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Existence of ground state solutions for a class of Choquard equations with local nonlinear perturbation and variable potential

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Abstract

In this paper, we focus on the existence of solutions for the Choquard equation

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + \lambda |u|^{p-2}u, & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $\lambda > 0$ is a parameter, $\alpha \in (0, N)$, $N \geq 3$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential. As usual, $\alpha/N + 1$ is the lower critical exponent in the Hardy–Littlewood–Sobolev inequality. Under some weak assumptions, by using minimax methods and Pohožaev identity, we prove that this problem admits a ground state solution if $\lambda > \lambda_*$ for some given number λ_* in three cases: (i) $2 < p < \frac{4}{N} + 2$, (ii) $p = \frac{4}{N} + 2$, and (iii) $\frac{4}{N} + 2 < p < 2^*$. Our result improves the previous related ones in the literature.

MSC: 34C37; 35A15; 37J45; 47J30

Keywords: Ground state solution; Variable potential; Choquard equation; Critical point

1 Introduction

In this paper, we mainly study the Choquard equation with a variable potential and a local nonlinearity:

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + \lambda |u|^{p-2}u, & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\lambda > 0$, $\alpha \in (0, N)$, $N \geq 3$, $2 < p < 2^*$, and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha \pi^{N/2}|x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

$V : \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumptions:

(V1) $V \in \mathcal{C}(\mathbb{R}^N, [0,\infty))$;
(V2) $V(x) \le V_\infty := \lim_{|y|\to\infty} V(y) < \infty$ for all $x \in \mathbb{R}^N$.
By the Hardy–Littlewood–Sobolev inequality (see Lemma 2.1), one has

$$\left[\int_{\mathbb{R}^N} \bigl(I_\alpha * |u|^{\frac{\alpha}{N}+1}\bigr)|u|^{\frac{\alpha}{N}+1}\,dx\right]^{\frac{N}{N+\alpha}} \le S^{-1}\int_{\mathbb{R}^N} u^2\,dx. \tag{1.2}$$

In the work of Lieb and Loss (see [1]), the sharp constant S is achieved by a function $u \in H^1(\mathbb{R}^N)$ if and only if, for every $x \in \mathbb{R}^N$,

$$u(x) = A\bigl(z^2 + |x-a|^2\bigr)^{-\frac{N}{2}} \tag{1.3}$$

for $a \in \mathbb{R}^N$, $A > 0$, and $z > 0$. In Lemma 2.9, we choose $A = A_0 > 0$, and A_0 is determined by

$$A_0^{\frac{2\alpha}{N}+2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{dxdz}{(1+|x|^2)^{\frac{N+\alpha}{2}}|x-z|^{N-\alpha}(1+|z|^2)^{\frac{N+2}{2}}} = \frac{2^\alpha \pi^{\frac{N}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N-\alpha}{2})}. \tag{1.4}$$

Under (V1), (V2), (1.2), and the Sobolev embedding theorem, the weak solutions of (1.1) correspond to the critical points of the energy functional $\mathcal{I}: H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{I}(u) = {}& \frac{1}{2}\int_{\mathbb{R}^N}\bigl[|\nabla u|^2 + V(x)u^2\bigr]\,dx - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}\bigl(I_\alpha * |u|^{\frac{\alpha}{N}+1}\bigr)|u|^{\frac{\alpha}{N}+1}\,dx \\ & - \frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,dx,\end{aligned} \tag{1.5}$$

which is continuously differentiable and

$$\begin{aligned}\bigl\langle \mathcal{I}'(u), v\bigr\rangle = {}& \int_{\mathbb{R}^N}\bigl(\nabla u\cdot\nabla v + V(x)uv\bigr)\,dx - \int_{\mathbb{R}^N}\bigl(I_\alpha * |u|^{\frac{\alpha}{N}+1}\bigr)|u|^{\frac{\alpha}{N}-1}uv\,dx \\ & - \lambda\int_{\mathbb{R}^N}|u|^{p-2}uv\,dx, \quad \forall v \in H^1\bigl(\mathbb{R}^N\bigr).\end{aligned} \tag{1.6}$$

If the potential $V(x) \equiv V_\infty$, then (1.1) reduces to the autonomous equation

$$\begin{cases} -\Delta u + V_\infty u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + \lambda|u|^{p-2}u, & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.7}$$

Similar to (1.5), the energy functional of (1.7) is defined by

$$\begin{aligned}\mathcal{I}^\infty(u) = {}& \frac{1}{2}\int_{\mathbb{R}^N}\bigl[|\nabla u|^2 + V_\infty u^2\bigr]\,dx - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}\bigl(I_\alpha * |u|^{\frac{\alpha}{N}+1}\bigr)|u|^{\frac{\alpha}{N}+1}\,dx \\ & - \frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,dx.\end{aligned} \tag{1.8}$$

Equation (1.1) is a special form of the following Choquard equation with a local nonlinear perturbation and a variable potential:

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^q)|u|^{q-2}u + f(u), & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.9}$$

where $1 + \frac{\alpha}{N} < q < \frac{N+\alpha}{N-2}$.

If $f = 0$ and $V(x) \equiv 1$, (1.9) appears under the background of various physical models. For example, as early as in 1954, Pekar [2] introduced (1.9) into the physical model to study the free electrons in a ionic lattice interact with phonons associated with deformations of the lattice. Choquard equation is also known as the Schrödinger–Newton equation after the addition of non-relativistic Newtonian gravity to some Schrödinger equations [3–6]. Lieb [7] first verified the positive solution of (1.9) in $\mathbb{R}^3$ when $f = 0$, $\alpha = 2$, $V(x) \equiv 1$, and $q = 2$. Later, Lions [8, 9] further improved the results of (1.9) and obtained the existence and multiplicity of normalized solution for (1.9). The existence of a ground state solution and the qualitative properties of the solution in the range of exponents q which satisfies

$$1 + \frac{\alpha}{N} < q < \frac{N+\alpha}{N-2}$$

were established in [10].

The endpoints $\frac{N+\alpha}{N-2}$ and $\frac{N+\alpha}{N}$ are critical exponents. It is known to all that $\frac{N+\alpha}{N-2}$ is an upper critical exponent which plays a similar role as the Sobolev critical exponent in the local semilinear equations [11–17]. The lower critical exponent $\frac{N+\alpha}{N}$ is strictly greater than 1 which comes from inequality (1.2). So far, many authors have investigated the existence of nontrivial solutions of many forms of (1.9) (see [18–21]). In addition, for some applications of the variational method in elliptic systems, we refer to [22–24]. If the potential $V(x) \equiv 1$, then (1.1) reduces to the following equation:

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u + \lambda|u|^{p-2}u, & x \in \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.10}$$

Tang, Wei, and Chen [25] proved that (1.10) has ground state solutions in the following assumptions:

(i) $2 < p < \frac{4}{N} + 2$ and $\lambda > 0$;
(ii) $p = \frac{4}{N} + 2$ and $\lambda > \frac{N^2}{A_0^{\frac{4}{N}} S^{\frac{2}{\alpha}}}$;
(iii) $\frac{4}{N} + 2 < p < 2^*$ and $\lambda > \frac{pN^4 \Gamma \frac{pN}{2} \Gamma \frac{N}{2}}{8(N+1)!(A_0 t_0)^{p-2} \Gamma \frac{(p-1)N}{2}}$.

By using the mountain pass lemma, they obtained a Palais–Smale sequence and the corresponding energy level m. Then, from these three assumptions, an estimate of the energy level m was given, which is very important to ensure the Sobolev compactness. We further improve these three hypotheses to be applicable to the research in this paper. This has certain enlightenment to our work.

Motivated by the work of [26, 27], we use a weaker decay assumption on ∇V to solve the trouble caused by variable potential.

(V3) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$, and there is $\theta \in [0, 1)$ such that

$$\nabla V(x) \cdot x \leq \frac{(N-2)^2\theta}{2|x|^2}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

Van Schaftingen and Xia [11], Chen and Tang [26] did a pretty good job, which gives us some inspiration. To our knowledge, there seems to be no results of (1.1). Motivated by the above works, especially [25, 26], in this paper, we establish the existence result of ground state solutions for (1.1). To state our result, inspired by [28], we define the following Pohožaev identity functional on $H^1(\mathbb{R}^N)$:

$$\begin{aligned}\mathcal{P}(u) &:= \frac{N-2}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}\bigl[NV(x) + \nabla V(x)\cdot x\bigr]u^2\,dx \\ &\quad - \frac{N}{2}\int_{\mathbb{R}^N}\bigl(I_\alpha * |u|^{\frac{\alpha}{N}+1}\bigr)|u|^{\frac{\alpha}{N}+1}\,dx - \frac{N\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,dx, \end{aligned}\tag{1.11}$$

and

$$\begin{aligned}\mathcal{P}^\infty(u) &:= \frac{N-2}{2}\|\nabla u\|_2^2 + \frac{NV_\infty}{2}\|u\|_2^2 - \frac{N}{2}\int_{\mathbb{R}^N}\bigl(I_\alpha * |u|^{\frac{\alpha}{N}+1}\bigr)|u|^{\frac{\alpha}{N}+1}\,dx \\ &\quad - \frac{N\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,dx.\end{aligned}\tag{1.12}$$

In view of [29, Prorosition 3.1], if $\bar{u}$ is a solution of (1.1), then it satisfies the Pohožaev identity $\mathcal{P}(u) = 0$. Let

$$\mathcal{M} := \bigl\{u \in H^1\bigl(\mathbb{R}^N\bigr)\setminus\{0\} : \mathcal{P}(u) = 0\bigr\}.\tag{1.13}$$

Our main result is as follows.

Theorem 1.1 *Assume that V satisfies $(V1)$–$(V3)$ and one of the following conditions:*
(i) $2 < p < \frac{4}{N} + 2$ *and* $\lambda > 0$;
(ii) $p = \frac{4}{N} + 2$ *and* $\lambda > \frac{(N+2)N^2}{2(N+1)A_0^{\frac{4}{N}}(SV_\infty)^{\frac{2}{\alpha}}}$;
(iii) $\frac{4}{N} + 2 < p < 2^*$ *and* $\lambda > \frac{25pN^4\Gamma(\frac{N}{2})\Gamma(\frac{pN}{2})}{256(N+1)!A_0^{p-2}\varepsilon^{2^*-2}(V_\infty S)^{\frac{2}{\alpha}}\Gamma(\frac{(p-1)N}{2})}$ *holds. Then problem (1.1) has a solution $\bar{u} \in H^1(\mathbb{R}^N)$ such that*

$$\mathcal{I}(\bar{u}) = \inf_{\mathcal{M}}\mathcal{I} = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}}\max_{t>0}\mathcal{I}(u_t) > 0,$$

where $u_t(x) := u(x/t)$.

In this paper, we use the following notations:

- $H^1(\mathbb{R}^N)$ denotes the usual Sobolev space equipped with the inner product and the norm

$$(u,v) = \int_{\mathbb{R}^N}(\nabla u\cdot\nabla v + uv)\,dx,\qquad \|u\| = (u,u)^{1/2},\quad \forall u,v\in H^1\bigl(\mathbb{R}^N\bigr).$$

- $L^s(\mathbb{R}^N)$ $(1 < s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^N}|u|^s\,dx)^{1/s}$.
- For any $u \in H^1(\mathbb{R}^N)$ and $r > 0, B_r(x) := \{y \in \mathbb{R} : |y - x| < r\}$.
- For any $u \in H^1(\mathbb{R}^N)\setminus\{0\}$, $u_t(x) := u(x/t)$ for $t > 0$.
- $C, C_1, C_2, \dots$ denote positive constants possibly different in different places.

2 Proof of the main result

Before proving the main result, we first give some key inequalities and lemmas. The following famous Hardy–Littlewood–Sobolev inequality [1, Theorem 4.3] is an origin of the variational approach to (1.1).

Lemma 2.1 *Let $\alpha \in (0,N)$ and $s \in (1, \frac{N}{\alpha})$. If $u \in L^s(\mathbb{R}^N)$, then $I_\alpha * u \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$, and*

$$\int_{\mathbb{R}^N} |I_\alpha * u|^{\frac{Ns}{N-\alpha s}}\,dx \le C\left(\int_{\mathbb{R}^N} |u|^s\,dx\right)^{\frac{N}{N-\alpha s}}, \tag{2.1}$$

where the constant $C > 0$ depends only on α, N, and s.

By a simple calculation, we have the following lemma.

Lemma 2.2 *The following two inequalities hold*:

$$g(t) := 2 - Nt^{N-2} + (N-2)t^N \ge 0, \quad \forall t \in [0,+\infty), \tag{2.2}$$

$$\beta(t) := \alpha - (N+\alpha)t^N + Nt^{N+\alpha} \ge \beta(1) = 0, \quad \forall t \in [0,+\infty). \tag{2.3}$$

Moreover, $(V3)$ implies that the following inequality holds:

$$\begin{aligned} &Nt^N\big[V(x) - V(tx)\big] + \big(t^N - 1\big)\nabla V(x)\cdot x \\ &\quad \ge -\frac{(N-2)^2\theta[2 - Nt^{N-2} + (N-2)t^N]}{4|x|^2}, \quad \forall t > 0, x \in \mathbb{R}^N \setminus \{0\}. \end{aligned} \tag{2.4}$$

Lemma 2.3 *Assume that $(V1)$ and $(V3)$ hold. Then*

$$\begin{aligned} \mathcal{I}(u) \ge{}& \mathcal{I}(u_t) + \frac{1-t^N}{N}\mathcal{P}(u) + \frac{(1-\theta)g(t)}{2N}\|\nabla u\|_2^2 \\ &+ \frac{\beta(t)}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{\frac{\alpha}{N}+1}\big)|u|^{\frac{\alpha}{N}+1}\,dx, \quad \forall u \in H^1\big(\mathbb{R}^N\big), t > 0. \end{aligned} \tag{2.5}$$

Proof According to Hardy's inequality, we obtain

$$\|\nabla u\|_2^2 \ge \frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\,dx, \quad \forall u \in H^1\big(\mathbb{R}^N\big). \tag{2.6}$$

Note that

$$\begin{aligned} \mathcal{I}(u_t) ={}& \frac{t^{N-2}}{2}\|\nabla u\|_2^2 + \frac{t^N}{2}\int_{\mathbb{R}^N} V(tx)u^2\,dx - \frac{t^N\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,dx \\ &- \frac{Nt^{N+\alpha}}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{\frac{\alpha}{N}+1}\big)|u|^{\frac{\alpha}{N}+1}\,dx. \end{aligned} \tag{2.7}$$

Thus, by (1.5), (1.11), (2.2), (2.3), (2.4), (2.6), and (V3), one has

$$\begin{aligned} \mathcal{I}(u) - \mathcal{I}(u_t) ={}& \frac{1-t^{N-2}}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}\big[V(x) - t^N V(tx)\big]u^2\,dx \\ &+ \frac{Nt^{N+\alpha} - N}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{\frac{\alpha}{N}+1}\big)|u|^{\frac{\alpha}{N}+1}\,dx + \frac{(t^N-1)\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,dx \\ ={}& \frac{1-t^N}{N}\left\{\frac{N-2}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}\big[NV(x) + \nabla V(x)\cdot x\big]u^2\,dx\right. \end{aligned}$$

$$
\begin{aligned}
&- \frac{N}{2}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{\frac{\alpha}{N}+1}\big)|u|^{\frac{\alpha}{N}+1}\,dx - \frac{N\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,dx\bigg\} + \frac{g(t)}{2N}\|\nabla u\|_2^2 \\
&+ \frac{1}{2}\int_{\mathbb{R}^N}\bigg\{t^N\big[V(x) - V(tx)\big] - \frac{1-t^N}{N}\nabla V(x)\cdot x\bigg\}u^2\,dx \\
&+ \frac{\beta(t)}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{\frac{\alpha}{N}+1}\big)|u|^{\frac{\alpha}{N}+1}\,dx \\
\geq{}& \frac{1-t^N}{N}\mathcal{P}(u) + \frac{(1-\theta)g(t)}{2N}\|\nabla u\|_2^2 \\
&+ \frac{\beta(t)}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{\frac{\alpha}{N}+1}\big)|u|^{\frac{\alpha}{N}+1}\,dx.
\end{aligned}
$$

□

From Lemma 2.3, we have the following corollary.

Corollary 2.4 *Assume that* $(V1)$ *and* $(V3)$ *hold. Then, for* $u \in \mathcal{M}$,

$$
\mathcal{I}(u) = \max_{t>0}\mathcal{I}(u_t). \tag{2.8}
$$

Based on the above results, we establish the following important property for $\mathcal{M}$.

Lemma 2.5 *For any* $u \in H^1(\mathbb{R}^N)\setminus\{0\}$, *there is unique* $t_u > 0$ *such that* $u_{t_u} \in \mathcal{M}$.

Proof Let $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ be fixed and define a function $\xi(t) := \mathcal{I}(u_t)$ on $(0,\infty)$. Clearly, by (1.11) and (2.7), we have

$$
\begin{aligned}
\xi'(t) = 0 \quad \Leftrightarrow \quad & \frac{N-2}{2}t^{N-2}\|\nabla u\|_2^2 + \frac{t^N}{2}\int_{\mathbb{R}^N}\big[NV(tx) + \nabla V(tx)\cdot(tx)\big]u^2\,dx \\
& - \frac{Nt^{N+\alpha}}{2}\int_{\mathbb{R}^N}\big(I_\alpha * u^{\frac{\alpha}{N}+1}\big)u^{\frac{\alpha}{N}+1}\,dx - \frac{N\lambda t^N}{p}\|u\|_p^p = 0 \\
\Leftrightarrow \quad & \mathcal{P}(u_t) = 0 \quad \Leftrightarrow \quad u_t \in \mathcal{M}.
\end{aligned} \tag{2.9}
$$

It is not hard to verify, using (V1), (V2), (1.2), and (2.7), that $\lim_{t\to 0}\xi(t) = 0$, $\xi(t) > 0$ for $t > 0$ small and $\xi(t) < 0$ for t large. Therefore $\max_{t\in(0,\infty)}\xi(t)$ is achieved at some $t_u > 0$ so that $\xi'(t_u) = 0$ and $u_{t_u} \in \mathcal{M}$.

Not unnaturally, we claim that t_u is unique for any $u \in H^1(\mathbb{R}^N)\setminus\{0\}$. As a matter of fact, for any given $u \in H^1(\mathbb{R}^N)\setminus\{0\}$, if there are two positive constants $t_1 \neq t_2$ such that $u_{t_1}, u_{t_2} \in \mathcal{M}$, then $\mathcal{P}(u_{t_1}) = \mathcal{P}(u_{t_2}) = 0$. Together with (2.3), (2.4), and (2.5), we have

$$
\begin{aligned}
\mathcal{I}(u_{t_1}) \geq{}& \mathcal{I}(u_{t_2}) + \frac{t_1^N - t_2^N}{Nt_1^N}\mathcal{P}(u_{t_1}) + \frac{(1-\theta)g(t_2/t_1)}{2N}\|\nabla u_{t_1}\|_2^2 \\
&+ \frac{\beta(t_2/t_1)}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * u_{t_1}^{\frac{\alpha}{N}+1}\big)u_{t_1}^{\frac{\alpha}{N}+1}\,dx \\
\geq{}& \mathcal{I}(u_{t_2}) + \frac{(1-\theta)[2t_1^N - Nt_1^2t_2^{N-2} + (N-2)t_2^N]}{2Nt_1^2}\|\nabla u\|_2^2.
\end{aligned} \tag{2.10}
$$

The same procedure may be easily adapted to obtain the following equation:

$$\mathcal{I}(u_{t_2}) \geq \mathcal{I}(u_{t_1}) + \frac{(1-\theta)[2t_2^N - Nt_2^2 t_1^{N-2} + (N-2)t_1^N]}{2Nt_2^2}\|\nabla u\|_2^2. \tag{2.11}$$

From (2.10) and (2.11), we have $u_{t_1} = u_{t_2}$, which shows that $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$. □

Lemma 2.6 ([19, Lemma 2.5]) *Assume that* $(V1)$–$(V3)$ *hold. Then there are two constants* $\gamma_1, \gamma_2 > 0$ *such that*

$$\gamma_1\|u\|^2 \leq (N-2)\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} \big[NV(x) + \nabla V(x) \cdot x\big] u^2\, dx \leq \gamma_2 \|u\|^2. \tag{2.12}$$

Proof The proof of Lemma 2.6 is routine, and we omit it. □

From Corollary 2.4 and Lemma 2.5, we have $\mathcal{M} \neq \emptyset$. Next, we apply the method introduced in [26] to prove the following lemma, which is key to verifying the minimax characterization.

Lemma 2.7 *Assume that* $(V1)$ *and* $(V2)$ *hold. Then*

$$\inf_{u\in\mathcal{M}} \mathcal{I}(u) := m = \inf_{u\in H^1(\mathbb{R})\setminus\{0\}} \max_{t>0} \mathcal{I}(u_t).$$

Lemma 2.8 *Assume that* $(V1)$ *and* $(V2)$ *hold. Then*
(*i*) *there exists* $\rho > 0$ *such that* $\|u\| > \rho$.
(*ii*) $m = \inf_{u\in\mathcal{M}} \mathcal{I}(u) > 0$.

Proof (i). Since $\mathcal{P}(u) = 0$ for all $u \in \mathcal{M}$, by (2.1), (1.11), Lemma 2.6, and Sobolev embedding inequality, one has

$$\begin{aligned}\frac{\gamma_1}{2}\|u\|^2 &\leq \frac{N-2}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}\big[NV(x) + \nabla V(x)\cdot x\big]u^2\,dx \\ &= \frac{N}{2}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{\frac{\alpha}{N}+1}\big)|u|^{\frac{\alpha}{N}+1}\,dx + \frac{N\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,dx \\ &\leq C_1\|u\|^{\frac{2(N+\alpha)}{N}} + C_2\|u\|^p.\end{aligned} \tag{2.13}$$

There are two cases to consider.
Case (1). When $\frac{2(N+\alpha)}{N} \geq p$, from (2.13), one has

$$\frac{\gamma_1}{2}\|u\|^2 \leq C_1\|u\|^{\frac{2(N+\alpha)}{N}} + C_2\|u\|^{\frac{2(N+\alpha)}{N}}, \tag{2.14}$$

which implies

$$\|u\| \geq \rho_0 := \min\left\{1, \left[\frac{\gamma_1}{2(C_1+C_2)}\right]^{\frac{N}{2\alpha}}\right\}. \tag{2.15}$$

Case (2). When $p > \frac{2(N+\alpha)}{N}$ and $\lambda > 0$, one has

$$\frac{\gamma_1}{2}\|u\|^2 \le C_1\|u\|^p + C_2\|u\|^p, \tag{2.16}$$

which implies

$$\|u\| \ge \rho_1 := \min\left\{1, \left[\frac{\gamma_1}{2(C_1+C_2)}\right]^{\frac{1}{p-2}}\right\}. \tag{2.17}$$

From (2.15) and (2.17), we know that (i) holds.

(ii). Let $\{u_n\} \subset \mathcal{M}$ be such that $\mathcal{I}(u_n) \to m$. There are two possible cases:

Case (i). $\inf_{n\in\mathbb{N}}\|\nabla u_n\|_2 := \sigma > 0$. From (1.5) and (1.11), one has

$$\begin{aligned}\mathcal{I}(u_n) - \frac{1}{N}\mathcal{P}(u_n) ={}& \frac{1}{N}\|\nabla u_n\|_2^2 - \frac{1}{2N}\int_{\mathbb{R}^N}\nabla V(x)\cdot x u_n^2\,dx \\ &+ \frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^{\frac{\alpha}{N}+1}\big)|u_n|^{\frac{\alpha}{N}+1}\,dx, \quad \forall u \in H^1(\mathbb{R}^N).\end{aligned} \tag{2.18}$$

From (V3), we have

$$\int_{\mathbb{R}^N}\nabla V(x)\cdot x u^2\,dx \le \frac{\theta(N-2)^2}{2}\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}\,dx \le 2\theta\|\nabla u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^N). \tag{2.19}$$

From (2.18) and (2.19), we obtain

$$\begin{aligned} m + o(1) = \mathcal{I}(u_n) &= \mathcal{I}(u_n) - \frac{1}{N}\mathcal{P}(u_n) \\ &\ge \frac{1}{N}\|\nabla u_n\|_2^2 - \frac{\theta}{N}\|\nabla u_n\|_2^2 + \frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^{\frac{\alpha}{N}+1}\big)|u_n|^{\frac{\alpha}{N}+1}\,dx \\ &\ge \frac{1-\theta}{N}\|\nabla u_n\|_2^2 = \frac{1-\theta}{N}\sigma^2. \end{aligned} \tag{2.20}$$

Case (ii). $\inf_{n\in\mathbb{N}}\|\nabla u_n\|_2 = 0$. In this case, by (2.15) and (2.17), passing to a subsequence, one has

$$\|\nabla u_n\|_2 \to 0, \qquad \|u_n\|_2 \ge \frac{1}{2}\max\{\rho_0, \rho_1\}. \tag{2.21}$$

By Lemma 2.1 and the Sobolev inequality, one has

$$\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^{\frac{\alpha}{N}+1}\big)|u_n|^{\frac{\alpha}{N}+1}\,dx \le C_3\|u_n\|_2^{\frac{2(N+\alpha)}{N}}. \tag{2.22}$$

By (V1), there exists $0 < r_0 < \left[\frac{S}{C_3 t_n^\alpha \omega_N^{2/N}\|u_n\|_2^{2\alpha/N} + \frac{4\lambda\omega_N^{2/N}}{p}\|u_n\|_2^{p-2}}\right]^{1/2}$ such that

$$V(x) \ge \frac{V_\infty}{2} > C_3 t_n^\alpha\|u_n\|_2^{2\alpha/N} + \frac{2\lambda}{p}\|u_n\|_2^{p-2} \tag{2.23}$$

for $|x| \ge r_0$. Then

$$\int_{|tx|\ge r_0} V(tx)u_n^2\,dx \ge \frac{V_\infty}{2}\int_{|tx|\ge r_0} u_n^2\,dx, \quad \forall t > 0, u_n \in H^1(\mathbb{R}^N). \tag{2.24}$$

By the Sobolev inequality and Hölder's inequality, we have

$$\begin{aligned} \int_{|tx|<r_0} u^2\,dx &\le \left(\frac{\omega_N r_0^N}{t^N}\right)^{(2^*-2)/2^*} \left(\int_{|tx|<r_0} u^{2^*}\,dx\right)^{2/2^*} \\ &\le \omega_N^{2/N} r_0^2 t^{-2} S^{-1}\|\nabla u\|_2^2, \quad \forall t>0, u\in H^1(\mathbb{R}^N). \end{aligned} \tag{2.25}$$

Let

$$\sigma = \min\left\{V_\infty, S r_0^{-2}\omega_N^{-\frac{2}{N}}\right\} \tag{2.26}$$

and

$$t_n = \left(\frac{\sigma - \frac{4\lambda}{p}\|u_n\|_2^{p-2}}{4C_3}\right)^{\frac{1}{\alpha}} \|u_n\|_2^{-\frac{2}{N}}. \tag{2.27}$$

Since (2.21) implies that $\{t_n\}$ is bounded, then it follows from (2.7), (2.8), (2.21)–(2.27), Corollary 2.4, and the Sobolev embedding inequality that

$$\begin{aligned} m =& \mathcal{I}(u_n) \\ \ge& \mathcal{I}\big((u_n)_{t_n}\big) \\ =& \frac{t_n^{N-2}}{2}\|\nabla u_n\|_2^2 + \frac{t_n^N}{2}\int_{\mathbb{R}^N} V(t_n x)u_n^2\,dx - \frac{N t_n^{N+\alpha}}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |u_n|^{\frac{\alpha}{N}+1}\big)|u_n|^{\frac{\alpha}{N}+1}\,dx \\ & - \frac{t_n^N\lambda}{p}\int_{\mathbb{R}^N}|u_n|^p\,dx \\ \ge& \frac{S t_n^N}{2r_0^2\omega_N^{2/N}}\int_{|t_n x|<r_0} u_n^2\,dx + \frac{t_n^N V_\infty}{4}\int_{|t_n x|\ge r_0} u_n^2\,dx - \frac{C_3 t_n^{N+\alpha}}{2}\|u_n\|_2^{\frac{2(N+\alpha)}{N}} - \frac{\lambda t_n^N}{p}\|u_n\|_2^p \\ \ge& \frac{\sigma}{4}\|u_n\|_2^2 - \frac{C_3 t_n^{N+\alpha}}{2}\|u_n\|_2^{\frac{2(N+\alpha)}{N}} - \frac{\lambda t_n^N}{p}\|u_n\|_2^p \\ =& \frac{t_n^N}{4}\|u_n\|_2^2\left(\sigma - 2C_3 t_n^\alpha\|u_n\|_2^{\frac{2\alpha}{N}} - \frac{4\lambda}{p}\|u_n\|_2^{p-2}\right) > 0. \end{aligned} \tag{2.28}$$

The two cases show that $m = \inf_{u\in\mathcal{M}}\mathcal{I}(u) > 0$. □

Inspired by Tang and Chen [25], we give an estimate on the energy level m, which is essential in ensuring compactness.

Lemma 2.9 $m < m_* := \frac{\alpha}{2(N+\alpha)}(V_\infty S)^{\frac{N}{\alpha}+1}$

Proof We set $U(x) = A_0(1+|x|^2)^{-\frac{N}{2}}$, where A_0 is defined by (1.4). By the calculation of integral, we get

$$\|U\|_p^p = \int_{\mathbb{R}^N}|U|^p\,dx = \omega_N A_0^p\int_0^{+\infty} r^{N-1}\big(1+r^2\big)^{-pN/2}\,dr = \frac{\omega_N A_0^p\Gamma(\frac{(p-1)N}{2})\Gamma(\frac{N}{2})}{\Gamma(\frac{pN}{2})}$$

and

$$\|\nabla U\|_2^2 = \int_{\mathbb{R}^N} |\nabla U|^2\,dx = \omega_N (NA_0)^2 \int_0^{+\infty} r^{N+1}\bigl(1+r^2\bigr)^{-(N+2)}\,dr = \frac{\omega_N (N^2A_0)^2[\Gamma(\frac{N}{2})]^2}{8(N+1)!}.$$

Let $t_* = (V_\infty S)^{\frac{1}{\alpha}}$. For any $\varepsilon > 0$, we define two functions $f(t)$ and $h_\varepsilon(t)$ as follows:

$$f(t) = \frac{V_\infty S}{2} t^N - \frac{N}{2(N+\alpha)} t^{N+\alpha} \tag{2.29}$$

and

$$h_\varepsilon(t) = \frac{\varepsilon^2 \|\nabla U\|_2^2}{2} t^{N-2} - \frac{\lambda \varepsilon^{\frac{Np}{2}-N}}{p} \|U\|_p^p t^N. \tag{2.30}$$

It is easy to know that $f(t) < f(t_*) = \frac{\alpha}{2(N+\alpha)}(V_\infty S)^{\frac{N}{\alpha}+1} := m_*$ for $t \in [0, t_*) \cup (t_*, \infty)$. We set $U_\varepsilon(x) = \varepsilon^{N/2} U(\varepsilon x)$. Then it follows from the definition of $\mathcal{S}$ that

$$\|U_\varepsilon\|_2^2 = S \quad \text{and} \quad \int_{\mathbb{R}^N} \bigl(I_\alpha * |u|^{\frac{\alpha}{N}+1}\bigr)|u|^{\frac{\alpha}{N}+1}\,dx = 1, \tag{2.31}$$

and

$$\|\nabla U_\varepsilon\|_2^2 = \varepsilon^2 \|\nabla U\|_2^2 \quad \text{and} \quad \|U_\varepsilon\|_p^p = \varepsilon^{\frac{pN}{2}-N} \|U\|_p^p. \tag{2.32}$$

From (2.7), (2.29), (2.30), (2.31), and (2.32), we obtain

$$\begin{aligned}
\mathcal{I}\bigl((U_\varepsilon)_t\bigr) &= \frac{t^{N-2}}{2}\|\nabla U_\varepsilon\|_2^2 + \frac{t^N}{2}\int_{\mathbb{R}^N} V(tx) U_\varepsilon^2\,dx \\
&\quad - \frac{Nt^{N+\alpha}}{2(N+\alpha)} \int_{\mathbb{R}^N} \bigl(I_\alpha * |U_\varepsilon|^{\frac{\alpha}{N}+1}\bigr)|U_\varepsilon|^{\frac{\alpha}{N}+1}\,dx \\
&\quad - \frac{t^N\lambda}{p}\int_{\mathbb{R}^N} |U_\varepsilon|^p\,dx \\
&\le \frac{V_\infty t^N}{2}\|U_\varepsilon\|_2^2 + \frac{t^{N-2}}{2}\|\nabla U_\varepsilon\|_2^2 - \frac{Nt^{N+\alpha}}{2(N+\alpha)}\int_{\mathbb{R}^N} \bigl(I_\alpha * |U_\varepsilon|^{\frac{\alpha}{N}+1}\bigr)|U_\varepsilon|^{\frac{\alpha}{N}+1}\,dx \\
&\quad - \frac{t^N\lambda}{p}\int_{\mathbb{R}^N} |U_\varepsilon|^p\,dx \\
&= \frac{V_\infty S}{2}t^N - \frac{N}{2(N+\alpha)}t^{N+\alpha} + \frac{\varepsilon^2\|\nabla U\|_2^2}{2}t^{N-2} - \frac{\lambda\varepsilon^{\frac{Np}{2}-N}}{p}\|U\|_p^p t^N \\
&= f(x) + h_\varepsilon(t).
\end{aligned} \tag{2.33}$$

There are three possible cases to distinguish.

Case 1. $2 < p < 2 + \frac{4}{N}$ and $\lambda > 0$. In this case, we choose $\varepsilon \in (0,1)$, then

$$h_\varepsilon(t) \le \frac{1}{2p}\varepsilon^2 t^{N-2}\bigl(p\|\nabla U\|_2^2 - 2\lambda \|U\|_p^p t^2\bigr). \tag{2.34}$$

Let

$$T_0 = \max\left\{\frac{6t_*}{5}, \left(\frac{p\|\nabla U\|_2^2}{2\lambda\|U\|_p^p}\right)^{\frac{1}{2}}\right\}. \tag{2.35}$$

We can choose $\varepsilon \in (0,1)$ such that

$$\lambda t_*^2 \geq \frac{25p\|\nabla U\|_2^2}{32\|U\|_p^p}, \tag{2.36}$$

and

$$\frac{1}{2}\varepsilon^2 T_0^{N-2}\|\nabla U\|_2^2 < m_* - f\left(\frac{6t_*}{5}\right), \qquad \frac{1}{2}\varepsilon^2 t_*^{N-2}\|\nabla U\|_2^2 < m_* - f\left(\frac{4t_*}{5}\right). \tag{2.37}$$

There are four possible subcases.

Subcase (i) $t \geq T_0$. Then it follows from (2.29), (2.33), (2.34), and (2.35) that

$$\begin{aligned}\max_{t\geq T_0}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{t\geq T_0}\big[f(t)+h_\varepsilon(t)\big]\\ &\leq f\left(\frac{6t_*}{5}\right) \leq f(t_*) = m_*.\end{aligned} \tag{2.38}$$

Subcase (ii) $\frac{6t_*}{5} \leq t \leq T_0$. Then it follows from (2.29), (2.33), (2.34), and (2.37) that

$$\begin{aligned}\max_{\frac{6t_*}{5}\leq t\leq T_0}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{\frac{6t_*}{5}\leq t\leq T_0}\big[f(t)+h_\varepsilon(t)\big]\\ &\leq f\left(\frac{6t_*}{5}\right) + \frac{1}{2}\varepsilon_2 T_0^{N-2}\|\nabla U\|_2^2 < m_*.\end{aligned} \tag{2.39}$$

Subcase (iii) $\frac{4t_*}{5} \leq t \leq \frac{6t_*}{5}$. Then it follows from (2.29), (2.33), and (2.35) that

$$\begin{aligned}\max_{\frac{4t_*}{5}\leq t\leq \frac{6t_*}{5}}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{\frac{4t_*}{5}\leq t\leq \frac{6t_*}{5}}\big[f(t)+h_\varepsilon(t)\big]\\ &\leq f(t_*) + h_\varepsilon\left(\frac{4t_*}{5}\right) \leq m_*.\end{aligned} \tag{2.40}$$

Subcase (iv) $0 \leq t \leq \frac{4t_*}{5}$. Then it follows from (2.29), (2.33), (2.34), and (2.37) that

$$\begin{aligned}\max_{0\leq t\leq \frac{4t_*}{5}}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{0\leq t\leq \frac{4t_*}{5}}\big[f(t)+h_\varepsilon(t)\big]\\ &\leq f\left(\frac{4t_*}{5}\right) + \frac{1}{2}\varepsilon_2 t_*^{N-2}\|\nabla U\|_2^2 < m_*.\end{aligned} \tag{2.41}$$

Case 2. $p = 2 + \frac{4}{N}$ and $\lambda > \frac{(N+2)N^2}{2(N+1)A_0^{\frac{4}{N}}(SV_\infty)^{\frac{2}{\alpha}}}$. In this case we choose $\varepsilon \in (0,1)$, then

$$h_\varepsilon(t) = \frac{1}{2}\varepsilon^2 t^{N-2}\left(\|\nabla U\|_2^2 - \frac{N}{N+2}\lambda\|U\|_{2+4/N}^{2+4/N}t^2\right). \tag{2.42}$$

Let

$$T_1 = \max\left\{t_* + \epsilon, \left(\frac{(N+2)\|\nabla U\|_2^2}{N\lambda\|U\|_{2+4/N}^{2+4/N}}\right)^{\frac{1}{2}}\right\}. \tag{2.43}$$

By assumption (ii) in Theorem 1.1, we can choose $\epsilon > 0$ such that

$$\lambda(t_* - \epsilon)^2 \geq \frac{(N+2)\|\nabla U\|_2^2}{N\|U\|_{2+4/N}^{2+4/N}}. \tag{2.44}$$

We choose $\varepsilon > 0$ such that

$$\frac{1}{2}\varepsilon^2 T_1^{N-2}\|\nabla U\|_2^2 < m_* - f(t_* + \epsilon), \qquad \frac{1}{2}\varepsilon^2 t_*^{N-2}\|\nabla U\|_2^2 < m_* - f(t_* - \epsilon). \tag{2.45}$$

There are also four possible subcases.

Subcase (i) $t \geq T_1$. Then it follows from (2.29), (2.33), (2.42), and (2.43) that

$$\begin{aligned}\max_{t\geq T_1}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{t\geq T_1}\big[f(t) + h_\varepsilon(t)\big] \\ &\leq f(t_* + \epsilon) \leq f(t_*) = m_*.\end{aligned} \tag{2.46}$$

Subcase (ii) $t_* + \epsilon \leq t \leq T_1$. Then it follows from (2.29), (2.33), (2.42), and (2.45) that

$$\begin{aligned}\max_{t_*+\epsilon\leq t\leq T_1}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{t_*+\epsilon\leq t\leq T_1}\big[f(t) + h_\varepsilon(t)\big] \\ &\leq f(t_* + \epsilon) + \frac{1}{2}\varepsilon_2 T_1^{N-2}\|\nabla U\|_2^2 < m_*.\end{aligned} \tag{2.47}$$

Subcase (iii) $t_* - \epsilon \leq t \leq t_* + \epsilon$. Then it follows from (2.29), (2.33), (2.42), and (2.44) that

$$\begin{aligned}\max_{t_*-\epsilon\leq t\leq t_*+\epsilon}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{t_*-\epsilon\leq t\leq t_*+\epsilon}\big[f(t) + h_\varepsilon(t)\big] \\ &\leq f(t_*) + h_\varepsilon(t_* - \epsilon) \leq m_*.\end{aligned} \tag{2.48}$$

Subcase (iv) $0 \leq t \leq t_* - \epsilon$. Then it follows from (2.29), (2.33), (2.42), and (2.45) that

$$\begin{aligned}\max_{0\leq t\leq t_*-\epsilon}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{0\leq t\leq t_*-\epsilon}\big[f(t) + h_\varepsilon(t)\big] \\ &\leq f(t_* - \epsilon) + \frac{1}{2}\varepsilon^2 t_*^{N-2}\|\nabla U\|_2^2 < m_*.\end{aligned} \tag{2.49}$$

Case 3. $2 + \frac{4}{N} < p < 2^*$ and $\lambda > \frac{25pN^4\Gamma(\frac{N}{2})\Gamma(\frac{pN}{2})}{256(N+1)!A_0^{p-2}\varepsilon^{2^*-2}(V_\infty S)^{\frac{2}{\alpha}}\Gamma(\frac{(p-1)N}{2})}$. In this case, we also choose $\varepsilon \in (0,1]$, then

$$h_\varepsilon(t) \leq \frac{\varepsilon^2\|\nabla U\|_2^2}{2}t^{N-2} - \frac{\lambda\varepsilon^{2^*}}{p}\|U\|_p^p t^N, \tag{2.50}$$

$$T_2 = \max\left\{\frac{6t_*}{5}, \left(\frac{p\|\nabla U\|_2^2}{2\lambda\varepsilon^{2^*-2}\|U\|_p^p}\right)^{1/2}\right\}, \tag{2.51}$$

and

$$\lambda > \frac{p\|\nabla U\|_2^2}{2\varepsilon^{\frac{(p-2)N}{2}-2}t_0^2}. \tag{2.52}$$

Now we can choose $\varepsilon \in (0,1]$ and $\lambda > 0$ such that

$$\lambda t_*^2 \geq \frac{25p\|\nabla U\|_2^2}{32\varepsilon^{2^*-2}\|U\|_p^p} \tag{2.53}$$

and

$$\frac{1}{2}\varepsilon^2 T_2^{N-2}\|\nabla U\|_2^2 < m_* - f\left(\frac{6t_*}{5}\right), \qquad \frac{1}{2}\varepsilon^2 t_*^{N-2}\|\nabla U\|_2^2 < m_* - f\left(\frac{4t_*}{5}\right). \tag{2.54}$$

There are four possible subcases.

Subcase (i) $t \geq T_2$. Then it follows from (2.29), (2.33), (2.51), and (2.50) that

$$\begin{aligned} \max_{t\geq T_2}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{t\geq T_2}\big[f(t)+h_\varepsilon(t)\big] \\ &\leq f\left(\frac{6t_*}{5}\right) \leq f(t_*) = m_*. \end{aligned} \tag{2.55}$$

Subcase (ii) $\frac{6t_*}{5} \leq t \leq T_2$. Then it follows from (2.29), (2.33), (2.52), (2.50), and (2.54) that

$$\begin{aligned} \max_{\frac{6t_*}{5}\leq t\leq T_2}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{\frac{6t_*}{5}\leq t\leq T_2}\big[f(t)+h_\varepsilon(t)\big] \\ &\leq f\left(\frac{6t_*}{5}\right) + \frac{1}{2}\varepsilon^2 T_2^{N-2}\|\nabla U\|_2^2 < m_*. \end{aligned} \tag{2.56}$$

Subcase (iii) $\frac{4t_*}{5} \leq t \leq \frac{6t_*}{5}$. Then it follows from (2.29), (2.50), and (2.53) that

$$\begin{aligned} \max_{\frac{4t_*}{5}\leq t\leq \frac{6t_*}{5}}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{\frac{4t_*}{5}\leq t\leq \frac{6t_*}{5}}\big[f(t)+h_\varepsilon(t)\big] \\ &\leq f(t_*) + h_\varepsilon\left(\frac{4t_*}{5}\right) \leq m_*. \end{aligned} \tag{2.57}$$

Subcase (iv) $0 \leq t \leq \frac{4t_*}{5}$. Then it follows from (2.29), (2.33), (2.50), and (2.54) that

$$\begin{aligned} \max_{0\leq t\leq \frac{4t_*}{5}}\mathcal{I}\big((U_\varepsilon)_t\big) &\leq \max_{0\leq t\leq \frac{4t_*}{5}}\big[f(t)+h_\varepsilon(t)\big] \\ &\leq f\left(\frac{4t_*}{5}\right) + \frac{1}{2}\varepsilon^2 t_*^{N-2}\|\nabla U\|_2^2 < m_*. \end{aligned} \tag{2.58}$$

The above three cases show that

$$m \leq \max_{t>0}\mathcal{I}\big((U_\varepsilon)_t\big) < m_*. \tag{2.59}$$

Lemma 2.10 *Assume that* $(V1)$–$(V3)$ *hold. Then* m *is achieved.*

Proof In view of Lemmas 2.5 and 2.8, we have $\mathcal{M} \neq \emptyset$ and $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\mathcal{I}(u_n) \to m$. Since $\mathcal{P}(u_n) = 0$, then it follows from (2.5), (2.18), and (2.20) that

$$\begin{aligned} m + o(1) = \mathcal{I}(u_n) &\geq \mathcal{I}(u_n) - \frac{1}{N}\mathcal{P}(u_n) \\ &\geq \frac{1-\theta}{N}\|\nabla u_n\|_2^2. \end{aligned} \tag{2.60}$$

This is to show that $\{\|\nabla u_n\|_2\}$ is bounded. Next, we prove that $\{\|u_n\|_2\}$ is also bounded. Arguing indirectly, assume that $\|u_n\|_2 \to \infty$, without loss of generality, we can assume that $\|u_n\|_2 \geq 1$. From (2.28), we have

$$\begin{aligned} m = \mathcal{I}(u_n) &\geq \mathcal{I}\big((u_n)_{\bar{t}_n}\big) \\ &= \frac{\bar{t}_n^{N-2}}{2}\|\nabla u_n\|_2^2 + \frac{\bar{t}_n^N}{2}\int_{\mathbb{R}^N} V(\bar{t}_n x)u_n^2\,dx - \frac{\bar{t}_n^N\lambda}{p}\int_{\mathbb{R}^N}|u_n|^p\,dx \\ &\quad - \frac{N\bar{t}_n^{N+\alpha}}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(\mathcal{I}_\alpha * |u_n|^{\frac{\alpha}{N}+1}\big)|u_n|^{\frac{\alpha}{N}+1}\,dx \\ &\geq \frac{S\bar{t}_n^N}{2r_0\omega_N^{2/N}}\int_{|\bar{t}_n x|<r_0} u_n^2\,dx + \frac{V_\infty \bar{t}_n^N}{4}\int_{|\bar{t}_n x|\geq r_0} u_n^2\,dx - \frac{C_3\bar{t}_n^{N+\alpha}}{2}\|u_n\|_2^{\frac{2(N+\alpha)}{N}} - \frac{\lambda\bar{t}_n^N}{p}\|u_n\|_2^p \\ &\geq \frac{\sigma\bar{t}_n^N}{4}\|u_n\|_2^2 - \frac{C_3\bar{t}^{N+\alpha}}{2}\|u_n\|_2^{\frac{2(N+\alpha)}{N}} - \frac{\lambda\bar{t}_n^N}{p}\|u_n\|_2^p \\ &= \frac{\bar{t}_n^N}{4}\|u_n\|_2^2\left(\sigma - 2C_3\bar{t}_n^\alpha\|u_n\|_2^{\frac{2\alpha}{N}} - \frac{4\lambda}{p}\|u_n\|_2^{p-2}\right), \quad \forall \bar{t}_n \geq 0. \end{aligned} \tag{2.61}$$

If $\frac{2\alpha}{N} \geq p-2$, we choose

$$\lambda = \frac{pC_3\bar{t}_n^\alpha}{4} > 0 \quad \text{and} \quad C_3 = \frac{\sigma}{4}\left(\frac{\sigma}{24m}\right)^{\frac{\alpha}{N}}. \tag{2.62}$$

Let

$$\bar{t}_n = \left(\frac{24m}{\sigma}\right)^{\frac{1}{N}}\|u_n\|_2^{-\frac{2}{N}}. \tag{2.63}$$

From (2.61), (2.62), and (2.63), we have

$$\begin{aligned} m &\geq \frac{\bar{t}_n^N}{4}\|u_n\|_2^2\left(\sigma - 2C_3\bar{t}_n^\alpha\|u_n\|_2^{\frac{2\alpha}{N}} - \frac{4\lambda}{p}\|u_n\|_2^{p-2}\right) \\ &\geq \frac{\bar{t}_n^N}{4}\|u_n\|_2^2\left(\sigma - 2C_3\bar{t}_n^\alpha\|u_n\|_2^{\frac{2\alpha}{N}} - \frac{4\lambda}{p}\|u_n\|_2^{\frac{2\alpha}{N}}\right) \\ &= \frac{3}{2}m. \end{aligned} \tag{2.64}$$

This is a contradiction.

When $\frac{2\alpha}{N} < p-2$, we choose

$$\lambda = \frac{pC_3\bar{t}_n^\alpha}{4} > 0 \quad \text{and} \quad C_3 = \frac{\sigma}{4}\left(\frac{\sigma}{24m\|u_n\|_2^{[N(p-2)-2\alpha]/\alpha}}\right)^{\alpha/N}. \tag{2.65}$$

Let

$$\bar{t}_n = \left(\frac{24m\|u_n\|_2^{[N(p-2)-2\alpha]/\alpha}}{\sigma}\right)^{1/N} \|u_n\|^{-(p-2)/\alpha}. \tag{2.66}$$

Then from (2.61), (2.65), and (2.66), we get

$$\begin{aligned} m &\geq \frac{\bar{t}_n^N}{4}\|u_n\|_2^2\left(\sigma - 2C_3\bar{t}_n^\alpha\|u_n\|_2^{\frac{2\alpha}{N}} - \frac{4\lambda}{p}\|u_n\|_2^{p-2}\right) \\ &\geq \frac{\bar{t}_n^N}{4}\|u_n\|_2^2\left(\sigma - 2C_3\bar{t}_n^\alpha\|u_n\|_2^{p-2} - \frac{4\lambda}{p}\|u_n\|_2^{p-2}\right) \\ &= \frac{3}{2}m, \end{aligned} \tag{2.67}$$

a contradiction. Hence, $\{\|u_n\|_2\}$ is also bounded. Therefore, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$. Then $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s \leq 2^*$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^N$. We obtain two possible cases.

Case (i) $\bar{u} = 0$, i.e., $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. Then $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s \leq 2^*$ and $u_n \to 0$ a.e. in $\mathbb{R}^N$. Let $t = 0$ in (2.4), one has

$$NV(x) + \nabla V(x)\cdot x \leq NV_\infty + \frac{(N-2)^2\theta}{2|x|^2}. \tag{2.68}$$

Let $t \to \infty$ in (2.4), one has

$$-\frac{(N-2)^3\theta}{4|x|^2} + NV_\infty \leq NV(x) + \nabla V(x)\cdot x. \tag{2.69}$$

By (V2), (2.68), and (2.69), it is easy to show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[V_\infty - V(x)\right]u_n^2\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}\nabla V(x)\cdot x u_n^2\,dx = 0. \tag{2.70}$$

From (1.5), (1.8), (1.11), and (2.70), one can get

$$\mathcal{I}(u_n) \to m, \qquad \mathcal{P}^\infty(u_n) \to 0. \tag{2.71}$$

From Lemma 2.8(i), (1.12), and (2.71), one has

$$\begin{aligned} \min\{N-2, NV_\infty\}\rho^2 &\leq \min\{N-2, NV_\infty\}\|u_n\|^2 \\ &\leq (N-2)\|\nabla u_n\|_2^2 + NV_\infty\|u_n\|_2^2 \\ &= N\int_{\mathbb{R}^N}\left(I_\alpha * |u_n|^{\frac{\alpha}{N}+1}\right)|u_n|^{\frac{\alpha}{N}+1}\,dx + \frac{2N\lambda}{p}\|u_n\|_p^p. \end{aligned} \tag{2.72}$$

Using (2.72) and the Lions concentration compactness principle [14, Lemma 1.21], we can prove that there exist $\sigma > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)}|u_n|^2\,dx > \sigma$. Let $\hat{u}_n(x) = u_n(x+y_n)$. Then we have $\|\hat{u}_n\| = \|u_n\|$ and

$$\mathcal{P}^\infty(\hat{u}_n) = o(1), \qquad \mathcal{I}(\hat{u}_n) \to m, \qquad \int_{B_1(0)}|\hat{u}_n|^2\,dx > \sigma. \tag{2.73}$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u} & \text{in } H^1(\mathbb{R}^N); \\ \hat{u}_n \to \hat{u} & \text{in } L^s_{\text{loc}}(\mathbb{R}^N), \forall s \in [1, 2^*); \\ \hat{u}_n \to \hat{u} & \text{a.e. on } \mathbb{R}^N. \end{cases} \tag{2.74}$$

Let $w_n = \hat{u}_n - \hat{u}$. Then (2.74) and the Brezis–Lieb type lemma (see [11, Lemmas 2.4]), [30, Lemmas 2.10] lead to

$$\mathcal{I}^\infty(\hat{u}_n) = \mathcal{I}^\infty(\hat{u}) + \mathcal{I}^\infty(w_n) + o(1) \tag{2.75}$$

and

$$\mathcal{P}^\infty(\hat{u}_n) = \mathcal{P}^\infty(\hat{u}) + \mathcal{P}^\infty(w_n) + o(1). \tag{2.76}$$

From (1.12), (1.8), and Lemma 2.3, one has

$$\mathcal{I}^\infty(u) \geq \mathcal{I}^\infty(u_t). \tag{2.77}$$

Moreover,

$$\frac{1}{N}\|\nabla w_n\|_2^2 = m - \frac{1}{N}\|\nabla \hat{u}\|_2^2 + \frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}\left(I_\alpha * |\hat{u}|^{\frac{\alpha}{N}+1}\right)|\hat{u}|^{\frac{\alpha}{N}+1}\,dx + o(1), \tag{2.78}$$

$$\mathcal{P}^\infty(w_n) = -\mathcal{P}^\infty(\hat{u}) + o(1). \tag{2.79}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then going to this subsequence, we have

$$\mathcal{I}^\infty(\hat{u}) = m, \qquad \mathcal{P}^\infty(\hat{u}) = 0. \tag{2.80}$$

Next we assume that $w_n \neq 0$. We claim that $\mathcal{P}^\infty(\hat{u}) \leq 0$. Otherwise, $\mathcal{P}^\infty(\hat{u}) > 0$ for large n. In view of Corollary 2.4 and Lemma 2.5, there exists $t_n > 0$ such that $(w_n)_{t_n} \in \mathcal{M}^\infty$. From (1.5), (1.12), (2.77), (2.78), and (2.80), we obtain

$$\begin{aligned} m - \frac{1}{N}\|\nabla \omega_n\|_2^2 + o(1) &= \frac{1}{N}\|\nabla \hat{u}\|_2^2 + \frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}\left(I_\alpha * |\hat{u}|^{\frac{\alpha}{N}+1}\right)|\hat{u}|^{\frac{\alpha}{N}+1}\,dx \\ &= \mathcal{I}^\infty(w_n) - \frac{1}{N}\mathcal{P}^\infty(w_n) \\ &\geq \mathcal{I}^\infty\left((w_n)_{t_n}\right) - \frac{t_n^N}{N}\mathcal{P}^\infty(w_n) \\ &\geq m^\infty - \frac{t_n^N}{N}\mathcal{P}^\infty(w_n) \geq m^\infty, \end{aligned} \tag{2.81}$$

which implies $\mathcal{P}^\infty(\hat{u}) \leq 0$ due to $\|\nabla \hat{u}\|_2 > 0$. Since $\hat{u} \neq 0$ and $\mathcal{P}^\infty(\hat{u}) \leq 0$, in view of Lemma 2.5, there exists $\hat{t} > 0$ such that $\hat{u}_{\hat{t}} \in \mathcal{M}^\infty$. From (1.8), (1.12), (2.77), (2.78), (2.80) and the weak semicontinuity of norm, one has

$$m = \lim_{n\to\infty}\left[\mathcal{I}^\infty(\hat{u}_n) - \frac{1}{N}\mathcal{P}^\infty(\hat{u}_n)\right]$$

$$\begin{aligned}
&= \frac{1}{N}\lim_{n\to\infty}\|\nabla\hat{u}_n\|_2^2 + \frac{\alpha}{2(N+\alpha)}\lim_{n\to\infty}\int_{\mathbb{R}^N}\bigl(I_\alpha * |\hat{u}_n|^{\frac{\alpha}{N}+1}\bigr)|\hat{u}_n|^{\frac{\alpha}{N}+1}\,dx \\
&\geq \mathcal{I}^\infty(\hat{u}) - \frac{1}{N}\mathcal{P}^\infty(\hat{u}) \\
&\geq \mathcal{I}^\infty(\hat{u}_{\hat{t}}) - \frac{\hat{t}^N}{N}\mathcal{P}^\infty(\hat{u}) \\
&\geq m^\infty - \frac{\hat{t}^N}{N}\mathcal{P}^\infty(\hat{u}) \geq m,
\end{aligned}\tag{2.82}$$

which implies that (2.80) also holds. In view of Lemma 2.5, there exists $\hat{t} > 0$ such that $\hat{u}_{\hat{t}} \in \mathcal{M}$; moreover, it follows from (V2), (1.5), (1.8), (2.81), and (2.82) that

$$m \leq \mathcal{I}(\hat{u}_{\hat{t}}) \leq \mathcal{I}^\infty(\hat{u}_{\hat{t}}) \leq \mathcal{I}^\infty(\hat{u}) = m. \tag{2.83}$$

This shows that m is achieved at $\hat{u}_{\hat{t}} \in \mathcal{M}$.

Case (ii). $\bar{u} \neq 0$. Let $v_n = u_n - \bar{u}$. If $u_n \rightharpoonup \bar{u}$, similar to [17] and [31], we have the following two equalities:

$$\mathcal{I}(u_n) = \mathcal{I}(\bar{u}) + \mathcal{I}(v_n) + o(1) \tag{2.84}$$

and

$$\mathcal{P}(u_n) = \mathcal{P}(\bar{u}) + \mathcal{P}(v_n) + o(1). \tag{2.85}$$

Set

$$\begin{aligned}
\Psi(u) &= \frac{1}{N}\|\nabla u\|_2^2 - \frac{1}{2N}\int_{\mathbb{R}^N}\bigl(\nabla V(x)\cdot x\bigr)u^2\,dx \\
&\quad + \frac{\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}\bigl(I_\alpha * u^{\frac{\alpha}{N}+1}\bigr)u^{\frac{\alpha}{N}+1}\,dx.
\end{aligned}\tag{2.86}$$

Then it follows from (1.2), (2.4) with $t = 0$, (2.6) and (2.86) that

$$\Psi(u) \geq \frac{1-\theta}{N}\|\nabla u\|_2^2, \quad \forall u \in H^1\bigl(\mathbb{R}^N\bigr). \tag{2.87}$$

Since $\mathcal{I}(u_n) \to m$ and $\mathcal{P}(u_n) = 0$, then it follows from (1.5), (1.11), (2.85), (2.86), and (2.87) that

$$\Psi(v_n) = m - \Psi(\bar{u}) + o(1) \tag{2.88}$$

and

$$\mathcal{P}(v_n) = -\mathcal{P}(\bar{u}) + o(1). \tag{2.89}$$

If there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} = 0$, then going to this subsequence, we have

$$\mathcal{I}(\bar{u}) = m, \qquad \mathcal{P}(\bar{u}) = 0, \tag{2.90}$$

which implies that the conclusion of Lemma 2.10 holds. Next, we assume that $v_n \not\equiv 0$. We claim that $\mathcal{P}(\bar{u}) \leq 0$. Otherwise, $\mathcal{P}(\bar{u}) > 0$, then (2.88) implies $\mathcal{P}(v_n) < 0$ for large n. In view of (2.8), there exists $t_n > 0$ such that $(v_n)_{t_n} \in \mathcal{M}$ for large n. From (1.5), (1.11), (2.5), (2.88), and (2.89), we obtain

$$\begin{aligned} m - \Psi(\bar{u}) + o(1) = \Psi(v_n) & \\ &= \mathcal{I}(v_n) - \frac{1}{N}\mathcal{P}(v_n) \\ &\geq \mathcal{I}\big((v_n)_{t_n}\big) - \frac{t_n^N}{N}\mathcal{P}(v_n) \\ &\geq m - \frac{t_n^N}{N}\mathcal{P}(v_n) \geq m, \end{aligned} \tag{2.91}$$

which implies $\mathcal{P}(\bar{u}) \leq 0$ due to $\Psi(\bar{u}) > 0$. Since $\bar{u} \not\equiv 0$ and $\mathcal{P}(\bar{u}) \leq 0$, in view of (2.8) and (2.77), there exists $\bar{t} > 0$ such that $\bar{u}_{\bar{t}} \in \mathcal{M}$. From (1.5), (1.11), (2.5), (2.87), (2.88) and the weak semicontinuity of norm, we obtain

$$\begin{aligned} m &= \lim_{n\to\infty}\left[\mathcal{I}(u_n) - \frac{1}{N}\mathcal{P}(u_n)\right] = \lim_{n\to\infty}\Psi(u_n) \geq \Psi(\bar{u}) \\ &\geq \mathcal{I}(\bar{u}) - \frac{1}{N}\mathcal{P}(\bar{u}) \geq \mathcal{I}(\bar{u}_{\bar{t}}) - \frac{\bar{t}^N}{N}\mathcal{P}(\bar{u}) \\ &\geq m - \frac{\bar{t}^N}{N}\mathcal{P}^\infty(\bar{u}) \geq m, \end{aligned} \tag{2.92}$$

which implies that (2.90) also holds. □

Lemma 2.11 *Assume that* $(V1)$–$(V3)$ *hold. If* $\bar{u} \in \mathcal{M}$ *and* $\mathcal{I}(\bar{u}) = m$, *then* $\bar{u}$ *is a critical point of* $\mathcal{I}$.

Proof Similar to the proof of [32, Lemma 2.12], we can clearly conclude the desired conclusion by using

$$\begin{aligned} \mathcal{I}(\bar{u}_t) &\leq \mathcal{I}(\bar{u}) - \frac{(1-\theta)g(t)}{2N}\|\nabla\bar{u}\|_2^2 - \frac{\beta(t)}{2(N+\alpha)}\int_{\mathbb{R}^N}\big(I_\alpha * |\bar{u}|^{\frac{\alpha}{N}+1}\big)|\bar{u}|^{\frac{\alpha}{N}+1}\,dx \\ &\leq \mathcal{I}(\bar{u}) - \frac{(1-\theta)g(t)}{2N}\|\nabla\bar{u}\|_2^2 < m, \quad \forall t > 0, \end{aligned} \tag{2.93}$$

and

$$\varepsilon_1 := \min\left\{\frac{(1-\theta)g(1-\epsilon_1)}{5(N+\alpha)}\|\nabla\bar{u}\|_2^2, \frac{(1-\theta)g(1+\epsilon_1)}{5(N+\alpha)}\|\nabla\bar{u}\|_2^2, 1, \frac{\rho\delta}{8}\right\} \tag{2.94}$$

instead of [26, (2.55) and ε], respectively. □

Proof of Theorem 1.1 In view of Lemma 2.7, Lemma 2.8, and Lemma 2.11, there exists $\bar{u} \in \mathcal{M}$ such that

$$\mathcal{I}(\bar{u}) = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}}\max_{t>0}\mathcal{I}(u_t) = m > 0, \quad \mathcal{I}'(\bar{u}) = 0. \tag{2.95}$$

This shows that $\bar{u}$ is a ground state solution of (1.1). □

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The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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