_x)\bigr\|_{L^2}^2\bigr) + \int_0^T \sigma(t)\bigl\|(u_t+uu_x)_x\bigr\|_{L^2}^2\,dt \le C, \tag{32}$$

where $\sigma(t) \triangleq \min\{1,t\}$.

Proof Taking the operator $\partial_t + (u\cdot)_x$ to (2), and multiplying it by $u_t + uu_x$, then integrating the resultant equality by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \rho|u_t+uu_x|^2\,dx + \int_0^1 \nu(\rho)\bigl|(u_t+uu_x)_x\bigr|^2\,dx \\
&\quad= \gamma\int_0^1 \rho^\gamma u_x(u_t+uu_x)_x\,dx + \int_0^1 \bigl[\nu(\rho)+\nu'(\rho)\rho\bigr]u_x^2(u_t+uu_x)_x\,dx \\
&\qquad+ \delta\int_0^1 \bigl(\chi_x\chi_{xt}(u_t+uu_x)_x + u\chi_x\chi_{xx}(u_t+uu_x)_x\bigr)\,dx \\
&\quad\le C\bigl(1+\|u_x\|_{L^\infty}\bigr)\|u_x\|_{L^2}\bigl\|(u_t+uu_x)_x\bigr\|_{L^2} \\
&\qquad+ C\|\chi_x\|_{L^\infty}\|\chi_{xt}\|_{L^2}\bigl\|(u_t+uu_x)_x\bigr\|_{L^2} \\
&\qquad+ C\|u\|_{L^\infty}\|\chi_x\|_{L^\infty}\|\chi_{xx}\|_{L^2}\bigl\|(u_t+uu_x)_x\bigr\|_{L^2} \\
&\quad\le C\bigl(1+\bigl\|\sqrt{\rho}(u_t+uu_x)\bigr\|_{L^2}\bigr)\|u_x\|_{L^2}\bigl\|(u_t+uu_x)_x\bigr\|_{L^2} \\
&\qquad+ C\|\chi_{xt}\|_{L^2}\bigl\|(u_t+uu_x)_x\bigr\|_{L^2} \\
&\quad\le \frac{1}{2}\bigl\|(u_t+uu_x)_x\bigr\|_{L^2}^2 + C\|u_x\|_{L^2}^2 + C\|u_x\|_{L^2}^2\bigl\|\sqrt{\rho}(u_t+uu_x)\bigr\|_{L^2}^2 + C\|\chi_{xt}\|_{L^2}^2,
\end{aligned} \tag{33}$$

due to (22), (25), and (26). Then, multiplying (33) by $\sigma(t)$ and integrating the resultant inequality over $(0,t)$, one has

$$\sup_{t\in[0,T]} \sigma(t)\int_0^1 \rho|u_t+uu_x|^2\,dx + \int_0^T\int_0^1 \nu(\rho)\bigl|(u_t+uu_x)_x\bigr|^2\,dx\,dt \le C, \tag{34}$$

due to (14), (22), and Gronwall's inequality. Furthermore, (34) together with (22) and (26) yields

$$\sup_{[0,T]} \sigma(t)\|u_x\|_{L^\infty} \le C. \tag{35}$$

Combining (34) and (35) leads to (32). This completes the proof. □

2.2 A priori estimates (II): higher order estimates

In this subsection, we derive the higher-order estimates of the smooth solution (ρ, u, χ) to system (1)–(7). Particularly, in this subsection the constant C may depend on the initial data (ρ_0, u_0, χ_0), γ, δ, and $\bar{\nu}$. Almost the a priori estimates obtained in this subsection could be obtained by similar arguments as in [2], we estimate them here to make the paper self-contained and satisfy the new assumptions on the initial data and compatibility condition (12).

Lemma 7 *Let* (ρ, u, χ) *be a smooth solution of* (1)– (7) *on* $(0,T)\times[0,1]$. *Then one has*

$$\sup_{t\in[0,T]}\left(\|\rho_t\|_{L^2}^2+\|\rho_x\|_{L^2}^2\right)\le C \tag{36}$$

and

$$\sup_{t\in[0,T]}\left(\sigma(t)\|\sqrt{\rho}u_t\|_{L^2}^2+\sigma(t)\|u_{xx}\|_{L^2}^2\right)+\int_0^T\left(\|u_{xx}\|_{L^2}^2+\sigma(t)\|u_{xt}\|_{L^2}^2\right)dt\le C. \tag{37}$$

Proof Differentiating (1) with respect to x leads to

$$\rho_{xt}+\rho_{xx}u+2\rho_x u_x+\rho u_{xx}=0. \tag{38}$$

Then, multiplying (38) by ρ_x and integrating the resultant equality by parts, by (15), one shows that

$$\frac{d}{dt}\|\rho_x\|_{L^2}\le C\|u_x\|_{L^\infty}\|\rho_x\|_{L^2}+C\|u_{xx}\|_{L^2}. \tag{39}$$

To prove the last term on the right-hand side of (39), let us rewrite (2) as

$$\nu(\rho)u_{xx}=\rho(u_t+uu_x)+\left(\rho^\gamma\right)_x-\nu'(\rho)\rho_x u_x-\delta\chi_x\chi_{xx}, \tag{40}$$

which together with (5), (15), (22), and (32) leads to

$$\begin{aligned}\|u_{xx}\|_{L^2}&\le C\left\|\sqrt{\rho}(u_t+uu_x)\right\|_{L^2}+C\left(1+\|u_x\|_{L^\infty}\right)\|\rho_x\|_{L^2}\\&\quad+C\|\chi_x\|_{L^\infty}\|\chi_{xx}\|_{L^2}\\&\le C\left(\sigma(t)\right)^{-1/2}\left(1+\|\rho_x\|_{L^2}\right).\end{aligned} \tag{41}$$

Therefore, substituting (41) into (39), then Gronwall's inequality gives

$$\sup_{t\in[0,T]}\|\rho_x\|_{L^2}\le C, \tag{42}$$

which together with (1), (15), and (22) yields

$$\sup_{t\in[0,T]}\|\rho_t\|_{L^2}\le C\|u\|_{L^\infty}\|\rho_x\|_{L^2}+C\|u_x\|_{L^2}\le C. \tag{43}$$

Next, it follows from (15), (22), and (41) that

$$\begin{aligned}\|\sqrt{\rho}u_t\|_{L^2}&\le C\left\|\sqrt{\rho}(u_t+uu_x)\right\|_{L^2}+C\|\sqrt{\rho}uu_x\|_{L^2}\\&\le C\left\|\sqrt{\rho}(u_t+uu_x)\right\|_{L^2}+C\end{aligned} \tag{44}$$

and

$$\begin{aligned}\|u_{xt}\|_{L^2}&\le C\left\|(u_t+uu_x)_x\right\|_{L^2}+C\|u_x\|_{L^4}^2+C\|u_x\|_{L^2}\|u_{xx}\|_{L^2}\\&\le C\left\|(u_t+uu_x)_x\right\|_{L^2}+C\|u_x\|_{L^2}^{3/4}\|u_{xx}\|_{L^2}^{1/4}+C\|u_{xx}\|_{L^2}\\&\le C\left\|(u_t+uu_x)_x\right\|_{L^2}+C,\end{aligned} \tag{45}$$

then (44) and (45) together with (32) and (41) yield (37). This completes the proof of Lemma 7. □

From now on, suppose that (ρ, u, χ) is a smooth solution of problem (1)–(7) with the smooth initial data satisfying the conditions in Theorem 2 and $\nu(\cdot) \in C^2[0,\infty)$.

Lemma 8 *Let* (ρ, u, χ) *be a smooth solution of* (1)– (7) *on* $(0,T) \times [0,1]$. *Then one has*

$$\sup_{t\in[0,T]} \left(\|u_{xx}\|_{L^2}^2 + \|u\|_{W^{1,\infty}} + \|\sqrt{\rho}u_t\|_{L^2}^2\right) + \int_0^t \|u_{xt}\|_{L^2}^2 \le C. \tag{46}$$

Proof Based on (33) and the compatibility condition (12), the initial data $\|\sqrt{\rho_0}(u_{0t} + u_0u_{0x})\|_{L^2}^2 \le C$. Therefore integrating (33) over $(0,t)$, one has

$$\sup_{t\in[0,T]} \int_0^1 \rho|u_t + uu_x|^2\,dx + \int_0^T \int_0^1 \nu(\rho)\left|(u_t + uu_x)_x\right|^2\,dx\,dt \le C,$$

which together with (22), (26), (41), (44), and (45) yields (46). This completes the proof. □

Lemma 9 *Let* (ρ, u, χ) *be a smooth solution of* (1)– (7) *on* $(0,T) \times [0,1]$. *Then one has*

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|\rho_{xx}\|_{L^2}^2 + \|\rho_{xt}\|_{L^2}^2 + \left\|\left(\rho^\gamma\right)_{xx}\right\|_{L^2}^2 + \left\|\left(\rho^\gamma\right)_{xt}\right\|_{L^2}^2 + \left\|\nu_{xt}(\rho)\right\|_{L^2}^2\right) \\ &\quad + \int_0^T \left(\|u_{xxx}\|_{L^2}^2 + \|\chi_{xxx}\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \left\|\left(\rho^\gamma\right)_{tt}\right\|_{L^2}^2 + \left\|\nu_{tt}(\rho)\right\|_{L^2}^2\right) \le C. \end{aligned} \tag{47}$$

Proof Differentiating (1) twice with respect to x, and multiplying the resultant equality by ρ_{xx}, then integrating it by parts, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int \rho_{xx}^2\,dx &\le C\|u_x\|_{L^\infty}\|\rho_{xx}\|_{L^2}^2 + C\|\rho_x\|_{L^\infty}\|\rho_{xx}\|_{L^2}\|u_{xx}\|_{L^2} \\ &\quad + C\|\rho_{xx}\|_{L^2}\|u_{xxx}\|_{L^2} \\ &\le C\|\rho_{xx}\|_{L^2}^2 + C\left(\|\rho_x\|_{L^2} + \|\rho_{xx}\|_{L^2}\right)\|\rho_{xx}\|_{L^2} + C\|u_{xxx}\|_{L^2}^2 \\ &\le C\|\rho_{xx}\|_{L^2}^2 + C\|u_{xxx}\|_{L^2}^2 + C \end{aligned} \tag{48}$$

due to (15), (36), and (46). Furthermore, it follows from (1) that

$$\left(\rho^\gamma\right)_t + \left(\rho^\gamma u\right)_x + (\gamma - 1)\rho^\gamma u_x = 0,$$

from which, by the same arguments as in (48), we deduce

$$\frac{1}{2}\frac{d}{dt}\int \left(\rho^\gamma\right)_{xx}^2\,dx \le C\left\|\left(\rho^\gamma\right)_{xx}\right\|_{L^2}^2 + C\|u_{xxx}\|_{L^2}^2 + C.$$

Combining the above inequality with (48) leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\rho_{xx}\|_{L^2}^2 + \left\|\left(\rho^\gamma\right)_{xx}\right\|_{L^2}^2\right) \\ &\quad \le C\left(\|\rho_{xx}\|_{L^2}^2 + \left\|\left(\rho^\gamma\right)_{xx}\right\|_{L^2}^2\right) + C\|u_{xxx}\|_{L^2}^2 + C. \end{aligned} \tag{49}$$

To obtain the second term on the right-hand side of (49), it follows from (2) that

$$\begin{aligned}\nu(\rho)u_{xxx} &= \rho_x u_t + \rho u_{xt} + \rho_x u u_x + \rho u_x^2 + \rho u u_{xx} + \left(\rho^\gamma\right)_{xx} + \delta\chi_{xx}^2 + \delta\chi_x\chi_{xxx} \\ &\quad - 2\nu'(\rho)\rho_x u_{xx} - \nu''(\rho)\rho_x^2 u_x - \nu'(\rho)\rho_{xx}u_x,\end{aligned}$$

which gives the following estimates:

$$\begin{aligned}\|u_{xxx}\|_{L^2} &\le C\|u_t\|_{L^\infty}\|\rho_x\|_{L^2} + C\|u_{xt}\|_{L^2} + C\|u_x\|_{L^2}\|u_x\|_{L^\infty}\|\rho_x\|_{L^2} \\ &\quad + C\|u_x\|_{L^4}^2 + C\|u_x\|_{L^2}\|u_{xx}\|_{L^2} + C\left\|\left(\rho^\gamma\right)_{xx}\right\|_{L^2} + C\|\chi_{xx}\|_{L^4}^2 \\ &\quad + C\|\chi_x\|_{L^\infty}\|\chi_{xxx}\|_{L^2} + C\|\rho_x\|_{L^\infty}\|u_{xx}\|_{L^2} + C\|\rho_x\|_{L^4}^2\|u_x\|_{L^\infty} \\ &\quad + C\|u_x\|_{L^\infty}\|\rho_{xx}\|_{L^2} \\ &\le C\left\|\left(\rho^\gamma\right)_{xx}\right\|_{L^2} + C\|\rho_{xx}\|_{L^2} + \|u_{xt}\|_{L^2}^2 + C\|\chi_{xxx}\|_{L^2} + C,\end{aligned} \tag{50}$$

where we have used (15), (22), (36), and (46). Furthermore, it follows from (24) that

$$\begin{aligned}\chi_{xxx} &= 2\rho\rho_x\chi_t + \rho^2\chi_{xt} + 2\rho\rho_x u\chi_x + \rho^2 u_x\chi_x + \rho^2 u\chi_{xx} \\ &\quad + \rho_x\left(\chi^3 - \chi\right) + \rho\left(3\chi^2 - 1\right)\chi_x,\end{aligned}$$

which, due to (14), (22), (36), and (46), implies that

$$\begin{aligned}\|\chi_{xxx}\|_{L^2} &\le C\left(1 + \|\chi_x\|_{L^\infty} + \|\chi_{xt}\|_{L^2}\right)\left(\|\rho_x\|_{L^2} + \|u_x\|_{L^2} + 1\right) + C \\ &\le C\|\chi_{xt}\|_{L^2} + C.\end{aligned} \tag{51}$$

Then, substituting (50) and (51), by using Gronwall's inequality, (22), and (46), we obtain

$$\sup_{t\in[0,T]}\left(\|\rho_{xx}\|_{L^2}^2 + \left\|\left(\rho^\gamma\right)_{xx}\right\|_{L^2}^2\right) \le C. \tag{52}$$

Next, it follows from (38) that

$$\|\rho_{xt}\|_{L^2} \le C\|\rho_{xx}\|_{L^2}\|u_x\|_{L^2} + C\|\rho_x\|_{L^2}\|u_x\|_{L^\infty} + C\|u_{xx}\|_{L^2} \le C \tag{53}$$

due to (22), (36), (46), and (52). Furthermore, it follows from (1) that

$$\rho_{tt} = -\rho_{xt}u + \rho_x u_t + \rho_t u_x - \rho u_{xt},$$

from which we have

$$\begin{aligned}&\int_0^T \|\rho_{tt}\|_{L^2}^2\,dt \\ &\quad\le C\int_0^T \left(\|u_x\|_{L^2}^2\|\rho_{xt}\|_{L^2}^2 + C\|u_{xt}\|_{L^2}^2\|\rho_x\|_{L^2}^2 + C\|u_x\|_{L^\infty}^2\|\rho_t\|_{L^2}^2 + C\|u_{xt}\|_{L^2}^2\right)dt \\ &\quad\le C\int_0^T \|u_{xt}\|_{L^2}^2\,dt + C\end{aligned} \tag{54}$$

due to (22), (36), (46), and (53). Similarly, we obtain

$$\sup_{t\in[0,T]}\left(\left\|\left(\rho^{\gamma}\right)_{xt}\right\|_{L^2}^2+\left\|v_{xt}(\rho)\right\|_{L^2}^2\right)+\int_0^T\left(\left\|\left(\rho^{\gamma}\right)_{tt}\right\|_{L^2}^2+\left\|v_{tt}(\rho)\right\|_{L^2}^2\right)dt\le C,$$

which together with (22) and (50)–(54) shows (47). Therefore, we complete the proof. □

Lemma 10 *Let (ρ,u,χ) be a smooth solution of (1)– (7) on $(0,T)\times[0,1]$. Then we have*

$$\begin{aligned}&\sup_{t\in[0,T]}\left(t\|u_{xt}\|_{L^2}^2+t\|\rho\chi_{tt}\|_{L^2}^2+\|\chi_{xt}\|_{L^2}^2+t\|u_{xxx}\|_{L^2}^2+\|\chi_{xxx}\|_{L^2}^2\right)\\&\quad+\int_0^T\left(t\|\sqrt{\rho}u_{tt}\|_{L^2}^2+t\|u_{xxt}\|_{L^2}^2+t\|\chi_{xxt}\|_{L^2}^2+t\|\chi_{xtt}\|_{L^2}^2+\|\rho\chi_{tt}\|_{L^2}^2\right)dt\le C.\end{aligned}\tag{55}$$

Proof Multiplying (28) by χ_{tt}, integrating the resultant equality by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\chi_{xt}^2\,dx+\int_0^1\rho^2\chi_{tt}^2\,dx\\&\quad=-\frac{d}{dt}\int_0^1\rho_t\left(\chi^3-\chi\right)\chi_t\,dx-\int_0^1\rho^2u\chi_{xt}\chi_{tt}\,dx-2\int_0^1\rho\rho_tu\chi_x\chi_{tt}\,dx\\&\qquad-\int_0^1\rho^2u_t\chi_x\chi_{tt}\,dx-2\int_0^1\rho\rho_t\chi_t\chi_{tt}\,dx-\int_0^1\rho\left(3\chi^2-1\right)\chi_t\chi_{tt}\,dx\\&\qquad+\int_0^1\rho_{tt}\left(\chi^3-\chi\right)\chi_t\,dx+\int_0^1\rho_t\left(3\chi^2-1\right)\chi_t^2\,dx\\&\quad\le-\frac{d}{dt}\int_0^1\rho_t\left(\chi^3-\chi\right)\chi_t\,dx+C\|u\|_{L^\infty}\|\rho\chi_{tt}\|_{L^2}\|\chi_{xt}\|_{L^2}\\&\qquad+C\|u\|_{L^\infty}\|\rho\chi_t\|_{L^2}\|\chi_x\|_{L^\infty}\|\rho_t\|_{L^2}+C\|\rho\chi_{tt}\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|\chi_x\|_{L^\infty}\\&\qquad+C\|\rho\chi_{tt}\|_{L^2}\left(\|\rho_t\|_{L^2}\|\chi_t\|_{L^\infty}+\|\chi_t\|_{L^\infty}\right)+C\|\rho_{tt}\|_{L^2}\|\chi_t\|_{L^\infty}\\&\qquad+C\|\rho_t\|_{L^2}\|\chi_t\|_{L^\infty}^2\\&\quad\le-\frac{d}{dt}\int_0^1\rho_t\left(\chi^3-\chi\right)\chi_t\,dx+\frac{1}{2}\|\rho\chi_{tt}\|_{L^2}^2+C\|\chi_{xt}\|_{L^2}^2+C,\end{aligned}$$

which together with the compatibility condition (12) yields

$$\sup_{t\in[0,T]}\|\chi_{xt}\|_{L^2}^2+\int_0^T\|\rho\chi_{tt}\|_{L^2}^2\,dt\le C,\tag{56}$$

by using the fact that

$$\int_0^1\rho_t\left(\chi^3-\chi\right)\chi_t\,dx\le\frac{1}{4}\|\chi_{xt}\|_{L^2}^2+C\|\rho_t\|_{L^2}^2.$$

Furthermore, (56) together with (51) leads to

$$\sup_{t\in[0,T]}\|\chi_{xxx}\|_{L^2}^2\le C.\tag{57}$$

To proceed, differentiating (28) with respect to t, multiplying the resultant equation by χ_{tt}, and integrating by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \rho^2\chi_{tt}^2\,dx + \int_0^1 \chi_{xtt}^2\,dx \\
&= -\frac{1}{2}\int_0^1 \rho^2 u_x\chi_{tt}^2\,dx - 4\int_0^1 \rho\rho_t\chi_{tt}^2\,dx - 2\int_0^1 \rho_t^2\chi_t\chi_{tt}\,dx \\
&\quad - 2\int_0^1 \rho\rho_{tt}\chi_t\chi_{tt}\,dx - 4\int_0^1 \rho\rho_t u\chi_{xt}\chi_{tt}\,dx - \int_0^1 \rho^2 u_t\chi_{xt}\chi_{tt}\,dx \\
&\quad - 2\int_0^1 \rho_t^2 u\chi_x\chi_{tt}\,dx - 2\int_0^1 \rho\rho_{tt}u\chi_x\chi_{tt}\,dx - 4\int_0^1 \rho\rho_t u_t\chi_x\chi_{tt}\,dx \\
&\quad - 2\int_0^1 \rho^2 u_{tt}\chi_x\chi_{tt}\,dx - 2\int_0^1 \rho^2 u_t\chi_{xt}\chi_{tt}\,dx - \int_0^1 \rho_{tt}\big(\chi^3-\chi\big)\chi_{tt}\,dx \\
&\quad - 2\int_0^1 \rho_t\big(3\chi^2-1\big)\chi_t\chi_{tt}\,dx - 6\int_0^1 \rho\chi\chi_t^2\chi_{tt}\,dx - \int_0^1 \rho\big(3\chi^2-1\big)\chi_{tt}^2\,dx \\
&\le C\|u_x\|_{L^\infty}\|\rho\chi_{tt}\|_{L^2}^2 + C\|\chi_{tt}\|_{L^\infty}\|\rho_t\|_{L^2}\|\rho\chi_{tt}\|_{L^2} + C\|\chi_t\|_{L^\infty}\|\chi_{tt}\|_{L^\infty}\|\rho_t\|_{L^2}^2 \\
&\quad + C\|\chi_t\|_{L^\infty}\|\rho_{tt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} + C\|\chi_{tt}\|_{L^\infty}\|\rho_t\|_{L^2}\|\chi_{xt}\|_{L^2} \\
&\quad + C\|u_t\|_{L^\infty}\|\chi_{xt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} + C\|\chi_{tt}\|_{L^\infty}\|\chi_x\|_{L^\infty}\|\rho_t\|_{L^2}^2 \\
&\quad + C\|\chi_x\|_{L^\infty}\|\rho_{tt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} + C\|u_t\|_{L^\infty}\|\rho_t\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \\
&\quad + C\|\chi_x\|_{L^\infty}\|\sqrt{\rho}u_{tt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} + C\|u_t\|_{L^\infty}\|\chi_{xt}\|_{L^2}\|\rho\chi_{tt}\|_{L^2} \\
&\quad + C\|\chi_{tt}\|_{L^2}\|\rho_{tt}\|_{L^2} + C\|\chi_{tt}\|_{L^\infty}\|\chi_t\|_{L^\infty}\|\rho_t\|_{L^2} \\
&\quad + C\|\chi_t\|_{L^\infty}^2\|\chi\|_{L^\infty}\|\rho\chi_{tt}\|_{L^2} + C\|\chi_{tt}\|_{L^\infty}\|\rho\chi_{tt}\|_{L^2} \\
&\le \frac{1}{2}\|\chi_{xtt}\|_{L^2}^2 + \varepsilon\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C\|\rho\chi_{tt}\|_{L^2}^2 + C\|u_{xt}\|_{L^2}^2 + C\|\rho_{tt}\|_{L^2}^2 + C.
\end{aligned}
$$

Multiplying the above inequality by t and using Gronwall's inequality, we obtain

$$
\sup_{t\in[0,T]} t\|\rho\chi_{tt}\|_{L^2}^2 + \int_0^T t\|\chi_{xtt}\|_{L^2}^2\,dt \le \varepsilon\int_0^T t\|\sqrt{\rho}u_{tt}\|_{L^2}^2\,dt + C, \tag{58}
$$

where we have used (46), (47), and (56).

Then, differentiating (2) with respect to t leads to

$$
\rho u_{tt} + \rho u u_{xt} - \big[\nu(\rho)u_x\big]_{xt} = -\rho_t(u_t + uu_x) - \rho u_t u_x - \frac{\delta}{2}\big(\chi_x^2\big)_{xt}. \tag{59}
$$

Multiplying the above equation by u_{tt} and integrating the resultant equality by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \nu(\rho)u_{xt}^2\,dx + \|\sqrt{\rho}u_{tt}\|_{L^2}^2 \\
&= \frac{1}{2}\int_0^1 \nu_t(\rho)u_{xt}^2\,dx - \int_0^1 \nu_t(\rho)u_x u_{xtt}\,dx - \int_0^1 \rho_t(u_t+uu_x)u_{tt}\,dx \\
&\quad - \int_0^1 \rho u_t u_x u_{tt}\,dx - \int_0^1 \big(\rho^\gamma\big)_{xt}u_{tt}\,dx - \frac{\delta}{2}\int_0^1 \big(\chi_x^2\big)_{xt}u_{tt}\,dx = \sum_{i=1}^6 J_i.
\end{aligned} \tag{60}
$$

Now, we focus on the estimates of the terms on the right-hand side of (60). First, due to (36) and (47), we obtain

$$J_1 \le C\big\|\nu_t(\rho)\big\|_{L^\infty}\|u_{xt}\|_{L^2}^2 \le C\|\rho_t\|_{L^2}^{1/2}\big\|\nu_{xt}(\rho)\big\|_{L^2}^{1/2}\|u_{xt}\|_{L^2}^2 \le C\|u_{xt}\|_{L^2}^2.$$

It follows from (46) and (47) that

$$\begin{aligned} J_2 &= -\frac{d}{dt}\int_0^1 \nu_t(\rho)u_xu_{xt}\,dx + \int_0^1 \nu_{tt}(\rho)u_xu_{xt}\,dx + \int_0^1 \nu_t(\rho)u_{xt}^2\,dx \\ &\le -\frac{d}{dt}\int_0^1 \nu_t(\rho)u_xu_{xt}\,dx + C\|u_x\|_{L^\infty}\big\|\nu_{tt}(\rho)\big\|_{L^2}\|u_{xt}\|_{L^2} \\ &\quad + C\big\|\nu_t(\rho)\big\|_{L^\infty}\|u_{xt}\|_{L^2}^2 \\ &\le -\frac{d}{dt}\int_0^1 \nu_t(\rho)u_xu_{xt}\,dx + C\|u_{xt}\|_{L^2}^2 + C\big\|\nu_{tt}(\rho)\big\|_{L^2}^2. \end{aligned}$$

Next, it follows from (1) and integration by parts that

$$\begin{aligned} J_3 &= \int_0^1 (\rho u)_x(u_t+uu_x)u_{tt}\,dx = -\int_0^1 \rho u(u_t+uu_x)_xu_{tt}\,dx \\ &\quad - \int_0^1 \rho u(u_t+uu_x)u_{xtt}\,dx \\ &\le -\frac{d}{dt}\int_0^1 \rho u(u_t+uu_x)u_{xt}\,dx + C\|\sqrt{\rho}u_{tt}\|_{L^2}\big\|(u_t+uu_x)_x\big\|_{L^2} \\ &\quad + \int_0^1 \rho_tu(u_t+uu_x)u_{xt}\,dx + \int_0^1 \rho u_t(u_t+uu_x)u_{xt}\,dx \\ &\quad + \int_0^1 \rho u(u_t+uu_x)_tu_{xt}\,dx \\ &\le -\frac{d}{dt}\int_0^1 \rho u(u_t+uu_x)u_{xt}\,dx + C\|\sqrt{\rho}u_{tt}\|_{L^2}\big\|(u_t+uu_x)_x\big\|_{L^2} \\ &\quad + C\|u_t+uu_x\|_{L^\infty}\|\rho_t\|_{L^2}\|u_{xt}\|_{L^2} \\ &\quad + C\|u_t+uu_x\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|u_{xt}\|_{L^2} + C\|u_{xt}\|_{L^2}^2 \\ &\le -\frac{d}{dt}\int_0^1 \rho u(u_t+uu_x)u_{xt}\,dx + \frac{1}{2}\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C\|u_{xt}\|_{L^2}^2 \\ &\quad + C\big\|(u_t+uu_x)_x\big\|_{L^2}^2, \end{aligned}$$

where we have used (15), (36), (46). Moreover, by (46), one easily shows that

$$J_4 \le C\|u_x\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u_{tt}\|_{L^2} \le \varepsilon\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C.$$

Furthermore, J_5 is estimated as

$$\begin{aligned} J_5 &= \frac{d}{dt}\int_0^1 \left(\rho^\gamma\right)_t u_{xt}\,dx - \int_0^1 \left(\rho^\gamma\right)_{tt}u_{xt}\,dx \\ &\le \frac{d}{dt}\int_0^1 \left(\rho^\gamma\right)_t u_{xt}\,dx + C\big\|\left(\rho^\gamma\right)_{tt}\big\|_{L^2}^2 + C\|u_{xt}\|_{L^2}^2. \end{aligned}$$

Similarly, we deduce that

$$
\begin{aligned}
J_6 &= \frac{\delta}{2}\frac{d}{dt}\int_0^1 (\chi_x^2)_t u_{xt}\,dx - \delta\int_0^1 \chi_{xt}^2 u_{xt}\,dx - \delta\int_0^1 \chi_x\chi_{xtt}u_{xt}\,dx \\
&\le \frac{\delta}{2}\frac{d}{dt}\int_0^1 (\chi_x^2)_t u_{xt}\,dx + C\|\chi_{xt}\|_{L^4}^2\|u_{xt}\|_{L^2} + C\|\chi_x\|_{L^\infty}\|\chi_{xtt}\|_{L^2}\|u_{xt}\|_{L^2} \\
&\le \frac{\delta}{2}\frac{d}{dt}\int_0^1 (\chi_x^2)_t u_{xt}\,dx + C\|u_{xt}\|_{L^2}^2 + C\|\chi_{xxt}\|_{L^2}^2 + C\|\chi_{xtt}\|_{L^2}^2.
\end{aligned}
$$

Substituting all the above estimates into (60), we obtain

$$
\begin{aligned}
&\frac{d}{dt}F(t) + \|\sqrt{\rho}u_{tt}\|_{L^2}^2 \\
&\quad\le C\|u_{xt}\|_{L^2}^2 + C\|\chi_{xxt}\|_{L^2}^2 + C\|\chi_{xtt}\|_{L^2}^2 + \varepsilon\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C\big\|(\rho^\gamma)_{tt}\big\|_{L^2}^2 \\
&\qquad + C\big\|(u_t + uu_x)_x\big\|_{L^2}^2 + C\big\|\nu_{tt}(\rho)\big\|_{L^2}^2 + C.
\end{aligned}
\tag{61}
$$

Multiplying the above inequality by t and adding (58) to the resultant inequality, then integrating it over $(0,t)$, after choosing ε small enough, and using (46), (47), (56), Gronwall's inequality, we show that

$$
\sup_{t\in[0,T]} t\big(\|\rho\chi_{tt}\|_{L^2}^2 + \|u_{xt}\|_{L^2}^2\big) + \int_0^T t\big(\|\chi_{xtt}\|_{L^2}^2 + \|\sqrt{\rho}u_{tt}\|_{L^2}^2\big)\,dt \le C,
\tag{62}
$$

due to

$$
\begin{aligned}
F(t) &= \frac{1}{2}\int_0^1 \nu(\rho)u_{xt}^2\,dx + \int_0^1 \nu_t(\rho)u_x u_{xt}\,dx + \int_0^1 \rho u(u_t + uu_x)u_{xt}\,dx \\
&\quad - \int_0^1 (\rho^\gamma)_t u_{xt}\,dx - \frac{\delta}{2}\int_0^1 (\chi_x^2)_t u_{xt}\,dx,
\end{aligned}
$$

satisfying

$$
\frac{\bar{\nu}}{2}\|u_{xt}\|_{L^2}^2 - C \le F(t) \le \|u_{xt}\|_{L^2}^2 + C.
$$

Moreover, due to (50), (51), (56), and (62), we obtain

$$
t\|u_{xxx}\|_{L^2}^2 \le C.
\tag{63}
$$

Furthermore, it follows from (59) that

$$
\begin{aligned}
\bar{\nu}\|u_{xxt}\|_{L^2}^2 &\le C\|\sqrt{\rho}u_{tt}\|_{L^2}^2 + C\|u_{xt}\|_{L^2}^2\big(1 + \|\rho_t\|_{L^2}^2 + \|\rho_x\|_{L^\infty}^2\big) + C\big\|(\rho^\gamma)_{xt}\big\|_{L^2}^2 \\
&\quad + C\|\chi_{xt}\|_{L^2}^2\|\chi_{xx}\|_{L^\infty}^2 + C\|\rho_{xt}\|_{L^2}^2 + C\|\chi_x\|_{L^\infty}^2\|\chi_{xxt}\|_{L^2}^2,
\end{aligned}
$$

which together with (62) leads to

$$
\int_0^T t\|u_{xxt}\|_{L^2}^2\,dt \le C.
\tag{64}
$$

Similarly, due to (28), one has

$$\int_0^T t\|\chi_{xxt}\|_{L^2}^2\,dt \le C,$$

which together with (57), (62), (63), (64), and (56) shows (55). This completes the proof. □

3 Proofs of the main theorems

In this section, based on the a priori estimates derived in Sect. 2, we extend the local classical solution obtained in Lemma 1 to a global one.

Proof of Theorem 1 To prove Theorem 1, we first construct a sequence of approximate solutions by giving the density without initial vacuum. Let $j_\delta(x)$ be a standard mollifier with width δ and define the initial density

$$\rho_0^\delta = j_\delta * \rho_0 + \delta, \qquad u_0^\delta = u_0 * j_\delta, \qquad \chi_0^\delta = \chi_0 * j_\delta,$$

where

$$\rho_0^\delta \to \rho_0, \qquad u_0^\delta \to u_0, \qquad \chi_0^\delta \to \chi_0, \quad \text{in } H^1, \text{as } \delta \to 0.$$

Due to Lemma 1, the initial boundary value problem (1)–(7) with initial data $(\rho_0^\delta, u_0^\delta, \chi_0^\delta)$ has a classical solution $(\rho^\delta, u^\delta, \chi^\delta)$ on $[0, T_0] \times [0, 1]$. Moreover, the estimates obtained in Lemmas 3–7 show that the solution $(\rho^\delta, u^\delta, \chi^\delta)$ satisfies, for any $0 < T < +\infty$,

$$\begin{aligned}&\sup_{t\in[0,T]}\big(\big\|\rho^\delta, (\rho^\delta)^\gamma\big\|_{H^1} + \big\|\rho_t^\delta\big\|_{L^2} + \big\|u^\delta\big\|_{H^1} + \big\|\chi^\delta\big\|_{H^2}\big)\\ &\quad + \int_0^T \big(\big\|u^\delta\big\|_{H^2}^2 + t\|u_{xt}\|_{L^2}^2 + \|\chi_{xt}\|_{L^2}^2\big) \le C,\end{aligned}$$

where C is independent of δ. With all the estimates at hand, one easily extracts subsequences to some limit (ρ, u, χ) in the weak sense. Then letting $\delta \to 0$, we deduce that (ρ, u, χ) is a strong solution to (1)–(7).

Furthermore, the uniqueness of the strong solution (ρ, u, χ) could be obtained by the similar argument as in [2]. For simplicity, we omit the details here.

Next, we will extend the local existence time T_0 of the strong solution to be infinity and therefore prove the global existence result. Let T^* be the maximal time of existence for the strong solution, thus, $T^* \ge T_0$. For any $0 < \tau < T \le T^*$ with T finite, we obtain

$$(u, \chi) \in L^\infty\big(0, T; H_0^1\big) \cap L^2\big(0, T; H^2\big), \qquad (u_t, \chi_t) \in L^2\big(\tau, T; H^1\big),$$

then

$$(u, \chi) \in C\big([\tau, T]; H_0^1\big). \tag{65}$$

Moreover, it follows from

$$\rho \in L^\infty\big(0, T; H^1\big), \qquad \rho_t \in L^\infty\big(0, T; L^2\big)$$

that

$$\rho \in C\big([0,T];H^1\big). \tag{66}$$

Let

$$\big(\rho^*,u^*,\chi^*\big) \triangleq (\rho,u,\chi)\big(T^*,x\big) = \lim_{t\to T^*}(\rho,u,\chi)(t,x),$$

it follows from (65) and (66) that (ρ^*,u^*,χ^*) satisfies the initial condition stated in Theorem 1.

Therefore, we take (ρ^*,u^*,χ^*) as the initial data at T^* and then use the local result, Lemma 1, to extend the strong solution beyond the maximum existence time T^*. This contradicts the assumption on T^*. We finally show that T^* could be infinity and complete the proof of the global existence of the strong solution.

It remains to prove (11), process of which is similar to that in in [13], we sketch it here for completeness. Due to integration by parts, we have

$$\begin{aligned}
\left|\frac{d}{dt}\|u_x\|_{L^2}^2\right| &= \left|2\int_0^1 u_x u_{xt}\,dx\right| \\
&= \left|2\int_0^1 u_x\big[(u_t+uu_x)_x-(uu_x)_x\big]\,dx\right| \\
&= \left|2\int_0^1 u_x(u_t+uu_x)_x\,dx-\int_0^1 u_x^3\,dx\right| \\
&\le C\big\|(u_t+uu_x)_x\big\|_{L^2}^2+C\big(1+\|u_x\|_{L^\infty}\big)\|u_x\|_{L^2}^2,
\end{aligned}$$

which together with (14) and (32) leads to

$$\int_1^{+\infty}\left(\|u_x\|_{L^2}^2+\left|\frac{d}{dt}\|u_x\|_{L^2}^2\right|\right)dt \le C.$$

Therefore, we obtain

$$\lim_{t\to+\infty}\|u_x\|_{L^2}^2 = 0,$$

which together with (32) yields

$$\lim_{t\to+\infty}\|u\|_{W^{1,p}} = 0, \quad \forall p\in[1,\infty).$$

This completes the proof. □

Proof of Theorem 2 With the higher-order estimates in Lemmas 8–10 at hand, the proof of Theorem 2 is similar to that of Theorem 1, and so it is omitted here for simplicity. □

Acknowledgements
The author is indebted to the editor's kind help and the anonymous reviewers' invaluable suggestions which improved the manuscript greatly.

Funding
This work was supported by the National Natural Science Foundation of China (no. 11671188).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by MLS. MLS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 July 2021 Accepted: 23 October 2021 Published online: 17 November 2021

References

1. Blesgen, T.: A generalization of the Navier–Stokes equations to two-phase flow. J. Phys. D, Appl. Phys. **32**, 1119–1123 (1999)
2. Chen, M., Guo, X.: Global large solutions for a coupled compressible Navier–Stokes/Allen–Cahn system with initial vacuum. Nonlinear Anal., Real World Appl. **37**, 350–373 (2017)
3. Chen, S., Wen, H., Zhu, C.: Global existence of weak solution to compressible Navier–Stokes/Allen–Cahn system in three dimensions. J. Math. Anal. Appl. **477**, 1265–1295 (2019)
4. Chen, S., Zhu, C.: Blow-up criterion and the global existence of strong/classical solutions to Navier–Stokes/Allen–Cahn system. Z. Angew. Math. Phys. **72**(1), Paper No. 14 (2021)
5. Ding, S., Li, Y., Luo, W.: Global solutions for a coupled compressible Navier–Stokes/Allen–Cahn system in 1D. J. Math. Fluid Mech. **15**, 335–360 (2013)
6. Ding, S., Li, Y., Tang, Y.: Strong solutions to 1D compressible Navier–Stokes/Allen–Cahn system with free boundary. Math. Methods Appl. Sci. **42**, 4780–4794 (2019)
7. Feireisl, E., Petcu, M., Pražák, D.: Relative energy approach to a diffuse interface model of a compressible two-phase flow. Math. Methods Appl. Sci. **42**, 1465–1479 (2019)
8. Feireisl, E., Petzeltová, H., Rocca, E., Schimperna, G.: Analysis of a phase-field model for two-phase compressible fluids. Math. Methods Appl. Sci. **31**, 1972–1995 (2008)
9. Feireisl, E., Petzeltová, H., Rocca, E., Schimperna, G.: Analysis of a phase-field model for two-phase compressible fluids. Math. Models Methods Appl. Sci. **20**, 1129–1160 (2010)
10. Hošek, R., Mácha, V.: Weak-strong uniqueness for Navier–Stokes/Allen–Cahn system. Czechoslov. Math. J. **69**(144), 837–851 (2019)
11. Kotschote, M.: Strong solutions of the Navier–Stokes equations for a compressible fluid of Allen–Cahn type. Arch. Ration. Mech. Anal. **206**, 489–514 (2012)
12. Liu, Z., Gao, C., Fang, Z.: A general decay estimate for the nonlinear transmission problem of weak viscoelastic equations with time-varying delay. J. Math. Phys. **60**, 101507 (2019)
13. Lü, B., Wang, Y., Wu, Y.: On global classical solutions to 1D compressible Navier–Stokes equations with density-dependent viscosity and vacuum. Math. Methods Appl. Sci. **43**, 5127–5150 (2020)
14. Luo, T., Yin, H., Zhu, C.: Stability of the composite wave for compressible Navier–Stokes/Allen–Cahn system. Math. Models Methods Appl. Sci. **30**, 343–385 (2020)
15. Luo, T., Yin, H., Zhu, C.: Stability of the rarefaction wave for a coupled compressible Navier–Stokes/Allen–Cahn system. Math. Methods Appl. Sci. **41**, 4724–4736 (2018)
16. Ma, L., Fang, Z.: Energy decay estimates and infinite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source. Math. Methods Appl. Sci. **41**, 2639–2653 (2018)
17. Yan, Y., Ding, S., Li, Y.: Strong solutions for 1D compressible Navier–Stokes/Allen–Cahn system with phase variable dependent viscosity. https://arxiv.org/pdf/2109.14529.pdf
18. Yin, H., Zhu, C.: Asymptotic stability of superposition of stationary solutions and rarefaction waves for 1D Navier–Stokes/Allen–Cahn system. J. Differ. Equ. **266**, 7291–7326 (2019)
19. Zheng, Y., Fang, Z.: Critical exponents for a non-Newtonian polytropic filtration equation with weighted nonlocal inner sources. Math. Methods Appl. Sci. **43**, 2403–2420 (2020)
20. Zlotnik, A.A.: Uniform estimates and stabilization of symmetric solutions of a system of quasilinear equations. Differ. Equ. **36**, 701–716 (2000)