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On Glassey's conjecture for semilinear wave equations in Friedmann–Lemaître–Robertson–Walker spacetime

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Abstract

Consider nonlinear wave equations in the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes. We show blow-up in finite time of solutions and upper bounds of the lifespan of blow-up solutions to give the FLRW spacetime version of Glassey's conjecture for the time derivative nonlinearity. We also show blow-up results for the space derivative nonlinear term.

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1 Introduction

The spatially flat FLRW metric is given by

$$g: \quad ds^2 = -dt^2 + a(t)^2 d\sigma^2,$$

where the speed of light is equal to 1, $d\sigma^2$ is the line element of an n -dimensional Euclidean space, and $a(t)$ is the scale factor, which describes expansion or contraction of the spatial metric. As in our earlier work [1–3], we treat the scale factor as

$$a(t) = ct^{\frac{2}{n(1+w)}}, \quad (1.1)$$

where c is a positive constant, and w is the proportionality constant in the range $-1 < w \leq 1$. The constant w appears in the equation of state relating the pressure to the density for the perfect fluid. See [1].

In the preceding papers [1–3], we have shown upper bounds of the lifespan for the equation $\square_g u = -|u|^p$. In this paper we consider the equation $\square_g u = -|u_t|^p$ as well as $\square_g u = -|\nabla_x u|^p$, where $\nabla_x u = (\partial u / \partial x^1, \dots, \partial u / \partial x^n)$, $(x^1, \dots, x^n) \in \mathbf{R}^n$.

For the flat FLRW metric with (1.1), the semilinear wave equation $\square_g u = |g|^{-1/2} \times \partial_\alpha (|g|^{1/2} g^{\alpha\beta} \partial_\beta u) = -|u_t|^p$ or $-|\nabla_x u|^p$ with $p > 1$ becomes

$$u_{tt} - \frac{1}{t^{4/(n(1+w))}} \Delta u + \frac{2}{(1+w)t} u_t = |u_t|^p \quad \text{or} \quad |\nabla_x u|^p, \quad x \in \mathbf{R}^n, \quad (1.2)$$

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where $\Delta = \partial_1^2 + \cdots + \partial_n^2$, $\partial_j = \partial/\partial x^j$, $j = 1, \dots, n$. Our aim of this paper is to show that blow-up in a finite time occurs for the equation above as well as upper bounds of the lifespan of the blow-up solutions.

We first consider the following Cauchy problem in order to compare with the related known results including the case of the Minkowski spacetime:

$$u_{tt} - \frac{1}{t^{2\alpha}} \Delta u + \frac{\mu}{t} u_t = |u_t|^p, \quad t > 1, x \in \mathbf{R}^n \quad (1.3)$$

with the initial data given at $t = 1$,

$$u(1, x) = \varepsilon u_0(x), \quad u_t(1, x) = \varepsilon u_1(x), \quad x \in \mathbf{R}^n, \quad (1.4)$$

where α and μ are nonnegative constants and $\varepsilon > 0$ is a small parameter.

Let T_ε be the lifespan of solutions of (1.3) and (1.4), say, T_ε is the supremum of T such that (1.3) and (1.4) have a solution for $x \in \mathbf{R}^n$ and $1 \leq t < T$.

Let $\alpha = \mu = 0$ and $p_G(n) = 1 + 2/(n-1)$. The so-called Glassey's conjecture [4] asserts that if $p > p_G(n)$, then there exist global solutions in time for small initial data; on the other hand, if $1 < p \leq p_G(n)$ with $n \geq 2$ or if $p > 1$ for $n = 1$, then blow-up in finite time occurs. This conjecture is proved to be almost true. Actually, blow-up results in low dimensions ($n = 2, 3$) or in high dimensions ($n \geq 4$) imposing radial symmetry were proved in, e.g., [5–10], and Zhou [11] finally gave a simple proof of the blow-up result for $1 < p \leq p_G(n)$ and $n \geq 2$ as well as for $p > 1$ and $n = 1$. Global existence of solutions in low dimensions ($n = 2, 3$) has been proved in, e.g., [12–14]. For high dimensions ($n \geq 3$), it was proved by [15] that there exist global solutions in the radial case for $p > p_G(n)$. They [15] also proved the lifespan of local solutions in time for $1 < p \leq p_G(n)$.

For the case $\alpha = 0$ and $\mu \geq 0$, it has been recently shown by Hamouda and Hamza [16] that blow-up in finite time occurs and the lifespan of the blow-up solutions satisfies

$$T_\varepsilon \leq C\varepsilon^{-(p-1)/(1-(n+\mu-1)(p-1)/2)} \quad \text{if } 1 < p < p_G(n+\mu), n \geq 1, \quad (1.5)$$

$$T_\varepsilon \leq \exp(C\varepsilon^{-(p-1)}) \quad \text{if } p = p_G(n+\mu), n \geq 1. \quad (1.6)$$

These results improve the ones in [17].

The present paper treats the case $\alpha \geq 0$ and $\mu \geq 0$. We first show blow-up in a finite time and upper estimates of the lifespan of solutions of (1.3) and (1.4) in the case $0 \leq \alpha < 1$. If $\alpha = 0$, our upper bounds of the lifespan coincide with the results above by [16]. Similar results are independently shown by [18] where energy solutions are treated. In our results, however, another exponent appears as a blow-up condition in some case. This is different from the results by [18]. We emphasize that the generalized exponent of $p_G(n+\mu)$ cannot always be the critical exponent for the global existence of solutions. Our proofs are based on the test function method with the modified Bessel function of the second kind and on a generalized Kato's lemma. We next treat the case $\alpha \geq 1$.

Moreover, we show blow-up results for the problem

$$\begin{cases} u_{tt} - \frac{1}{t^{2\alpha}} \Delta u + \frac{\mu}{t} u_t = |\nabla_x u|^p, & t > 1, x \in \mathbf{R}^n, \\ u(1, x) = \varepsilon u_0(x), & u_t(1, x) = \varepsilon u_1(x), \quad x \in \mathbf{R}^n. \end{cases} \quad (1.7)$$

Unlike equation (1.3), our blow-up conditions are related to exponents that originate from the Strauss and Fujita ones and to another exponent like in the case of the time derivative nonlinearity. Hence, upper bounds of the lifespan have to do with those exponents.

We then apply our results for (1.3) and (1.7) to the original equation (1.2). Our aim is especially to clarify the difference with the case of the Minkowski spacetime and also how the scale factor affects the lifespan of the solution. Since global existence of solutions has not been obtained yet, *critical* exponent used in this paper means a candidate of the true critical exponent.

The paper is organized as follows. In Sect. 2, we state our first main result for (1.3) and (1.4) in the case $0 \leq \alpha < 1$. Theorem 2.1 presents that it is possible in some case to improve the estimate of the lifespan affected by Glassey's exponent. To prove the theorem, we use the test function method following the way in [19], and also generalized Kato's lemma for a first-order differential inequality, which is applied to the wave equation with the scale-invariant damping. This is proved by John's iteration argument [20]. We then show our second result which is for the case $\alpha \geq 1$. In Sect. 3 we treat (1.7) and divide results into several cases $0 \leq \alpha < 1$, wavelike and heatlike cases, critical and subcritical cases, and $\alpha \geq 1$. Finally in Sect. 4, we apply the theorems in Sects. 2 and 3 to the original equation (1.2). We discuss the effect of the scale factor to the solutions.

2 Time derivative nonlinearity

2.1 Case $0 \leq \alpha < 1$

We first consider problem (1.3) with (1.4) for $0 \leq \alpha < 1$. Our first result is the following theorem.

Theorem 2.1 *Let $n \geq 2$, $0 \leq \alpha < 1$, $\mu \geq 0$, and*

$$1 < p \leq p'_G(n, \alpha, \mu) \equiv 1 + \frac{2}{(1-\alpha)(n-1) + \mu + \alpha}$$

or

$$1 < p < p_0(n, \alpha, \mu) \equiv 1 + \frac{1}{n(1-\alpha) + \mu}.$$

Assume that $u_0 \in C^2(\mathbf{R}^n)$ and $u_1 \in C^1(\mathbf{R}^n)$ are nontrivial and satisfy $u_1(x) \geq u_0(x) \geq 0$, $\text{supp } u_0, \text{supp } u_1 \subset \{|x| \leq R\}$ with $R > 0$. Suppose that problem (1.3) with (1.4) has a classical solution $u \in C^2([1, T) \times \mathbf{R}^n)$. Then $T < \infty$, and there exists a constant $\varepsilon_0 > 0$ depending on $p, \alpha, \mu, R, u_0, u_1$ such that T_ε has to satisfy

$$T_\varepsilon \leq C\varepsilon^{\frac{-(p-1)}{1-[(1-\alpha)(n-1)+\mu+\alpha](p-1)/2}} \quad \text{if } p < p'_G(n, \alpha, \mu), \quad (2.1)$$

$$T_\varepsilon \leq \exp(C\varepsilon^{-(p-1)}) \quad \text{if } p = p'_G(n, \alpha, \mu), \quad (2.2)$$

$$T_\varepsilon \leq C\varepsilon^{\frac{-(p-1)}{1-(p-1)[n(1-\alpha)+\mu]}} \quad \text{if } p < p_0(n, \alpha, \mu) \quad (2.3)$$

for $0 < \varepsilon \leq \varepsilon_0$, where $C > 0$ is a constant independent of ε .

Remark (1) If $\alpha = 0$, then (2.1) and (2.2) are the same with the upper bounds (1.5) and (1.6).

(2) By the theorem, the exponent $p'_G(n, \alpha, \mu)$ cannot always be the critical exponent for the global existence of solutions. We discuss more details at the end of this subsection.

(3) If $p < p_0(n, \alpha, \mu)$, then the above assumption $u_1(x) \geq u_0(x) \geq 0$ can be replaced just by $u_1(x) \geq 0$.

Proof Multiplying (1.3) by a test function $\phi(t, x)$ and t^μ , and integrating over $\mathbf{R}^n$, we have

$$\begin{aligned} & \frac{d}{dt} \int t^\mu (u\phi)_t dx - 2 \frac{d}{dt} \int t^\mu u \phi_t dx + \int t^\mu u \left(\phi_{tt} - \frac{1}{t^{2\alpha}} \Delta \phi + \frac{\mu}{t} \phi_t \right) dx \\ &= \int t^\mu |u_t|^p \phi dx. \end{aligned} \quad (2.4)$$

Integrating over $[1, t]$, we obtain

$$\begin{aligned} & \int t^\mu (u\phi)_t(t, x) dx - 2 \int t^\mu u \phi_t(t, x) dx + \int_1^t s^\mu \int u \left(\phi_{ss} - \frac{1}{s^{2\alpha}} \Delta \phi + \frac{\mu}{s} \phi_s \right) dx ds \\ &= \varepsilon \int (u_1(x) \phi(1, x) - u_0(x) \phi_t(1, x)) dx + \int_1^t s^\mu \int |u_s|^p \phi dx ds. \end{aligned} \quad (2.5)$$

We remark that the C^2 -solution u of (1.3) and (1.4) has the property of finite speed of propagation and satisfies

$$\text{supp } u(t, \cdot) \subset \{|x| \leq A(t) + R\}, \quad A(t) = \int_1^t s^{-\alpha} ds = \frac{t^{1-\alpha} - 1}{1-\alpha}, \quad (2.6)$$

provided that $\text{supp } u_0, \text{supp } u_1 \subset \{|x| \leq R\}$. See [1] for its proof.

We now define a smooth test function by

$$\begin{aligned} \phi(t, x) &= \lambda(t) \int_{|\omega|=1} e^{x \cdot \omega} dS_\omega, \\ \lambda(t) &= t^{(1-\mu)/2} K_\nu \left(\frac{1}{1-\alpha} t^{1-\alpha} \right), \quad \nu = \frac{\mu-1}{2(1-\alpha)}, \end{aligned}$$

where $K_\nu(t)$ is the modified Bessel function of the second kind, which is given by

$$K_\nu(t) = \int_0^\infty e^{-t \cosh z} \cosh \nu z dz, \quad t > 0, \nu \in \mathbf{R}. \quad (2.7)$$

It is well known that the Bessel function K_ν satisfies the following properties (see, e.g., [21]):

$$t^2 K_\nu''(t) + t K_\nu'(t) - (t^2 + \nu^2) K_\nu(t) = 0, \quad (2.8)$$

$$K_\nu(t) = \sqrt{\frac{\pi}{2}} \frac{e^{-t}}{\sqrt{t}} \left(1 + O\left(\frac{1}{t}\right) \right) \quad (t \rightarrow \infty), \quad (2.9)$$

$$K_\nu'(t) = \frac{\nu}{t} K_\nu(t) - K_{\nu+1}(t). \quad (2.10)$$

We can verify by (2.8)–(2.10) that there holds

$$\phi_{tt} - \frac{1}{t^{2\alpha}} \Delta \phi + \frac{\mu}{t} \phi_t = 0. \quad (2.11)$$

The following estimate is shown in [1]:

$$\int_{|x| \leq A(t)+R} |\phi(t, x)| dx \lesssim (t+R)^{((1-\alpha)(n-1)-(\mu-\alpha))/2}. \quad (2.12)$$

See (3.23) in [1]. We also see by [1] that

$$C_d \equiv \varepsilon \int (u_1(x)\phi(1, x) - u_0(x)\phi_t(1, x)) dx = C_0 \varepsilon > 0 \quad (2.13)$$

under assumption on the initial data. Moreover, we have

$$\begin{aligned} \phi_t(t, x) &= \lambda'(t) \int_{|\omega|=1} e^{x \cdot \omega} dS_\omega \\ &= \left\{ \frac{1-\mu}{2} t^{-1} + t^{-\alpha} \frac{K'_v(\frac{1}{1-\alpha} t^{1-\alpha})}{K_v(\frac{1}{1-\alpha} t^{1-\alpha})} \right\} \phi(t, x) \end{aligned} \quad (2.14)$$

$$= -t^{-\alpha} \frac{K_{v+1}(\frac{1}{1-\alpha} t^{1-\alpha})}{K_v(\frac{1}{1-\alpha} t^{1-\alpha})} \phi(t, x), \quad (2.15)$$

where we have used (2.10) and $v = (\mu - 1)/(2(1 - \alpha))$ for the last equality.

Set

$$F_1(t) = \int u\phi(t, x) dx.$$

From (2.5), proceeding as in [1], we have

$$\frac{t^{\mu-1}}{K_v(\frac{1}{1-\alpha} t^{1-\alpha})^2} F_1(t) - \frac{1}{K_v(\frac{1}{1-\alpha})^2} F_1(1) \geq C_d \int_1^t \frac{s^{-1}}{K_v(\frac{1}{1-\alpha} s^{1-\alpha})^2} ds.$$

Since $F_1(1)/K_v(\frac{1}{1-\alpha})^2 > 0$ by assumption, we obtain

$$F_1(t) = \int u\phi(t, x) dx > 0 \quad \text{for } t \geq 1. \quad (2.16)$$

We next go back to (2.4), which becomes

$$\frac{d}{dt} \int t^\mu u_t \phi dx - \int t^\mu u_t \phi_t dx - \frac{1}{t^{2\alpha}} \int t^\mu u \Delta \phi dx = \int t^\mu |u_t|^p \phi dx.$$

Using (2.15) and $\Delta \phi = \phi$ yields

$$\begin{aligned} &\frac{d}{dt} \int t^\mu u_t \phi dx + t^{-\alpha} \frac{K_{v+1}(\frac{1}{1-\alpha} t^{1-\alpha})}{K_v(\frac{1}{1-\alpha} t^{1-\alpha})} \int t^\mu u_t \phi dx - \frac{1}{t^{2\alpha}} \int t^\mu u \phi dx \\ &= \int t^\mu |u_t|^p \phi dx. \end{aligned} \quad (2.17)$$

On the other hand, by (2.5), (2.11), (2.13), and (2.15),

$$\int t^\mu u_t \phi(t, x) dx + t^{-\alpha} \frac{K_{v+1}(\frac{1}{1-\alpha} t^{1-\alpha})}{K_v(\frac{1}{1-\alpha} t^{1-\alpha})} \int t^\mu u \phi(t, x) dx$$

$$= C_d + \int_1^t s^\mu \int |u_s|^p \phi \, dx \, ds. \quad (2.18)$$

We see that there exists a constant $M \geq 1$ such that

$$\frac{K_{v+1}(\frac{1}{1-\alpha}t^{1-\alpha})}{K_v(\frac{1}{1-\alpha}t^{1-\alpha})} \leq M \quad (2.19)$$

for $t \geq 1$ by (2.9). Combining (2.17) and (2.18) multiplied by $M^{-1}t^{-\alpha}$, we have

$$\begin{aligned} & \frac{d}{dt} \int t^\mu u_t \phi \, dx + \left[\frac{K_{v+1}(\frac{1}{1-\alpha}t^{1-\alpha})}{K_v(\frac{1}{1-\alpha}t^{1-\alpha})} + M^{-1} \right] t^{-\alpha} \int t^\mu u_t \phi \, dx \\ & + \left[M^{-1} \frac{K_{v+1}(\frac{1}{1-\alpha}t^{1-\alpha})}{K_v(\frac{1}{1-\alpha}t^{1-\alpha})} - 1 \right] t^{-2\alpha} \int t^\mu u \phi(t, x) \, dx \\ & = C_d M^{-1} t^{-\alpha} + \int t^\mu |u_t|^p \phi \, dx + M^{-1} t^{-\alpha} \int_1^t s^\mu \int |u_s|^p \phi \, dx \, ds. \end{aligned}$$

We note that

$$t^{-2\alpha+\mu} \int u \phi(t, x) \, dx > 0$$

for all $t \geq 1$ by (2.16). Using (2.19), we obtain

$$\begin{aligned} & \frac{d}{dt} \int t^\mu u_t \phi \, dx + \left(M + \frac{1}{M} \right) t^{-\alpha} \int t^\mu u_t \phi \, dx \\ & \geq C_d M^{-1} t^{-\alpha} + \int t^\mu |u_t|^p \phi \, dx + M^{-1} t^{-\alpha} \int_1^t s^\mu \int |u_s|^p \phi \, dx \, ds \quad \text{for } t \geq 1. \end{aligned} \quad (2.20)$$

We now set

$$G(t) = \int t^\mu u_t \phi \, dx - \frac{1}{M^2 + 1} \int_1^t s^\mu \int |u_s|^p \phi \, dx \, ds - \frac{C_d}{2(M^2 + 1)} \quad \text{for } t \geq 1.$$

Then

$$G'(t) = \frac{d}{dt} \int t^\mu u_t \phi \, dx - \frac{1}{M^2 + 1} \int t^\mu |u_t|^p \phi \, dx,$$

hence (2.20) becomes

$$G'(t) + \frac{M^2 + 1}{M} t^{-\alpha} G(t) \geq \frac{C_d}{2M} t^{-\alpha} + \frac{M^2}{M^2 + 1} \int t^\mu |u_t|^p \phi \, dx > 0 \quad \text{for } t \geq 1.$$

Multiplying the inequality above by $\exp[(M^2 + 1)t^{1-\alpha}/(M(1-\alpha))]$ and integrating over $[1, t]$, we obtain

$$\exp\left(\frac{M^2 + 1}{M(1-\alpha)}t^{1-\alpha}\right) G(t) - \exp\left(\frac{M^2 + 1}{M(1-\alpha)}\right) G(1) > 0 \quad \text{for } t \geq 1. \quad (2.21)$$

Note that

$$\begin{aligned} G(1) &= \varepsilon \int u_1(x) \phi(1, x) dx - \frac{\varepsilon}{2(M^2 + 1)} \int (u_1(x) \phi(1, x) - u_0(x) \phi_t(1, x)) dx \\ &\geq \varepsilon \int \left(\frac{2M^2 + 1}{2(M^2 + 1)} u_1(x) - \frac{M}{2(M^2 + 1)} u_0(x) \right) \phi(1, x) dx > 0 \end{aligned}$$

by (2.15), (2.19), and the assumption on the initial data. It holds from (2.21) that $G(t) > 0$ for $t \geq 1$. Thus, we see that

$$\int t^\mu u_t \phi dx > \frac{1}{M^2 + 1} \int_1^t s^\mu \int |u_s|^p \phi dx ds + \frac{C_d}{2(M^2 + 1)} \quad \text{for } t \geq 1. \quad (2.22)$$

We now define

$$H(t) = \int_1^t s^\mu \int |u_s|^p \phi dx ds + \frac{C_d}{2} \quad \text{for } t \geq 1.$$

By Hölder's inequality, (2.12), and (2.22), we have

$$\begin{aligned} H'(t) &= \int t^\mu |u_t|^p \phi dx \geq \left(\int t^\mu u_t \phi dx \right)^p \left(\int_{|x| \leq A(t)+R} t^\mu \phi dx \right)^{1-p} \\ &\gtrsim t^{-\{(1-\alpha)(n-1)+\mu+\alpha\}(p-1)/2} H(t)^p \quad \text{for } t \geq 1. \end{aligned} \quad (2.23)$$

We also have

$$H(1) = \frac{C_d}{2} = \frac{1}{2} C_0 \varepsilon > 0 \quad (2.24)$$

by (2.13). By integrating (2.23) multiplied by $H(t)^{-p}$ over $[1, t]$ and using (2.24), we therefore obtain the desired results (2.1) and (2.2).

It remains to prove (2.3). We use the following lemma, which is generalized Kato's lemma.

Lemma 2.2 *Let $p > 1, a \geq 0, b \geq 0, q \geq 0, r \geq 0, \mu \geq 0, c > 0$, and*

$$M \equiv (p-1)(c-a) - q + 1 > 0.$$

Let $T \geq T_1 > T_0 \geq 1$. Assume that $F \in C^1([T_0, T])$ satisfies the following three conditions:

- (i) $F(t) \geq A_0 t^{-a} (\ln t)^{-b} (t - T_1)^c$ for $t \geq T_1$,
- (ii) $F'(t) + \frac{\mu}{t} F(t) \geq A_1 (t + R)^{-q} (\ln t)^{-r} |F(t)|^p$ for $t \geq T_0$,
- (iii) $F(T_0) > 0$,

where A_0, A_1 , and R are positive constants. Then T has to satisfy

$$T^{M/(p-1)} (\ln t)^{-b-r/(p-1)} < C A_0^{-1},$$

where C is a constant depending on R, A_1, μ, p, q , and c .

Proof Multiplying assumption (ii) by t^μ , we have

$$t^\mu F' + \mu t^{\mu-1} F \geq A_1 t^\mu (t+R)^{-q} (\ln t)^{-r} |F|^p.$$

Integrating the above inequality over $[T_0, t]$ yields

$$t^\mu F(t) - F(T_0) \geq A_1 C_{R,q} \int_{T_0}^t s^{\mu-q} (\ln s)^{-r} |F(s)|^p ds \geq 0, \quad C_{R,q} = (1+R)^{-q}. \quad (2.25)$$

By assumption (iii), we see that $F(t) > 0$ for $t \geq T_0$. Hence, by assumption (i), we have

$$\begin{aligned} F(t) &\geq A_0^p A_1 C_{R,q} t^{-\mu} \int_{T_1}^t s^{\mu-ap-q} (\ln s)^{-bp-r} (s-T_1)^{cp} ds \\ &\geq A_0^p A_1 C_{R,q} t^{-\mu-ap-q} (\ln t)^{-bp-r} \int_{T_1}^t (s-T_1)^{\mu+cp} ds \\ &= \frac{A_0^p A_1 C_{R,q}}{\mu+cp+1} t^{-\mu-ap-q} (\ln t)^{-bp-r} (t-T_1)^{\mu+cp+1} \quad \text{for } t \geq T_1. \end{aligned}$$

Based on the fact above, we define the sequences a_j, b_j, c_j, D_j for $j = 0, 1, 2, \dots$ by

$$\begin{aligned} a_{j+1} &= p a_j + \mu + q, & b_{j+1} &= p b_j + r, & c_{j+1} &= p c_j + \mu + 1, \\ D_{j+1} &= \frac{A_1 C_{R,q} D_j^p}{p c_j + \mu + 1}, \end{aligned} \quad (2.26)$$

$$a_0 = a, \quad b_0 = b, \quad c_0 = c, \quad D_0 = A_0. \quad (2.27)$$

Solving (2.26) and (2.27), we obtain

$$\begin{aligned} a_j &= p^j \left(a + \frac{\mu+q}{p-1} \right) - \frac{\mu+q}{p-1}, & b_j &= p^j \left(b + \frac{r}{p-1} \right) - \frac{r}{p-1}, \\ c_j &= p^j \left(c + \frac{\mu+1}{p-1} \right) - \frac{\mu+1}{p-1}, \end{aligned}$$

and thus

$$D_{j+1} = \frac{A_1 C_{R,q} D_j^p}{c_{j+1}} \geq \left(c + \frac{\mu+1}{p-1} \right)^{-1} \frac{A_1 C_{R,q} D_j^p}{p^{j+1}}.$$

Then

$$\begin{aligned} D_j &\geq \frac{B D_{j-1}^p}{p^j} \\ &\geq \frac{B}{p^j} \left(\frac{B D_{j-2}^p}{p^{j-1}} \right)^p = \frac{B^{1+p}}{p^{j+p(j-1)}} D_{j-2}^{p^2} \\ &\geq \frac{B^{1+p}}{p^{j+p(j-1)}} \left(\frac{B D_{j-3}^p}{p^{j-2}} \right)^{p^2} = \frac{B^{1+p+p^2}}{p^{j+p(j-1)+p^2(j-2)}} D_{j-3}^{p^3} \\ &\geq \dots \geq \frac{B^{1+p+p^2+\dots+p^{j-1}}}{p^{j+p(j-1)+p^2(j-2)+\dots+p^{j-1}}} D_0^{p^j}, \end{aligned}$$

and

$$\ln D_j \geq \frac{\ln B}{p-1} (p^j - 1) - p^j \sum_{k=0}^j \frac{k}{p^k} \ln p + p^j \ln D_0,$$

where $B = \{c + (\mu + 1)/(p - 1)\}^{-1} A_1 C_{R,q}$. For sufficiently large j , we have

$$D_j \geq \exp(E p^j),$$

where

$$E = \frac{1}{p-1} \min(0, \ln B) - \sum_{k=0}^{\infty} \frac{k}{p^k} \ln p + \ln A_0. \quad (2.28)$$

Thus, since $F(t) \geq D_j t^{-a_j} (\ln t)^{-b_j} (t - T_1)^{c_j}$ holds for $t \geq T_1$, we obtain

$$\begin{aligned} F(t) &\geq t^{(\mu+q)/(p-1)} (\ln t)^{r/(p-1)} (t - T_1)^{-(\mu+1)/(p-1)} \\ &\quad \times \exp \left[\left\{ E + \left(c + \frac{\mu+1}{p-1} \right) \ln(t - T_1) - \left(a + \frac{\mu+q}{p-1} \right) \ln t \right. \right. \\ &\quad \left. \left. - \left(b + \frac{r}{p-1} \right) \ln(\ln t) \right\} p^j \right] \end{aligned} \quad (2.29)$$

for $t \geq T_1$. Since

$$\left(c + \frac{\mu+1}{p-1} \right) - \left(a + \frac{\mu+q}{p-1} \right) = c - a + \frac{1-q}{p-1} > 0$$

by assumption, choosing t large enough, we can find a positive δ such that

$$E + \left(c + \frac{\mu+1}{p-1} \right) \ln(t - T_1) - \left(a + \frac{\mu+q}{p-1} \right) \ln t - \left(b + \frac{r}{p-1} \right) \ln(\ln t) \geq \delta > 0.$$

It then follows from (2.29) that $F(t) \rightarrow \infty$ as $j \rightarrow \infty$ for sufficiently large t . We therefore see that the lifespan T of $F(t)$ has to satisfy

$$T^{M/(p-1)} (\ln T)^{-b-r/(p-1)} < C A_0^{-1},$$

where $M = (p-1)(c-a) - q + 1 > 0$, and C is a constant depending on A_1, R, μ, p, q , and c . This completes the proof of the proposition. $\square$

We now prove (2.3) by applying Lemma 2.2. Set

$$F(t) = \int u_t(t, x) dx. \quad (2.30)$$

Integrating equation (1.3) and using Hölder's inequality, we have, by (2.6),

$$F'(t) + \frac{\mu}{t} F(t) = \int |u_t|^p dx \quad (2.31)$$

$$\begin{aligned}
&\geq \frac{1}{(A(t) + R)^{n(p-1)}} |F(t)|^p \\
&\geq \frac{C}{(t + R)^{n(1-\alpha)(p-1)}} |F(t)|^p.
\end{aligned} \tag{2.32}$$

Multiplying (2.32) by t^μ and integrating imply

$$t^\mu F(t) - F(1) \gtrsim \int_1^t s^{\mu-n(1-\alpha)(p-1)} |F(s)|^p ds \geq 0. \tag{2.33}$$

We hence see that

$$F(t) \geq t^{-\mu} F(1) = C\varepsilon t^{-\mu} > 0 \quad \text{for } t \geq 1 \tag{2.34}$$

by assumption. From (2.33) and (2.34), we have

$$\begin{aligned}
F(t) &\geq C\varepsilon^p t^{-\mu} \int_1^t s^{\mu-n(1-\alpha)(p-1)-\mu p} ds \\
&\geq C\varepsilon^p t^{-\mu(p+1)-n(1-\alpha)(p-1)} \int_1^t (s-1)^\mu ds.
\end{aligned}$$

Therefore, we obtain

$$F(t) \geq C\varepsilon^p t^{-\mu(p+1)-n(1-\alpha)(p-1)} (t-1)^{\mu+1} \quad \text{for } t \geq 1. \tag{2.35}$$

Finally, by (2.32) and (2.35), applying Lemma 2.2 with $q = n(1-\alpha)(p-1)$, $a = \mu(p+1) + n(1-\alpha)(p-1)$, $b = r = 0$, $c = \mu + 1$, and $A_0 = C\varepsilon^p$, we obtain the desired result (2.3) since

$$\begin{aligned}
M &= (p-1) \{ \mu + 1 - \mu(p+1) - n(1-\alpha)(p-1) \} - n(1-\alpha)(p-1) + 1 \\
&= p \{ 1 - (p-1)(\mu + n(1-\alpha)) \} > 0.
\end{aligned}$$

This completes the proof of Theorem 2.1. $\square$

At the end of this subsection, we discuss the blow-up conditions and the estimates of the lifespan in the two subcritical cases in Theorem 2.1. We note that if $p'_G(n, \alpha, \mu) = p_0(n, \alpha, \mu)$, then

$$\mu = \mu_{n,\alpha} \equiv \alpha(n+2) - (n+1).$$

Figure 1 and Fig. 2 show the regions of blow-up conditions in the cases $n = 3, \alpha = 0.2$ and $n = 3, \alpha = 0.9$, respectively.

For $\mu > \max\{0, \mu_{n,\alpha}\}$, the exponent $p'_G(n, \alpha, \mu)$ is bigger than $p_0(n, \alpha, \mu)$. See Region (G) in Figs. 1 and 2. On the other hand, if $(n+1)/(n+2) \leq \alpha < 1$, then for $0 \leq \mu \leq \mu_{n,\alpha}$, the condition $1 < p < p_0(n, \alpha, \mu)$ includes the one $1 < p < p'_G(n, \alpha, \mu)$. This means that if α is close to 1, then a better region $1 < p < p_0(n, \alpha, \mu)$ than $1 < p < p'_G(n, \alpha, \mu)$ appears for $0 \leq \mu < 1$, as shown by Region (O) in Fig. 2. Hence, the exponent $p'_G(n, \alpha, \mu)$ cannot always be the critical exponent for the global existence of solutions.

If we compare the two upper bounds (2.1) and (2.3), then in Region (G) in Fig. 1 and Fig. 2, (2.1) is better than (2.3), while in Region (O) in Fig. 2, this relation becomes reverse.

2.2 Case $\alpha \geq 1$

We next consider the same problem for the case $\alpha \geq 1$.

Theorem 2.3 Let $n \geq 2, \alpha \geq 1, \mu \geq 0$, and $1 < p < 1 + 1/\mu$. Assume that $u_0 \in C^2(\mathbb{R}^n)$ and $u_1 \in C^1(\mathbb{R}^n)$ are nontrivial and satisfy $u_1(x) \geq 0$, $\text{supp } u_0, \text{supp } u_1 \subset \{|x| \leq R\}$ with $R > 0$. Suppose that problem (1.3) with (1.4) has a classical solution $u \in C^2([1, T) \times \mathbb{R}^n)$. Then $T < \infty$, and there exists a constant $\varepsilon_0 > 0$ depending on $p, \alpha, \mu, R, u_0, u_1$ such that T_ε has to satisfy

$$T_\varepsilon^{1-\mu(p-1)} (\ln T_\varepsilon)^{-n(p-1)} \leq C\varepsilon^{-(p-1)} \quad \text{if } \alpha = 1, \quad (2.36)$$

$$T_\varepsilon \leq C\varepsilon^{-(p-1)/(1-\mu(p-1))} \quad \text{if } \alpha > 1 \quad (2.37)$$

for $0 < \varepsilon \leq \varepsilon_0$, where $C > 0$ is a constant independent of ε .

Proof We first remark that the C^2 -solution u of (1.3) and (1.4) has the property of finite speed of propagation and satisfies

$$\operatorname{supp} u(t, \cdot) \subset \{|x| \leq A(t) + R\},$$

where

$$A(t) = \begin{cases} \int_1^t s^{-1} ds = \ln t & \text{if } \alpha = 1, \\ \int_1^t s^{-\alpha} ds = \frac{1}{\alpha-1}(1 - t^{1-\alpha}) & \text{if } \alpha > 1, \end{cases} \quad (2.38)$$

provided that $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset \{|x| \leq R\}$. See [3] for its proof.

Define $F(t)$ by (2.30), and let $\alpha = 1$. From (2.31) and (2.38),

$$\begin{aligned} F'(t) + \frac{\mu}{t} F(t) &\geq \frac{1}{(A(t) + R)^{n(p-1)}} |F(t)|^p \\ &\geq \frac{C}{(\ln t)^{n(p-1)}} |F(t)|^p. \end{aligned} \quad (2.39)$$

Proceeding as in the proof of Theorem 2.1 for the case $0 \leq \alpha < 1$, we have, by (2.34),

$$\begin{aligned} F(t) &\geq C\varepsilon^p t^{-\mu} \int_1^t s^{\mu-\mu p} (\ln s)^{-n(p-1)} ds \\ &\geq C\varepsilon^p t^{-\mu(p+1)} (\ln t)^{-n(p-1)} \int_1^t (s-1)^\mu ds. \end{aligned}$$

Therefore, we obtain

$$F(t) \geq C\varepsilon^p t^{-\mu(p+1)} (\ln t)^{-n(p-1)} (t-1)^{\mu+1} \quad \text{for } t \geq 1. \quad (2.40)$$

Finally, by (2.39) and (2.40), applying Lemma 2.2 with $q = 0$, $a = \mu(p+1)$, $b = r = n(p-1)$, $c = \mu+1$, and $A_0 = C\varepsilon^p$, we obtain the desired result (2.36) since

$$M = (p-1)(1 - \mu p) + 1 = p(1 - \mu(p-1)) > 0. \quad (2.41)$$

On the other hand, if $\alpha > 1$, then from (2.31) and (2.38)

$$F'(t) + \frac{\mu}{t} F(t) \geq C |F(t)|^p. \quad (2.42)$$

Proceeding as before, by (2.34), we have

$$\begin{aligned} F(t) &\geq C\varepsilon^p t^{-\mu} \int_1^t s^{\mu-\mu p} ds \\ &\geq C\varepsilon^p t^{-\mu(p+1)} \int_1^t (s-1)^\mu ds. \end{aligned}$$

Therefore, we obtain

$$F(t) \geq C\varepsilon^p t^{-\mu(p+1)} (t-1)^{\mu+1} \quad \text{for } t \geq 1. \quad (2.43)$$

Finally, by (2.42) and (2.43), applying Lemma 2.2 with $q = 0$, $a = \mu(p+1)$, $b = r = 0$, $c = \mu + 1$, and $A_0 = C\varepsilon^p$, we obtain the desired result (2.37) since (2.41). This completes the proof of Theorem 2.3. $\square$

3 Space derivative nonlinearity

In this section we consider problem (1.7). Let u_0 and u_1 be nonnegative and satisfy $\text{supp } u_0, \text{supp } u_1 \subset \{|x| \leq R\}$ with $R > 0$.

We prepare several basic inequalities which will be used repeatedly. Let

$$F(t) = \int u(t, x) dx.$$

Then integrating equation (1.7) over $\mathbf{R}^n$ and using Poincaré's and Hölder's inequalities imply that

$$F''(t) + \frac{\mu}{t} F'(t) = \int |\nabla_x u|^p dx \quad (3.1)$$

$$\geq \frac{1}{(A(t) + R)^p} \int |u|^p dx \quad (3.2)$$

$$\geq \frac{1}{(A(t) + R)^{p+n(p-1)}} |F(t)|^p. \quad (3.3)$$

On the other hand, multiplying (3.1) by t^μ and integrating imply

$$t^\mu F'(t) - F'(1) = \int_1^t s^\mu \int |\nabla_x u|^p dx ds. \quad (3.4)$$

Since $F'(1) > 0$, by assumption,

$$F'(t) \geq t^{-\mu} \int_1^t s^\mu \int |\nabla_x u|^p dx ds.$$

Integrating again, we have from $F(1) > 0$ by assumption

$$F(t) \geq \int_1^t \tau^{-\mu} \int_1^\tau s^\mu \int |\nabla_x u|^p dx ds d\tau \quad \text{for } t \geq 1. \quad (3.5)$$

3.1 Case $0 \leq \alpha < 1$

We call wavelike and heatlike cases if a blow-up condition is concerned with exponents similar to the Strauss and Fujita ones, respectively.

3.1.1 Wavelike and subcritical case

Let $p'_c(n, \alpha, \mu)$ be the positive root of the equation¹

$$\gamma'(n, p, \alpha, \mu) \equiv -p^2 \left(n + 1 + \frac{\mu - \alpha}{1 - \alpha} \right) + p \left(n + 1 + \frac{\mu + 3\alpha}{1 - \alpha} \right) + 2 = 0, \quad (3.6)$$

and let

$$p'_0(n, \alpha, \mu) = 1 + \frac{1 + \alpha}{(n + 1)(1 - \alpha) + \mu - 1}.$$

Theorem 3.1 *Let $n \geq 2$, $0 \leq \alpha < 1$, $\mu \geq 0$, and $1 < p < p'_c(n, \alpha, \mu)$ or $1 < p < p'_0(n, \alpha, \mu)$. Assume that $u_0 \in C^2(\mathbf{R}^n)$ and $u_1 \in C^1(\mathbf{R}^n)$ are nonnegative, nontrivial, and $\text{supp } u_0, \text{supp } u_1 \subset \{|x| \leq R\}$ with $R > 0$. Suppose that problem (1.7) has a classical solution $u \in C^2([1, T] \times \mathbf{R}^n)$. Then $T < \infty$, and there exists a constant $\varepsilon_0 > 0$ depending on $p, \alpha, \mu, R, u_0, u_1$ such that T_ε has to satisfy*

$$T_\varepsilon \leq C\varepsilon^{\frac{-2p(p-1)}{(1-\alpha)\gamma'(n,p,\alpha,\mu)}} \quad \text{if } 1 < p < p'_c(n, \alpha, \mu), \quad (3.7)$$

$$T_\varepsilon \leq C\varepsilon^{-\frac{p-1}{\{1-\mu-(n+1)(1-\alpha)\}(p-1)+1+\alpha}} \quad \text{if } 1 < p < p'_0(n, \alpha, \mu) \quad (3.8)$$

for $0 < \varepsilon \leq \varepsilon_0$, where $C > 0$ is a constant independent of ε .

Remark (1) The upper bound of the lifespan in (3.8) is better than that in (3.7) if $1 < p < 2(1 - \alpha)/\{(n + 1)(1 - \alpha) + \mu + \alpha - 2\}$ since

$$\{1 - \mu - (n + 1)(1 - \alpha)\}(p - 1) + 1 + \alpha > \frac{1 - \alpha}{2p} \gamma'(n, p, \alpha, \mu).$$

Note that in this case, if $p < p'_c(n, \alpha, \mu)$, then $\{1 - \mu - (n + 1)(1 - \alpha)\}(p - 1) + 1 + \alpha > 0$.

(2) If $p < 2(1 - \alpha)/\{(n + 1)(1 - \alpha) + \mu + \alpha - 2\}$, then the condition $\mu < 1$ is necessary since $p > 1$ and $n \geq 2$. Hence, the theorem in this case is not applied to the original equation (1.2) for $|\nabla_x u|^p$ in the FLRW spacetime since $\mu = 2/(1 + w) \geq 1$. This is covered later in Sect. 4 in more detail.

Proof We have proved in [1], by choosing large $T_1 > 0$,

$$\int |u|^p dx \geq C\varepsilon^p t^{-(\mu-\alpha)p/2+(1-\alpha)(n-1)(1-p/2)}$$

for $t \geq T_1$. By (2.6) and (3.2), we obtain

$$\int |\nabla_x u|^p dx \geq C\varepsilon^p t^{-(\mu-\alpha)p/2+(1-\alpha)(n-1)(1-p/2)-p(1-\alpha)} \quad \text{for } t \geq T_1. \quad (3.9)$$

¹For the equation $u_{tt} - \frac{1}{t^2\alpha} \Delta u + \frac{\mu}{t} u_t = |u|^p$, the critical exponents as blow-up conditions are $\rho_c(n, \alpha, \mu)$ and $\rho_F(n, \alpha)$, where $\rho_c(n, \alpha, \mu)$ is the positive root of

$$\gamma(n, p, \alpha, \mu) = -p^2 \left(n - 1 + \frac{\mu - \alpha}{1 - \alpha} \right) + p \left(n + 1 + \frac{\mu + 3\alpha}{1 - \alpha} \right) + 2 = 0,$$

and $\rho_F(n, \alpha) = 1 + 2/\{n(1 - \alpha)\}$. We remark that if $\mu = \alpha = 0$, then $\rho_c(n, 0, 0)$ and $\rho_F(n, 0)$ coincide with the Strauss and Fujita exponents, respectively. See [1, 2].

Using (3.5) implies

$$\begin{aligned} F(t) &\geq C\varepsilon^p \int_{T_1}^t \tau^{-\mu} \int_{T_1}^{\tau} s^{\mu-(\mu-\alpha)p/2+(1-\alpha)(n-1)(1-p/2)-p(1-\alpha)} ds d\tau \\ &\geq C\varepsilon^p \int_{T_1}^t \tau^{-\mu(1+p/2)-(1-\alpha)(n-1)p/2-p(1-\alpha)} \int_{T_1}^{\tau} (s-T_1)^{\mu+\alpha p/2+(1-\alpha)(n-1)} ds d\tau \\ &\geq C\varepsilon^p t^{-\mu(1+p/2)-(1-\alpha)(n-1)p/2-p(1-\alpha)} \int_{T_1}^t (\tau-T_1)^{\mu+\alpha p/2+(1-\alpha)(n-1)+1} d\tau. \end{aligned}$$

Therefore,

$$F(t) \geq C\varepsilon^p t^{-\mu(1+p/2)-(1-\alpha)(n-1)p/2-p(1-\alpha)} (t-T_1)^{\mu+\alpha p/2+(1-\alpha)(n-1)+2} \quad \text{for } T \geq T_1. \quad (3.10)$$

We also have

$$F''(t) + \frac{\mu}{t} F'(t) \geq \frac{C}{(t+R)^{(p+n(p-1))(1-\alpha)}} |F(t)|^p \quad (3.11)$$

by (2.6) and (3.3). The following lemma is proved in [1].

Lemma 3.2 *Let $p > 1$, $a \geq 0$, $b > 0$, $q > 0$, $\mu \geq 0$, and*

$$M \equiv (p-1)(b-a) - q + 2 > 0.$$

Let $T \geq T_1 > T_0 \geq 1$. Assume that $F \in C^2([T_0, T])$ satisfies the following three conditions:

- (i) $F(t) \geq A_0 t^{-a} (t-T_1)^b$ for $t \geq T_1$,
- (ii) $F''(t) + \frac{\mu F'(t)}{t} \geq A_1 (t+R)^{-q} |F(t)|^p$ for $t \geq T_0$,
- (iii) $F(T_0) \geq 0$, $F'(T_0) > 0$,

where A_0, A_1 , and R are positive constants. Then T has to satisfy

$$T < CA_0^{-(p-1)/M},$$

where C is a constant depending on R, A_1, μ, p, q, a , and b .

From (3.10) and (3.11), applying Lemma 3.2 with $q = (p+n(p-1))(1-\alpha)$, $a = \mu(1+p/2) + (1-\alpha)(n-1)p/2 + p(1-\alpha)$, $b = \mu + \alpha p/2 + (1-\alpha)(n-1) + 2$, and $A_0 = C\varepsilon^p$, we obtain the desired result since

$$\begin{aligned} M &= (p-1) \left\{ 2 - \frac{\mu-\alpha}{2} p + (1-\alpha) \left((n-1) \left(1 - \frac{p}{2} \right) - p \right) \right\} - (p+n(p-1))(1-\alpha) + 2 \\ &= \frac{1-\alpha}{2} \gamma'(n, p, \alpha, \mu) > 0. \end{aligned}$$

It remains to prove (3.8). From (3.4), we have $F'(t) \geq t^{-\mu} F'(1)$, hence integrating and using $F(1) > 0$ by assumption imply

$$F(t) \geq F'(1)t^{1-\mu} = C\varepsilon t^{1-\mu} \quad \text{for } t \geq T'$$

with some $T' > 1$. By (2.6) and (3.3),

$$\int |\nabla_x u|^p dx \geq C\varepsilon^p t^{-(p+n(p-1))(1-\alpha)+p(1-\mu)} \quad \text{for } t \geq T'. \quad (3.12)$$

From (3.5),

$$\begin{aligned} F(t) &\gtrsim \varepsilon^p \int_{T'}^t \tau^{-\mu} \int_{T'}^{\tau} s^{\mu-(p+n(p-1))(1-\alpha)+p(1-\mu)} ds d\tau \\ &\gtrsim \varepsilon^p \int_{T'}^t \tau^{-(p+n(p-1))(1-\alpha)-p\mu} \int_{T'}^{\tau} s^p ds d\tau \\ &\gtrsim \varepsilon^p \int_{T'}^t \tau^{-(p+n(p-1))(1-\alpha)-p\mu} (\tau - T')^{p+1} d\tau \\ &\geq C\varepsilon^p t^{-(p+n(p-1))(1-\alpha)-p\mu} (t - T')^{p+2} \quad \text{for } t \geq T'. \end{aligned} \quad (3.13)$$

From (3.11) and (3.13), applying Lemma 3.2 with $q = (p + n(p - 1))(1 - \alpha)$, $a = (p + n(p - 1))(1 - \alpha) + p\mu$, $b = p + 2$, and $A_0 = C\varepsilon^p$, we obtain the desired result. This completes the proof of Theorem 3.1. $\square$

In Theorem 3.1 the coefficient $n + 1 + (\mu - \alpha)/(1 - \alpha)$ of p^2 in (3.6) is positive since it is assumed that the positive root $p'_c(n, \alpha, \mu)$ exists. If $\mu < 1$ and $3/4 \leq \alpha < 1$, then $n + 1 + (\mu - \alpha)/(1 - \alpha) \leq 0$ can happen; hence, $\gamma'(n, p, \alpha, \mu) > 0$ for all $p > 1$ and the following corollary holds.

Corollary 3.3 *Let $n \geq 2$, $0 \leq \alpha < 1$, $\mu \geq 0$, $n + 1 + (\mu - \alpha)/(1 - \alpha) \leq 0$, and $p > 1$. Under the assumptions on the initial data of Theorem 3.1, there holds (3.7).*

Remark One cannot apply Corollary 3.3 to the original equation (1.2) for $|\nabla_x u|^p$ in the FLRW spacetime since $n + 1 + (\mu - \alpha)/(1 - \alpha) > 0$. This is covered later in Sect. 4 in more detail.

3.1.2 Wavelike and critical case

We next consider the critical case $p = p'_c(n, \alpha, \mu)$.

Theorem 3.4 *Let $0 \leq \alpha < 1$ for $n \geq 3$ and $2/7 < \alpha < 1$ for $n = 2$, and let $\mu \geq 0$ and*

$$p = p'_c(n, \alpha, \mu) > \begin{cases} p'_F(n, \alpha) = 1 + \frac{1+\alpha}{(n+1)(1-\alpha)} & \text{if } n \geq 3, \\ \max\{p'_F(2, \alpha), 2\} & \text{if } n = 2. \end{cases} \quad (3.14)$$

Assume that $u_0 \in C^2(\mathbf{R}^n)$ and $u_1 \in C^1(\mathbf{R}^n)$ are nonnegative, nontrivial, and $\text{supp } u_0, \text{supp } u_1 \subset \{|x| \leq R\}$ with some $0 < R \leq 1/(2(1 - \alpha))$. Suppose that problem (1.7) has a classical solution $u \in C^2([1, T) \times \mathbf{R}^n)$. Then $T < \infty$, and there exists a constant $\varepsilon_0 > 0$ depending

on $p, \alpha, \mu, R, u_0, u_1$ such that T_ε has to satisfy

$$T_\varepsilon \leq \exp(C\varepsilon^{-p(p-1)})$$

for $0 < \varepsilon \leq \varepsilon_0$, where $C > 0$ is a constant independent of ε .

Remark (1) In case $n = 2$, we note that

$$\max\{p'_F(2, \alpha), 2\} = \begin{cases} p'_F(2, \alpha) & \text{if } \alpha \geq 1/2, \\ 2 & \text{if } \alpha < 1/2. \end{cases}$$

(2) When $n = 2$, the condition $\alpha > 2/7$ is necessary for the existence of p satisfying (3.14).

(3) If $p'_F(n, \alpha) = p'_c(n, \alpha, \mu)$, then

$$\mu = \mu^* \equiv (n+1)(1-\alpha) + \alpha - \frac{2(n+1)(1-\alpha)^2}{(n+1)(1-\alpha) + 1 + \alpha}.$$

(4) The case $p'_c(n, \alpha, \mu) < p \leq p'_F(n, \alpha)$ is considered in the next sub-subsection.

Proof Let

$$\begin{aligned} \lambda_\eta(t) &= \lambda(\eta t) = (\eta t)^{(1-\mu)/2} K_\nu \left(\frac{1}{1-\alpha} (\eta t)^{1-\alpha} \right), \\ \psi_\eta(t, x) &= \lambda_\eta(t) \int_{|\omega|=1} e^{\eta^{1-\alpha} x \cdot \omega} dS_\omega, \end{aligned} \quad (3.15)$$

where $K_\nu(t)$ is the modified Bessel function given by (2.7). We now define a test function $\phi_q(t, x)$ by

$$\phi_q(t, x) = \int_0^1 \psi_\eta(t, x) \eta^{q-1+\mu} d\eta. \quad (3.16)$$

Let q satisfy

$$q > -\frac{\mu + \alpha}{2} \quad (3.17)$$

and

$$q + \frac{\mu - 1}{2} - (1 - \alpha)|v| > -1. \quad (3.18)$$

It is proved in [1] that the function $\phi_q(t, x)$ satisfies the following properties.

Lemma 3.5 *Let $\phi_q(t, x)$ be defined by (3.16). Assume that q satisfies (3.17) and (3.18).*

(i) *Then there exists $T_0 > 0$ such that ϕ_q satisfies*

$$\phi_q(t, x) \sim \begin{cases} t^{(-\mu+\alpha)/2} (t^{1-\alpha} + |x|)^{-(q+\frac{\mu+\alpha}{2})/(1-\alpha)} \\ \quad (-\frac{\mu+\alpha}{2} < q < \frac{(n-1)(1-\alpha)-(\mu+\alpha)}{2}), \\ t^{(-\mu+\alpha)/2} (t^{1-\alpha} + |x|)^{-(n-1)/2} (t^{1-\alpha} - (1-\alpha)|x|)^{(n-1)/2-(q+\frac{\mu+\alpha}{2})/(1-\alpha)} \\ \quad (q > \frac{(n-1)(1-\alpha)-(\mu+\alpha)}{2}), \end{cases}$$

for $t \geq T_0$ and $|x| \leq (t^{1-\alpha} - 1)/(1 - \alpha) + R$ with $0 < R \leq 1/(2(1 - \alpha))$.

(ii) Moreover, if $q + 1 - \alpha > \{(n-1)(1-\alpha) - (\mu + \alpha)\}/2$, then there exists $T_1 > 0$ such that ϕ_q satisfies

$$\partial_t \phi_q(t, x) \sim t^{-(\mu+\alpha)/2} (t^{1-\alpha} + |x|)^{-(n-1)/2} (t^{1-\alpha} - (1-\alpha)|x|)^{(n-1)/2 - (q + \frac{\mu+\alpha}{2})/(1-\alpha) - 1}$$

for $t \geq T_1$ and $|x| \leq (t^{1-\alpha} - 1)/(1-\alpha) + R$ with $0 < R \leq 1/(2(1-\alpha))$.

We now prove the following key lemma to prove the theorem.

Lemma 3.6 Assume that u, u_0 , and u_1 satisfy the conditions in Theorem 3.4. Let $n \geq 2$, $v = (\mu - 1)/(2(1 - \alpha))$, and

$$q = \frac{(n-1)(1-\alpha) - (\mu + \alpha)}{2} - \frac{1-\alpha}{p},$$

and let p satisfy (3.14). Define

$$G(t) = \int_1^t (t-\tau) \tau^{1+\mu} \int |\nabla_x u|^p \phi_q(\tau, x) dx d\tau.$$

Then $G(t)$ satisfies

$$G'(t) \geq C(\ln t)^{1-p} t \left(\int_1^t \tau^{-3} G(\tau) d\tau \right)^p \quad \text{for } t \geq T_2$$

with some T_2 sufficiently large, where C is a constant independent of ε .

Proof We first verify that q satisfies the required conditions (3.17) and (3.18) to use Lemma 3.5. We claim that there holds

$$-\frac{\mu + \alpha}{2} < q < \frac{(n-1)(1-\alpha) - (\mu + \alpha)}{2}. \quad (3.19)$$

The second inequality is clearly true. To show the first inequality or (3.17), we remark that (3.14) implies after some calculation $p > 2/(n-1)$, which is equivalent to $q > -(\mu + \alpha)/2$.

We can also show that q satisfies (3.18) with $v = (\mu - 1)/(2(1 - \alpha))$, i.e., $q > -\min\{\mu, 1\}$. In fact, (i) the assumption $p > p'_F(n, \alpha)$ is equivalent to $q > -1$ since

$$q = n(1-\alpha) - \frac{2}{p-1} - 1 + p'(1-\alpha), \quad \frac{1}{p} + \frac{1}{p'} = 1 \quad \text{if } p = p'_c; \quad (3.20)$$

(ii) the critical case $p = p'_c$ satisfies $\gamma'(n, p, \alpha, \mu) = 0$ in (3.6), and this equality yields

$$\begin{aligned} p &= \frac{(n+1)(1-\alpha) + \mu + 3\alpha}{(n+1)(1-\alpha) + \mu - \alpha} + \frac{2(1-\alpha)}{p\{(n+1)(1-\alpha) + \mu - \alpha\}} \\ &> \frac{(n+1)(1-\alpha) + \mu + \alpha}{(n+1)(1-\alpha) + \mu - \alpha} \\ &= 1 + \frac{1+\alpha}{(n+1)(1-\alpha) + \mu - 1}, \end{aligned}$$

which is equivalent to $q > -\mu$ by (3.20).

In addition, q satisfies the condition of Lemma 3.5(ii) since

$$\begin{aligned} q &< \frac{(n-1)(1-\alpha) - (\mu + \alpha)}{2} \\ &< \frac{(n-1)(1-\alpha) - (\mu + \alpha)}{2} + (1-\alpha) \left(1 - \frac{1}{p}\right) = q + 1 - \alpha. \end{aligned} \quad (3.21)$$

Let us now show the inequality of the lemma. Multiplying the equation in (1.7) by a test function $\phi(t, x)$ and t^μ , and integrating over $\mathbf{R}^n$, we have

$$\begin{aligned} &\frac{d}{dt} \int t^\mu (u\phi)_t dx - 2 \frac{d}{dt} \int t^\mu u \phi_t dx + \int t^\mu u \left(\phi_{tt} - \frac{1}{t^{2\alpha}} \Delta \phi + \frac{\mu}{t} \phi_t \right) dx \\ &= \int t^\mu |\nabla_x u|^p \phi dx. \end{aligned} \quad (3.22)$$

As shown in [1], the test function ϕ_q given in (3.16) satisfies

$$\left(\partial_t^2 - \frac{1}{t^{2\alpha}} \Delta + \frac{\mu}{t} \partial_t \right) \phi_q(t, x) = 0. \quad (3.23)$$

Applying (3.22) with $\phi = \phi_q$ and (3.23), we have

$$\frac{d^2}{dt^2} \int t^\mu u \phi_q dx - \mu \frac{d}{dt} \int t^{\mu-1} u \phi_q dx - 2 \frac{d}{dt} \int t^\mu u \partial_t \phi_q dx = \int t^\mu |\nabla_x u|^p \phi_q dx.$$

Moreover, integrating over $[1, t]$ three times, we obtain

$$\begin{aligned} &\int_1^t \tau^\mu \int u \phi_q dx d\tau - \mu \int_1^t (t-\tau) \tau^{\mu-1} \int u \phi_q dx d\tau - 2 \int_1^t (t-\tau) \tau^\mu \int u \partial_\tau \phi_q dx d\tau \\ &= C_{\text{data}}(t) + \frac{1}{2} \int_1^t (t-\tau)^2 \tau^\mu \int |\nabla_x u|^p \phi_q dx d\tau, \end{aligned}$$

where

$$C_{\text{data}}(t) = \varepsilon(t-1) \int u_0(x) \phi_q(1, x) dx + \frac{\varepsilon}{2}(t-1)^2 \int (u_1(x) \phi_q(1, x) - u_0(x) \partial_t \phi_q(1, x)) dx.$$

We note that $\partial_t \phi_q(1, x) \leq 0$, which is shown in [1]. Hence, by the positivity assumption on u_0 and u_1 , it holds that $C_{\text{data}}(t) \geq 0$ for $t \geq 1$. Thus,

$$\begin{aligned} &\int_1^t \tau^\mu \int u \phi_q(\tau, x) dx d\tau - \mu \int_1^t (t-\tau) \tau^{\mu-1} \int u \phi_q(\tau, x) dx d\tau \\ &\quad - 2 \int_1^t (t-\tau) \tau^\mu \int u \partial_\tau \phi_q(\tau, x) dx d\tau \\ &\geq \frac{1}{2} \int_1^t (t-\tau)^2 \tau^\mu \int |\nabla_x u|^p \phi_q(\tau, x) dx d\tau. \end{aligned} \quad (3.24)$$

Since

$$G'(t) = \int_1^t \tau^{1+\mu} \int |\nabla_x u|^p \phi_q dx d\tau \quad \text{and} \quad G''(t) = t^{1+\mu} \int |\nabla_x u|^p \phi_q dx d\tau,$$

the right-hand side of (3.24) becomes

$$\begin{aligned} \frac{1}{2} \int_1^t (t-\tau)^2 \tau^\mu \int |\nabla_x u|^p \phi_q dx d\tau &= \frac{1}{2} \int_1^t (t-\tau)^2 \tau^{-1} G''(\tau) d\tau \\ &= \int_1^t t^2 \tau^{-3} G(\tau) d\tau. \end{aligned} \quad (3.25)$$

We now estimate the left-hand side of (3.24). By Poincaré's and Hölder's inequalities, the first integral is estimated by

$$\begin{aligned} I &\equiv \int_1^t \tau^\mu \int u \phi_q(\tau, x) dx d\tau \\ &\leq \int_1^t \tau^\mu \|\phi_q(\tau)\|_{L^\infty(|x| \leq A(\tau)+R)} \int_{|x| \leq A(\tau)+R} |u(\tau, x)| dx d\tau \\ &\leq \int_1^t \tau^\mu \|\phi_q(\tau)\|_{L^\infty(|x| \leq A(\tau)+R)} (A(\tau) + R) \int_{|x| \leq A(\tau)+R} |\nabla_x u(\tau, x)| dx d\tau \\ &\leq \int_1^t \tau^\mu \|\phi_q(\tau)\|_{L^\infty(|x| \leq A(\tau)+R)} (A(\tau) + R) \\ &\quad \times \left(\int_{|x| \leq A(\tau)+R} |\nabla_x u(\tau, x)|^p \phi_q(\tau, x) dx \right)^{1/p} \left(\int_{|x| \leq A(\tau)+R} \phi_q(\tau, x)^{1-p'} dx \right)^{1/p'} d\tau \\ &\leq \left(\int_1^t \tau^{1+\mu} \int |\nabla_x u|^p \phi_q(\tau, x) dx d\tau \right)^{1/p} \\ &\quad \times \left(\int_1^t \tau^{1+\mu-p'} \|\phi_q(\tau)\|_{L^\infty(|x| \leq A(\tau)+R)}^{p'} (A(\tau) + R)^{p'} \int_{|x| \leq A(\tau)+R} \phi_q(\tau, x)^{1-p'} dx d\tau \right)^{1/p'}. \end{aligned}$$

We remark that q satisfies (3.19). Applying Lemma 3.5 to the last integral above, we have

$$\begin{aligned} &\left(\int_1^{T_0} + \int_{T_0}^t \right) \tau^{1+\mu-p'} \|\phi_q(\tau)\|_{L^\infty(|x| \leq A(\tau)+R)}^{p'} (A(\tau) + R)^{p'} \int_{|x| \leq A(\tau)+R} \phi_q(\tau, x)^{1-p'} dx d\tau \\ &\lesssim C_{T_0} + \int_{T_0}^t \tau^{1+\mu-p'+p'(-\mu+\alpha)/2-p'(q+(\mu+\alpha)/2)+p'(1-\alpha)} \\ &\quad \times \int_{|x| \leq A(\tau)+R} \left\{ \tau^{(-\mu+\alpha)/2} (\tau^{1-\alpha} + |x|)^{-(q+(\mu+\alpha)/2)/(1-\alpha)} \right\}^{1-p'} dx d\tau \\ &\lesssim C_{T_0} + \int_1^t \tau^{n(1-\alpha)-q-p'/p+p'(1-\alpha)} d\tau \\ &\lesssim C_{T_0} t^{n(1-\alpha)-q-p'/p+p'(1-\alpha)+1}, \end{aligned}$$

where we have used $A(\tau) + R = (\tau^{1-\alpha} - 1)/(1 - \alpha) + R$, with $R \leq 1/(2(1 - \alpha))$. Note that if $p = p'_c$, then q satisfies by (3.20)

$$n(1 - \alpha) - q - \frac{p'}{p} + p'(1 - \alpha) = p'. \quad (3.26)$$

Thus, we obtain

$$I \lesssim G'(t)^{1/p} t^{1+1/p'} \quad \text{for } t \geq T_0. \quad (3.27)$$

The second integral on the left-hand side of (3.24) can be estimated as before by

$$\begin{aligned} II &\equiv -\mu \int_1^t (t-\tau) \tau^{\mu-1} \int u \phi_q(\tau, x) dx d\tau \\ &\lesssim \left(\int_1^t \tau^{1+\mu} \int |\nabla_x u|^p \phi_q(\tau, x) dx d\tau \right)^{1/p} \\ &\quad \times \left(\int_1^t (t-\tau)^{p'} \tau^{1+\mu-2p'} \|\phi_q(\tau)\|_{L^\infty(|x| \leq A(\tau)+R)}^{p'} (A(\tau)+R)^{p'} \right. \\ &\quad \left. \times \int_{|x| \leq A(\tau)+R} \phi_q(\tau, x)^{1-p'} dx d\tau \right)^{1/p'}. \end{aligned}$$

Using Lemma 3.5,

$$II \lesssim G'(t)^{1/p} \left(C_{T_0} + \int_{T_0}^t (t-\tau)^{p'} \tau^{1-2p'-q+n(1-\alpha)+p'(1-\alpha)} d\tau \right)^{1/p'}.$$

Since q satisfies $n(1-\alpha) - q + 1 - 2p' + p'(1-\alpha) = 0$, by (3.26), we obtain

$$II \lesssim G'(t)^{1/p} t^{1+1/p'} \quad \text{for } t \geq T_0. \quad (3.28)$$

We finally estimate the third integral on the left-hand side of (3.24),

$$III \equiv -2 \int_1^t (t-\tau) \tau^\mu \int u \partial_\tau \phi_q(\tau, x) dx d\tau.$$

Set $T_2 = \max\{T_0, T_1\}$ to apply Lemma 3.5(i) and (ii). Proceeding in a similar way as before, we have

$$\begin{aligned} III &\lesssim \left(\int_1^t \tau^{1+\mu} \int |\nabla_x u|^p \phi_q(\tau, x) dx d\tau \right)^{1/p} \\ &\quad \times \left(C_{T_2} + \int_{T_2}^t (t-\tau)^{p'} \tau^{1+\mu-p'+p'(1-\alpha)} \int_{|x| \leq A(\tau)+R} \phi_q \left(\frac{\partial_\tau \phi_q}{\phi_q} \right)^{p'}(\tau, x) dx d\tau \right)^{1/p'}. \end{aligned}$$

We remark here that q satisfies (3.21). By Lemma 3.5,

$$\phi_q \left(\frac{\partial_\tau \phi_q}{\phi_q} \right)^{p'} \sim \tau^{(-\mu+\alpha)/2-\alpha p'} (\tau^{1-\alpha} + |x|)^{-(n-1)/2-(p'-1)/p} (\tau^{1-\alpha} - (1-\alpha)|x|)^{-1} \quad \text{for } t \geq T_2.$$

Hence,

$$\begin{aligned} &\tau^{1+\mu-p'+p'(1-\alpha)} \int_{|x| \leq A(\tau)+R} \phi_q \left(\frac{\partial_\tau \phi_q}{\phi_q} \right)^{p'}(\tau, x) dx \\ &\leq \tau^{1+\mu-p'+p'(1-\alpha)+(-\mu+\alpha)/2-\alpha p'-(n-1)(1-\alpha)/2-(p'-1)(1-\alpha)/p} \int_{|x| \leq A(\tau)+R} (\tau^{1-\alpha} - (1-\alpha)|x|)^{-1} dx \\ &\lesssim \tau^{-(n-1)(1-\alpha)} \int_0^{A(\tau)+R} (\tau^{1-\alpha} - (1-\alpha)r)^{-1} r^{n-1} dr \\ &\lesssim \ln \tau, \end{aligned}$$

where we note that since $(n-1)(1-\alpha)/2 = q + (\mu + \alpha)/2 + (1-\alpha)/p$ and (3.26),

$$\begin{aligned} 1 + \mu - p' + p'(1-\alpha) + \frac{-\mu + \alpha}{2} - \alpha p' - \frac{(n-1)(1-\alpha)}{2} - \frac{(p'-1)(1-\alpha)}{p} \\ = 1 - p' + p'(1-\alpha) - \alpha p' - q - \frac{p'(1-\alpha)}{p} \\ = -(n-1)(1-\alpha). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} III &\lesssim G'(t)^{1/p} \left(C_{T_2} + \int_{T_2}^t (t-\tau)^{p'} \ln \tau \, d\tau \right)^{1/p'} \\ &\lesssim G'(t)^{1/p} t^{1+1/p'} (\ln t)^{1/p'} \quad \text{for } t \geq T_2. \end{aligned} \quad (3.29)$$

Combining (3.24), (3.25), and (3.27)–(3.29) all together, we obtain the desired inequality. This completes the proof of Lemma 3.6. $\square$

Then the rest of the proof is the same as that of Theorem 2.3 in [1]. $\square$

3.1.3 Heatlike case

Theorem 3.7 *Let $n \geq 2$, $0 \leq \alpha < 1$, $\mu \geq 0$, and $1 < p \leq p'_F(n, \alpha) = 1 + (1 + \alpha)/\{(n + 1)(1 - \alpha)\}$. Assume that $u_0 \in C^2(\mathbf{R}^n)$ and $u_1 \in C^1(\mathbf{R}^n)$ are nonnegative, nontrivial, and $\text{supp } u_0, \text{supp } u_1 \subset \{|x| \leq R\}$ with $R > 0$. Suppose that problem (1.7) has a classical solution $u \in C^2([1, T) \times \mathbf{R}^n)$. Then $T < \infty$, and there exists a constant $\varepsilon_0 > 0$ depending on $p, \alpha, \mu, R, u_0, u_1$ such that T_ε has to satisfy*

$$T_\varepsilon \leq C\varepsilon^{-\frac{p-1}{2-(n(p-1)+p)(1-\alpha)}} \quad \text{if } p < p'_F(n, \alpha), \quad (3.30)$$

$$T_\varepsilon \leq \exp(C\varepsilon^{-p(p-1)/(p+1)}) \quad \text{if } p = p'_F(n, \alpha) \text{ and } 0 \leq \mu \leq 1, \quad (3.31)$$

$$T_\varepsilon \leq \exp(C\varepsilon^{-(p-1)}) \quad \text{if } p = p'_F(n, \alpha) \text{ and } \mu > 1$$

for $0 < \varepsilon \leq \varepsilon_0$, where $C > 0$ is a constant independent of ε .

Remark In the critical case $p = p'_F(n, \alpha)$, if $0 \leq \mu \leq 1$, then the estimate above is better than that for the case $\mu > 1$. However, (3.7) and (3.8) are applicable for $p = p'_F(n, \alpha)$ and $0 \leq \mu \leq 1$. These estimates are much better than $T_\varepsilon \leq \exp(C\varepsilon^{-p(p-1)/(p+1)})$ above. See Figs. 3, 4, and 5.

Proof Proceeding as in [2], we have

$$F(t) \geq F(1) = C\varepsilon > 0 \quad \text{for } t \geq 1. \quad (3.32)$$

By (2.6) and (3.3), we have

$$\int |\nabla_x u|^p \, dx \geq C\varepsilon^p t^{-(p+n(p-1))(1-\alpha)} \quad \text{for } t \geq 1. \quad (3.33)$$

Figure 3 Range of blow-up conditions in case $n = 3$ and $\alpha = 0$

Figure 4 Range of blow-up conditions in case $n = 3$ and $\alpha = 0.3$

Let $p < p'_F(n, \alpha)$. From (3.5),

$$\begin{aligned}
 F(t) &\gtrsim \varepsilon^p \int_1^t \tau^{-\mu} \int_1^\tau s^{\mu-(p+n(p-1))(1-\alpha)} ds d\tau \\
 &\gtrsim \varepsilon^p \int_1^t \tau^{-\mu-(p+n(p-1))(1-\alpha)} \int_1^\tau (s-1)^\mu ds d\tau \\
 &\gtrsim \varepsilon^p \int_1^t \tau^{-\mu-(p+n(p-1))(1-\alpha)} (\tau-1)^{\mu+1} d\tau \\
 &\geq C\varepsilon^p t^{-\mu-(p+n(p-1))(1-\alpha)} (t-1)^{\mu+2} \quad \text{for } t \geq 1.
 \end{aligned} \tag{3.34}$$

From (3.11) and (3.34), applying Lemma 3.2 with $q = (p+n(p-1))(1-\alpha)$, $a = \mu + (p+n(p-1))(1-\alpha)$, $b = \mu + 2$, and $A_0 = C\varepsilon^p$, we obtain the desired results since

$$\begin{aligned}
 M &= (p-1)\{2 - (p+n(p-1))(1-\alpha)\} - (p+n(p-1))(1-\alpha) + 2 \\
 &= p\{2 - (p+n(p-1))(1-\alpha)\} > 0.
 \end{aligned}$$

Let next $p = p'_F(n, \alpha)$. Since $(p+n(p-1))(1-\alpha) = 2$, from (3.33),

$$\int |\nabla_x u|^p dx \geq C\varepsilon t^{-2} \quad \text{for } t \geq 1.$$

Hence, by (3.5),

$$\begin{aligned}
 F(t) &\gtrsim \varepsilon^p \int_1^t \tau^{-\mu} \int_1^\tau s^{\mu-2} ds d\tau \\
 &\gtrsim \varepsilon^p \int_1^t \tau^{-\mu-2} \int_1^\tau (s-1)^\mu ds d\tau \gtrsim \varepsilon^p \int_2^t \tau^{-1} d\tau \\
 &\geq C\varepsilon^p \ln \frac{t}{2} \quad \text{for } t \geq 2.
 \end{aligned} \tag{3.35}$$

The following lemma is proved in [2].

Lemma 3.8 *Let $p > 1, b > 0, \mu \geq 0$, and $T \geq T_1 > T_0 \geq 1$. Assume that $F \in C^2([T_0, T])$ satisfies the following three conditions:*

- (i) $F(t) \geq A_0 \left(\ln \frac{t}{T_1} \right)^b$ for $t \geq T_1$,
- (ii) $F''(t) + \frac{\mu F'(t)}{t} \geq A_1 (t+R)^{-2} |F(t)|^p$ for $t \geq T_0$,
- (iii) $F(T_0) \geq 0, \quad F'(T_0) > 0$,

where A_0, A_1 , and R are positive constants. Then T has to satisfy

$$T < \begin{cases} \exp(CA_0^{-(p-1)/\{b(p-1)+2\}}) & \text{if } \mu \leq 1, \\ \exp(CA_0^{-(p-1)/\{b(p-1)+1\}}) & \text{if } \mu > 1, \end{cases}$$

where C is a constant depending on R, A_1, μ, p , and b .

By (3.11) with $(p + n(p - 1))(1 - \alpha) = 2$ and (3.35), using Lemma 3.8 with $b = 1$ and $A_0 = C\varepsilon^p$, we obtain the desired results. This completes the proof of Theorem 3.7. $\square$

At the end of this subsection, we discuss the blow-up conditions and the estimates of the lifespan in the subcritical cases in Theorems 3.1 and 3.7.

We recall that $p'_c(n, \alpha, \mu)$ is the positive root of the equation $\gamma'(n, p, \alpha, \mu) = 0$ given in (3.6). Figures 3, 4, and 5 show the regions of blow-up conditions in the cases $\alpha = 0, \alpha = 0.3$, and $\alpha = 0.7$, respectively, each for $n = 3$. We recall that if $p'_c(n, \alpha, \mu) = p'_F(n, \alpha)$, then

$$\mu = \mu^* = (n + 1)(1 - \alpha) + \alpha - \frac{2(n + 1)(1 - \alpha)^2}{(n + 1)(1 - \alpha) + 1 + \alpha}.$$

Note that if $p'_c(n, \alpha, \mu) = p'_0(n, \alpha, \mu)$, then

$$\mu = \mu_0 \equiv -(n - 1)(1 - \alpha) + \sqrt{3\alpha^2 - 4\alpha + 2}.$$

We easily see that $\mu^* > 1$ and $\mu_0 < 1$. We also note that $2p(p - 1)/\{(1 - \alpha)\gamma'(n, p, \alpha, \mu)\} = (p - 1)/[2 - \{n(p - 1) + p\}(1 - \alpha)]$ yields $p = 2(1 - \alpha)/\{(n + 1)(1 - \alpha) - \mu + \alpha\}$, and that $2p(p - 1)/\{(1 - \alpha)\gamma'(n, p, \alpha, \mu)\} = (p - 1)/\{[1 - \mu - (n + 1)(1 - \alpha)](p - 1) + 1 + \alpha\}$ yields $p = 2(1 - \alpha)/\{(n + 1)(1 - \alpha) + \mu + \alpha - 2\}$.

Among the three upper bounds (3.7), (3.8), and (3.30), if $1 < p < \min\{p'_0(n, \alpha, \mu), 2(1 - \alpha)/\{(n + 1)(1 - \alpha) + \mu + \alpha - 2\}\}$, then (3.8) is the best. This is Region (O) shown in Figs. 4 and 5.

If $\max\{2(1 - \alpha)/\{(n + 1)(1 - \alpha) + \mu + \alpha - 2\}, 2(1 - \alpha)/\{(n + 1)(1 - \alpha) - \mu + \alpha\}, 1\} < p < p'_c(n, \alpha, \mu)$, then (3.7) is the best (Region (C) in Figs. 3, 4, and 5).

On the other hand, if $1 < p \leq 2(1 - \alpha)/\{(n + 1)(1 - \alpha) - \mu + \alpha\}$ and $p < p'_F(n, \alpha)$, then (3.30) is the best (Region (F) in Figs. 3, 4, and 5).

After some calculation, we see the following facts. If $(n - 3)/(n - 2) \leq \alpha < 1$ for $n \geq 3$ and $0 \leq \alpha < 1$ for $n = 2$, Region (O) appears in $0 \leq \mu < 1$. Moreover, if $\alpha \geq \max\{0, (n^2 - 2n - 1 - \sqrt{n^2 - 2n - 1})/(n^2 - 2n - 2)\}$, then $p'_0(n, \alpha, \mu) > p'_c(n, \alpha, \mu) > p'_F(n, \alpha)$ for $0 \leq \mu < \mu_0$, hence the blow-up condition $1 < p \leq p'_0(n, \alpha, \mu)$ is the best for $0 \leq \mu < \mu_0$. This is unlike the case of the equation with $|u|^p$ -nonlinearity for which the blow-up condition is related to only $p_c(n, \alpha, \mu)$ and $p_F(n, \alpha)$.

3.2 Case $\alpha \geq 1$

Theorem 3.9 *Let $n \geq 2, \alpha \geq 1, \mu \geq 0$, and $p > 1$. Assume that $u_0 \in C^2(\mathbf{R}^n)$ and $u_1 \in C^1(\mathbf{R}^n)$ are nonnegative, nontrivial, and $\text{supp } u_0, \text{supp } u_1 \subset \{|x| \leq R\}$ with $R > 0$. Suppose that problem (1.7) has a classical solution $u \in C^2([1, T) \times \mathbf{R}^n)$. Then $T < \infty$, and there exists a constant $\varepsilon_0 > 0$ depending on $p, \alpha, \mu, R, u_0, u_1$ such that T_ε has to satisfy*

$$\begin{aligned} T_\varepsilon^2 (\ln T_\varepsilon)^{-(p+n(p-1))} &\leq C\varepsilon^{-(p-1)} \quad \text{if } \alpha = 1, \\ T_\varepsilon &\leq C\varepsilon^{-(p-1)/2} \quad \text{if } \alpha > 1 \end{aligned}$$

for $0 < \varepsilon \leq \varepsilon_0$, where $C > 0$ is a constant independent of ε .

Proof We first prove the theorem for the case $\alpha = 1$. By (2.38) and (3.3),

$$F''(t) + \frac{\mu}{t} F'(t) \geq \frac{C}{(\ln t)^{p+n(p-1)}} |F(t)|^p. \quad (3.36)$$

Proceeding as in [3], we have (3.32).

$$F(t) \geq F(1) = C\varepsilon > 0 \quad \text{for } t \geq 1.$$

By (2.38) and (3.3),

$$\int |\nabla_x u|^p dx \geq C\varepsilon^p (\ln t)^{-(p+n(p-1))} \quad \text{for } t \geq 1. \quad (3.37)$$

Hence, from (3.5),

$$\begin{aligned} F(t) &\geq C\varepsilon^p \int_1^t (\ln \tau)^{-(p+n(p-1))} \tau^{-\mu} (\tau-1)^{\mu+1} d\tau \\ &\geq C\varepsilon^p (\ln t)^{-(p+n(p-1))} t^{-\mu} \int_1^t (\tau-1)^{\mu+1} d\tau \\ &\geq C\varepsilon^p (\ln t)^{-(p+n(p-1))} t^{-\mu} (t-1)^{\mu+2} \quad \text{for } t \geq 1. \end{aligned} \quad (3.38)$$

We here use another Kato's lemma which is Lemma 2.3 in [3].

Lemma 3.10 *Let $p > 1$, $a \geq 0$, $b \geq 0$, $c > 0$, $q \geq 0$, $\mu \geq 0$, and*

$$M \equiv (p-1)(c-a) + 2 > 0.$$

Let $T \geq T_1 > T_0 \geq 1$. Assume that $F \in C^2([T_0, T])$ satisfies the following three conditions:

- (i) $F(t) \geq A_0 t^{-a} (\ln t)^{-b} (t-T_1)^c$ for $t \geq T_1$,
- (ii) $F''(t) + \frac{\mu}{t} F'(t) \geq A_1 (\ln t)^{-q} |F(t)|^p$ for $t \geq T_0$,
- (iii) $F(T_0) \geq 0$, $F'(T_0) > 0$,

where A_0 and A_1 are positive constants. Then T has to satisfy

$$T^{M/(p-1)} (\ln t)^{-b-q/(p-1)} < C A_0^{-1},$$

where C is a constant depending on A_1 , μ , p , and c .

By (3.36) and (3.38), applying Lemma 3.10 with $a = \mu$, $b = q = p + n(p-1)$, $c = \mu + 2$, and $A_0 = C\varepsilon^p$, we obtain the desired result for $\alpha = 1$ since

$$M = 2(p-1) + 2 = 2p > 0.$$

It remains to prove the theorem for $\alpha > 1$. By (2.38) and (3.3),

$$F''(t) + \frac{\mu}{t} F'(t) \geq C |F(t)|^p. \quad (3.39)$$

As above, we also have

$$F(t) \geq C\varepsilon^p t^{-\mu} (t-1)^{\mu+2} \quad \text{for } t \geq 1.$$

Using Lemma 3.10 again, we obtain the desired results. This completes the proof of Theorem 3.9. $\square$

4 Wave equations in FLRW

We now apply Theorems 2.1, 2.3, 3.1, 3.4, 3.7, and 3.9 to the original equation (1.2), which is equivalent to (1.3) and (1.7) with $\alpha = 2/(n(1+w))$ and $\mu = 2/(1+w)$. We treat the case $-1 < w \leq 1$ and $n \geq 2$ so that $\alpha \geq 1/n$ and $\mu \geq 1$ in (1.3) and (1.7). Observe that the cases $-1 < w \leq 2/n - 1$ and $2/n - 1 < w \leq 1$ correspond to accelerating and decelerating expanding universes, respectively.

Consider the equation with the time derivative nonlinear term $|u_t|^p$. Denote $p'_G(n, 2/(n(1+w)), 2/(1+w))$ by $p'_G(n, w)$. Figure 6 shows the range of blow-up conditions in terms of w and p in the case $n = 3$.

For $2/n - 1 < w \leq 1$ and $n \geq 2$, applying Theorem 2.1 to (1.2) for $|u_t|^p$, we obtain the following upper bounds of the lifespan:

$$\begin{cases} T_\varepsilon \leq C\varepsilon^{\frac{-(p-1)}{1-(n-1+4/(n(1+w)))(p-1)/2}} & \text{if } 1 < p < p'_G(n, w), \\ T_\varepsilon \leq \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p'_G(n, w), \end{cases}$$

where $C > 0$ is a constant independent of ε . We note that estimate (2.3) in Theorem 2.1 is not applied to (1.2) since $\mu \geq 1$ in our case. See Region (G) in Fig. 6.

For $-1 < w \leq 2/n - 1$ and $n \geq 2$, we obtain from Theorem 2.3

$$\begin{cases} T_\varepsilon^{1-2(p-1)/(1+w)} (\ln T_\varepsilon)^{-n(p-1)} \leq C\varepsilon^{-(p-1)} & \text{if } 1 < p < 1 + \frac{1}{n} \text{ and } w = \frac{2}{n} - 1, \\ T_\varepsilon \leq C\varepsilon^{-(p-1)/(1-2(p-1)/(1+w))} & \text{if } 1 < p < \frac{3+w}{2} \text{ and } -1 < w < \frac{2}{n} - 1. \end{cases}$$

See Region (A) in Fig. 6.

From these results, we see that the blow-up range of p in the flat FLRW spacetime is smaller than that in the Minkowski spacetime because $p'_G(n, w) < p_G(n) = 1 + 2/(n-1)$. Moreover, in the subcritical case $p < p'_G(n, w)$, the lifespan of the blow-up solutions in the FLRW spacetime is longer than that in the Minkowski spacetime since

$\varepsilon^{-(p-1)/(1-(n-1)(p-1)/2)} < \varepsilon^{-(p-1)/(1-(n-1+4/(n(1+w)))(p-1)/2)}$ for sufficiently small ε . Although the critical value has not been established, we can say at the present time that global solutions in the FLRW spacetime exist more easily than in the Minkowski spacetime.

We next consider the equation with the space derivative nonlinear term $|\nabla_x u|^p$. We define here $\gamma'_0(n, p, w)$ corresponding to $\gamma'(n, p, \alpha, \mu)$ in (3.6) by

$$\gamma'_0(n, p, w) = \left(1 - \frac{2}{n(1+w)}\right) \gamma'\left(n, p, \frac{2}{n(1+w)}, \frac{2}{1+w}\right).$$

Then we obtain

$$\gamma'_0(n, p, w) = -\left(n+1 - \frac{4}{n(1+w)}\right)p^2 + \left(n+1 + \frac{4}{n(1+w)}\right)p + 2 - \frac{4}{n(1+w)}.$$

Let $p'_c(n, w)$ be the positive root of the equation $\gamma'_0(n, p, w) = 0$. We also denote $p'_F(n, 2/(n(1+w)))$ by $p'_F(n, w)$.

For $2/n - 1 < w \leq 1$ and $n \geq 2$, say a decelerated expanding universe, applying Theorems 3.1, 3.4, and 3.7 to (1.2) for $|\nabla_x u|^p$, we obtain the following upper bounds of the lifespan:

$$T_\varepsilon \leq C\varepsilon^{\frac{-2p(p-1)}{\gamma'_0(n, p, w)}} \quad \text{if } 1 < p < p'_c(n, w), \quad (4.1)$$

$$T_\varepsilon \leq \exp(C\varepsilon^{-p(p-1)}) \quad \text{if } p = p'_c(n, w) > p'_F(n, w)$$

$$T_\varepsilon \leq C\varepsilon^{\frac{-(p-1)}{2-\{(n+1)p-n\}(1-2/(n(1+w)))}} \quad \text{if } 1 < p < p'_F(n, w), \quad (4.2)$$

$$T_\varepsilon \leq \exp(C\varepsilon^{-(p-1)}) \quad \text{if } p = p'_F(n, w).$$

We note that estimates (3.8) in Theorem 3.1 and (3.31) in Theorem 3.7 are not applied to (1.2) since $\mu \geq 1$ in our case, and also that if $n = 2$, then $p'_F(2, w) \geq 2$ and $\alpha = 2/(n(1+w)) > 2/7$ in Theorem 3.4 since $w \leq 1$.

For $-1 < w \leq 2/n - 1$ and $n \geq 2$, say an accelerated expanding universe, we obtain from Theorem 3.9

$$\begin{aligned} T_\varepsilon^2 (\ln T_\varepsilon)^{-(n(p+1)-n)} &\leq C\varepsilon^{-(p-1)} \quad \text{if } p > 1 \text{ and } w = \frac{2}{n} - 1, \\ T_\varepsilon &\leq C\varepsilon^{-(p-1)/2} \quad \text{if } p > 1 \text{ and } -1 < w < \frac{2}{n} - 1. \end{aligned} \quad (4.3)$$

We see that blow-up in finite time can happen to occur for all $p > 1$. This is in contrast to the case of decelerated expansion above.

Figure 7 shows the range of blow-up conditions in terms of w and p in the case $n = 3$. Note that if $p'_F(n, w) = p'_c(n, w)$, then w is the larger root w^* of the equation

$$\begin{aligned} n^3(n+1)w^2 + 2n\{n^2(n+1) - (3n+4)(n-1)\}w + n^3(n+1) \\ - 2n(3n+4)(n-1) + 8(n^2 - n - 1) = 0. \end{aligned}$$

Region (A) is for the case of the accelerated expanding universe, where the lifespan of blow-up solutions is dominated by (4.3). In contrast, each Region (F) and (C) represents the decelerated expanding one. In Region (F) estimate (4.2) is better than (4.1); on the other hand, this relation becomes reverse in Region (C).

Figure 7 Range of blow-up conditions in case $n = 3$

Finally, let us compare the results for the term $|u_t|^p$ with those for $|\nabla_x u|^p$, especially in the decelerated expanding universe, say (G), (F), and (C). We observe that which has a larger blow-up range depends on the value of w for each n . If $n = 2$, then $\max\{p'_F(2, w), p'_C(2, w)\} > p'_G(2, w)$ for $0 = 2/n - 1 < w \leq 1$. The higher the dimension n becomes, however, the larger the w -interval such that $p'_G(n, w) > \max\{p'_F(n, w), p'_C(n, w)\}$ becomes.

We will treat the remaining case $w = -1$ in future papers.

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Authors' contributions

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