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Existence of groundstates for Choquard type equations with Hardy–Littlewood–Sobolev critical exponent

Xiaowei Li[1] and Feizhi Wang[1*]

*Correspondence:
wangfz@ytu.edu.cn
[1]School of Mathematics and Information Sciences, Yantai University, Yantai 264005, Shandong, P.R. China

Abstract

In this paper, we consider a class of Choquard equations with Hardy–Littlewood–Sobolev lower or upper critical exponent in the whole space $\mathbb{R}^N$. We combine an argument of L. Jeanjean and H. Tanaka (see (Proc. Am. Math. Soc. 131:2399–2408, 2003) with a concentration–compactness argument, and then we obtain the existence of ground state solutions, which extends and complements the earlier results.

Keywords: Choquard equation; Nonlocal critical growth; Pohožaev–Palais–Smale sequence; Hardy–Littlewood–Sobolev inequality

1 Introduction

In this paper, we consider the following nonlinear Choquard problem:

$$-\Delta u + u = \big(I_\alpha * F(u)\big)f(u), \quad u \in H^1(\mathbb{R}^N), \tag{P}$$

where $N \geq 3$, $0 < \alpha < N$, I_α is a Riesz potential

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha |x|^{N-\alpha}} := \frac{\hat{C}}{|x|^{N-\alpha}}$$

with $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}\,dx$, $s > 0$, $F \in C^1(\mathbb{R},\mathbb{R})$, and $f := F'$. Problem (P) can be studied by the variational method. It is the Euler–Lagrange equation of the functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N}\big[|\nabla u|^2 + u^2\big]\,dx - \frac{1}{2}\int_{\mathbb{R}^N}\big(I_\alpha * F(u)\big)F(u)\,dx.$$

As we know, a large number of works have been devoted to the problem like (P). We refer the readers to [2, 4, 5, 7–10, 12–15] and the references therein.

Especially, in [8], under the following conditions:

(MS_1) $f \in C(\mathbb{R},\mathbb{R})$ and there exists $C > 0$ such that, for every $s \in \mathbb{R}$,

$$\big|sf(s)\big| \leq C\big(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}\big);$$

(MS_2)

$$\lim_{s\to 0}\frac{F(s)}{|s|^{\frac{N+\alpha}{N}}}=0 \quad\text{and}\quad \lim_{s\to\infty}\frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}}=0;$$

(MS_3) There exists $s_0 \in \mathbb{R}\setminus\{0\}$ such that $F(s_0)\neq 0$,

Moroz and Schaftingen proved the existence of a ground state solution. They employed a method introduced by L. Jeanjean, where a key step is to construct Palais–Smale sequences that satisfy asymptotically the Pohozǎev identity [3]. Note that assumption (MS_2) is called subcritical. The constant $\frac{N+\alpha}{N}$ is termed the lower-critical exponent and $\frac{N+\alpha}{N-2}$ is termed the upper-critical exponent in the sense of Hardy–Littlewood–Sobolev inequality. In [14], Seok considered that problem (P) with $F(u)$ is doubly critical, i.e.,

$$F(u)=\frac{1}{p}|u|^p+\frac{1}{q}|u|^q,$$

where $p=\frac{N+\alpha}{N}$ and $q=\frac{N+\alpha}{N-2}$. In this situation,

$$\lim_{s\to 0}\frac{F(s)}{|s|^{\frac{N+\alpha}{N}}}=\frac{1}{p}\neq 0 \quad\text{and}\quad \lim_{s\to\infty}\frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}}=\frac{1}{q}\neq 0.$$

He showed the existence of nontrivial solutions of the nonlinear Choquard equation if $\alpha+4<N$.

In the following we give our main result.

Theorem 1.1 *Suppose that* $N\geq 3$, $p,q\in[\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}]$ *and*

$$F(u)=\frac{1}{p}|u|^p+\frac{1}{q}|u|^q.$$

Then problem (P) *has at least a ground state solution* $u\in H^1(\mathbb{R}^N)$ *provided one of the following conditions holds*:

(1) $q=\frac{N+\alpha}{N-2}$, $N\geq 4$, *and* $p\in(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2})$ *or* $N=3$ *and* $p\in(1+\frac{\alpha}{N-2},\frac{N+\alpha}{N-2})$;

(2) $p=\frac{N+\alpha}{N}$, $N>4+\alpha$, *and* $q\in(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}]$ *or* $N<4+\alpha$ *and* $q\in(\frac{N+\alpha}{N},\frac{N+\alpha+4}{N})$.

Remark 1.1 By conditions (1) and (2), it is easy to see that

$$\lim_{s\to 0}\frac{F(s)}{|s|^{\frac{N+\alpha}{N}}}=0 \quad\text{and}\quad \lim_{s\to\infty}\frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}}=\frac{1}{q}\neq 0$$

and

$$\lim_{s\to 0}\frac{F(s)}{|s|^{\frac{N+\alpha}{N}}}=\frac{1}{p}\neq 0 \quad\text{and}\quad \lim_{s\to\infty}\frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}}=0 \text{ or } \frac{1}{q},$$

respectively.

We denote the strong and the weak convergence in $H^1(\mathbb{R}^N)$ by $\to$ and $\rightharpoonup$, respectively. Set $\|u\|:=[\int_{\mathbb{R}^N}(|\nabla u|^2+u^2)\,dx]^{1/2}$ and $|u|_q:=[\int_{\mathbb{R}^N}|u|^q\,dx]^{1/q}$ for $1<q<\infty$. As for the

Choquard equation, the Hardy–Littlewood–Sobolev inequality (see [6] and [7]) implies that the nonlocal term is well defined for $u \in H^1(\mathbb{R}^N)$ and Φ is continuously differentiable on $H^1(\mathbb{R}^N)$. Clearly, $u = 0$ is a trivial solution of (P). The solutions of (P) must verify the Pohozǎev identity, as was proved in [8, Corollary 3.5]. In our case, the Pohozǎev identity reads as follows:

$$\mathcal{P}(u) := \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} u^2\,dx - \frac{N+\alpha}{2}\int_{\mathbb{R}^N}\big(I_\alpha * F(u)\big)F(u)\,dx = 0. \tag{1}$$

We call any weak solution $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ of (P) a groundstate of (P) if

$$\Phi(u) = c_0 := \inf\big\{\Phi(v) : v \in H^1\big(\mathbb{R}^N\big) \setminus \{0\} \text{ is a solution of (P)}\big\}.$$

Because problem (P) contains nonlocal critical nonlinearities in $\mathbb{R}^N$, there are more difficulties to overcome. One difficulty is the embedding of $H^1(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ which is not compact, where $2 \le q \le 2^*$. As a consequence, the corresponding functional of (P) does not satisfy the Palais–Smale condition; we overcome the lack of compactness by studying the problem in $H_r^1(\mathbb{R}^N)$:

$$H_r^1\big(\mathbb{R}^N\big) = \big\{u \in H^1\big(\mathbb{R}^N\big) : u(x) = u\big(|x|\big)\big\},$$

which embeds compactly into $L^q(\mathbb{R}^N)$. By standard arguments (the principle of symmetric criticality; see [11] or [16, Theorem 1.28]), one has that a critical point $u \in H_r^1(\mathbb{R}^N)$ for the functional $\Phi(u)$ of (P) is also a critical point in $H^1(\mathbb{R}^N)$. We say that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a Pohozǎev–Palais–Smale sequence for $\Phi \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ at level $c \in \mathbb{R}$ if and only if $\{u_n\}$ satisfies $\Phi(u_n) \to c$, $\Phi'(u_n) \to 0$, and $\mathcal{P}(u_n) \to 0$ as $n \to \infty$. Following the strategy in [3], we obtain that there exists a Pohozǎev–Palais–Smale sequence for Φ, with c confined in a suitable range. To ensure that the mini-max levels stay in a certain range, we make some careful computation in Sect. 2, which is crucial in our approach. Then, we make full use of three limit formulas in the Pohozǎev–Palais–Smale sequence and prove that this sequence has a strongly convergent subsequence.

2 Preliminaries

In the following, we recall the well-known Hardy–Littlewood–Sobolev inequality (see in [6, Theorem 4.3]).

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality) *Let $r, s > 1$ and $\alpha \in (0, N)$ with $\frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$. Then there exists $C > 0$ depending only on N, α, r such that, for any $f \in L^r(\mathbb{R}^N)$ and $g \in L^s(\mathbb{R}^N)$,*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}}\,dx\,dy\right| \le C(N,\alpha,r)\|f\|_{L^r(\mathbb{R}^N)}\|g\|_{L^s(\mathbb{R}^N)}.$$

Lemma 2.1 *Suppose that $N \ge 3$ and $p, q \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$. Let $\{v_n\} \subset H_r^1(\mathbb{R}^N)$ be a sequence converging weakly to 0 as $n \to \infty$. If $\frac{2(N+\alpha)}{N} < p + q < \frac{2(N+\alpha)}{N-2}$, then*

$$\int_{\mathbb{R}^N}\big(I_\alpha * |v_n|^p\big)|v_n|^q\,dx \to 0, \quad \text{as } n \to \infty.$$

Proof Since $v_n \rightharpoonup 0$ in $H_r^1(\mathbb{R}^N)$ and $2 < \frac{N(p+q)}{N+\alpha} < \frac{2N}{N-2}$, we have

$$v_n \to 0 \quad \text{in } L^{\frac{N(p+q)}{N+\alpha}}(\mathbb{R}^N),$$

see [16, Corollary 1.25]. By the Hardy–Littlewood–Sobolev inequality with $r = \frac{p+q}{p}\frac{N}{N+\alpha}$ and $t = \frac{p+q}{q}\frac{N}{N+\alpha}$, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\big(I_\alpha * |v_n|^p\big)|v_n|^q\,dx\right| &\le C\big\||v_n|^p\big\|_{L^r(\mathbb{R}^N)}\big\||v_n|^q\big\|_{L^t(\mathbb{R}^N)}\\ &= C\left(\int_{\mathbb{R}^N}|v_n|^{\frac{N(p+q)}{N+\alpha}}\,dx\right)^{\frac{N+\alpha}{N}} \to 0,\end{aligned}$$

where C is a positive constant. The proof is finished. □

Remark 2.1 p, q that appear in Theorem 1.1 satisfy

$$\frac{2(N+\alpha)}{N} < p+q < \frac{2(N+\alpha)}{N-2}.$$

The constant $\mathcal{S}_1$ is defined by

$$\mathcal{S}_1 := \inf\left\{\frac{|\nabla u|_2^2}{[\int_{\mathbb{R}^N}(I_\alpha * |u|^q)|u|^q\,dx]^{1/q}} : u \in H^1(\mathbb{R}^N)\setminus\{0\}\right\}$$

and is attained by the functions

$$U_\varepsilon(x) = \frac{C\varepsilon^{(N-2)/2}}{(\varepsilon^2+|x|^2)^{(N-2)/2}},$$

where $\varepsilon > 0$ (see in [6]). We define a cutoff function $\varphi(x)$ by

$$\varphi(x) = \begin{cases} 1, & x \in B_1;\\ 2-|x|, & x \in B_2\setminus B_1;\\ 0, & x \in \mathbb{R}^N\setminus B_2,\end{cases}$$

where $B_1 = \{x \in \mathbb{R}^N : |x| \le 1\}$ and $B_2 = \{x \in \mathbb{R}^N : |x| \le 2\}$. Set

$$u_\varepsilon = u_\varepsilon(x) = \varphi(x)\cdot U_\varepsilon(x).$$

Then we have the following lemma.

Lemma 2.2 *Suppose that* $N \ge 3$ *and* $p, q \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$. *Then there exists a positive constant* ε_0 *such that if* $\varepsilon \in (0, \varepsilon_0)$ *then*

$$\Phi(tu_\varepsilon) < \frac{1}{2}\left(1-\frac{1}{q}\right)q^{\frac{1}{q-1}}\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} \quad \textit{for all } t \ge 0,$$

provided $q = \frac{N+\alpha}{N-2}$ *and one of the following conditions holds*:

(1) $N \geq 4$ *and* $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$;
(2) $N = 3$ *and* $p \in (1 + \frac{\alpha}{N-2}, \frac{N+\alpha}{N-2})$.

Proof According to the definition of Φ and u_ε, we have

$$\begin{aligned}
\Phi(tu_\varepsilon) &= \frac{t^2}{2}\int_{\mathbb{R}^N}\big[|\nabla u_\varepsilon|^2 + u_\varepsilon^2\big]\,dx - \frac{1}{2}\int_{\mathbb{R}^N}\big(I_\alpha * F(tu_\varepsilon)\big)F(tu_\varepsilon)\,dx \\
&= \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\,dx + \frac{t^2}{2}\int_{\mathbb{R}^N}u_\varepsilon^2\,dx - \frac{t^{2p}}{2p^2}\int_{\mathbb{R}^N}\big(I_\alpha * |u_\varepsilon|^p\big)|u_\varepsilon|^p\,dx \\
&\quad - \frac{t^{p+q}}{pq}\int_{\mathbb{R}^N}\big(I_\alpha * |u_\varepsilon|^p\big)|u_\varepsilon|^q\,dx - \frac{t^{2q}}{2q^2}\int_{\mathbb{R}^N}\big(I_\alpha * |u_\varepsilon|^q\big)|u_\varepsilon|^q\,dx \\
&= \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\,dx + \frac{t^2}{2}\int_{\mathbb{R}^N}u_\varepsilon^2\,dx - \frac{t^{2p}}{2p^2}\mathcal{L} - \frac{t^{p+q}}{pq}\mathcal{H} - \frac{t^{2q}}{2q^2}\mathcal{M},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L} &:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}|u_\varepsilon(x)|^p|u_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy, \\
\mathcal{H} &:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}|u_\varepsilon(x)|^p|u_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy
\end{aligned}$$

and

$$\mathcal{M} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}|u_\varepsilon(x)|^q|u_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy.$$

By [1] (see also [16]), the following asymptotic estimates hold as ε is small enough:

$$\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\,dx = \mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} + O\big(\varepsilon^{N-2}\big) \tag{2}$$

and

$$h(\varepsilon) := \int_{\mathbb{R}^N}|u_\varepsilon|^2\,dx = \begin{cases} c\varepsilon^2 + O\big(\varepsilon^{N-2}\big), & \text{if } N \geq 5; \\ c\varepsilon^2|\log\varepsilon| + O\big(\varepsilon^2\big), & \text{if } N = 4; \\ c\varepsilon + O\big(\varepsilon^2\big), & \text{if } N = 3, \end{cases} \tag{3}$$

where c is a positive constant.

In the following we estimate the convolution terms $\mathcal{L}$, $\mathcal{H}$, and $\mathcal{M}$, respectively.

Case $\mathcal{L}$:

$$\begin{aligned}
\mathcal{L} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}|u_\varepsilon(x)|^p|u_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
&= \int_{B_2}\int_{B_2}\frac{\hat{C}|u_\varepsilon(x)|^p|u_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\geq \int_{B_1}\int_{B_1}\frac{\hat{C}|u_\varepsilon(x)|^p|u_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_{B_1}\int_{B_1} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
&= \int_{B_2}\int_{B_2} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy - \int_{B_2\setminus B_1}\int_{B_1} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\quad - \int_{B_2\setminus B_1}\int_{B_2\setminus B_1} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
&= \mathcal{L}_1 - 2\mathcal{L}_2 - \mathcal{L}_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}_1 &:= \int_{B_2}\int_{B_2} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy, \\
\mathcal{L}_2 &:= \int_{B_2\setminus B_1}\int_{B_1} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy,
\end{aligned}
$$

and

$$
\mathcal{L}_3 := \int_{B_2\setminus B_1}\int_{B_2\setminus B_1} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy.
$$

By direct computation, we have, for $\varepsilon < 1$,

$$
\begin{aligned}
\mathcal{L}_1 &= O\bigl(\varepsilon^{-(N-2)p}\bigr)\int_{B_2}\int_{B_2} \frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(1+|\frac{y}{\varepsilon}|^2)^{\frac{(N-2)p}{2}}}\,dx\,dy \\
&= O\bigl(\varepsilon^{N+\alpha-(N-2)p}\bigr)\int_{B_{2/\varepsilon}}\int_{B_{2/\varepsilon}} \frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(1+|y|^2)^{\frac{(N-2)p}{2}}}\,dx\,dy \\
&\geq O\bigl(\varepsilon^{N+\alpha-(N-2)p}\bigr)\int_{B_{\frac{1}{2}}}\biggl[\int_{B_{x,\frac{1}{2}}} \frac{1}{(1+|y|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}}\,dy\biggr]\frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}}\,dx \\
&\geq O\bigl(\varepsilon^{N+\alpha-(N-2)p}\bigr)\int_{B_{\frac{1}{2}}}\biggl[\int_{B_{x,\frac{1}{2}}} \frac{1}{|x-y|^{N-\alpha}}\,dy\biggr]\frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}}\,dx \\
&= O\bigl(\varepsilon^{N+\alpha-(N-2)p}\bigr)\int_{B_{\frac{1}{2}}} \frac{1}{|y|^{N-\alpha}}\,dy\cdot\int_{B_{\frac{1}{2}}} \frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}}\,dx = O\bigl(\varepsilon^{N+\alpha-(N-2)p}\bigr).
\end{aligned}
\tag{4}
$$

We also have

$$
\mathcal{L}_2 = O\bigl(\varepsilon^{(N-2)p}\bigr)\int_{B_1}\int_{B_2\setminus B_1} \frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(\varepsilon^2+|y|^2)^{\frac{(N-2)p}{2}}}\,dx\,dy.
$$

By the Hardy–Littlewood–Sobolev inequality with $\frac{1}{r_1}+\frac{1}{s_1} = 1+\frac{\alpha}{N}$ (see Proposition 2.1) and

$$
s_1 \in \biggl(\frac{N}{(N-2)p}, \frac{2N}{N+\alpha-(N-2)p}\biggr),
\tag{5}
$$

we have

$$\begin{aligned}
\mathcal{L}_2 &\le O\bigl(\varepsilon^{(N-2)p}\bigr)\left(\int_{B_2\setminus B_1}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)pr_1}{2}}}\,dx\right)^{\frac{1}{r_1}} \\
&\quad\times\left(\int_{B_1}\frac{1}{(\varepsilon^2+|y|^2)^{\frac{(N-2)ps_1}{2}}}\,dy\right)^{\frac{1}{s_1}} \\
&\le O\bigl(\varepsilon^{\frac{N}{s_1}}\bigr)\left(\int_0^{\frac{1}{\varepsilon}}\frac{z^{N-1}}{(1+|z|^2)^{\frac{(N-2)ps_1}{2}}}\,dz\right)^{\frac{1}{s_1}} \\
&\le O\bigl(\varepsilon^{\frac{N}{s_1}}\bigr)\left(1+\int_1^{\frac{1}{\varepsilon}}|z|^{N-1-(N-2)ps_1}\,dz\right)^{\frac{1}{s_1}} = O\bigl(\varepsilon^{\frac{N}{s_1}}\bigr).
\end{aligned} \tag{6}$$

We also get

$$\begin{aligned}
\mathcal{L}_3 &= O\bigl(\varepsilon^{(N-2)p}\bigr)\int_{B_2\setminus B_1}\int_{B_2\setminus B_1}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(\varepsilon^2+|y|^2)^{\frac{(N-2)p}{2}}}\,dx\,dy \\
&\le O\bigl(\varepsilon^{(N-2)p}\bigr)\int_{B_2\setminus B_1}\int_{B_2\setminus B_1}\frac{1}{|x|^{(N-2)p}|x-y|^{N-\alpha}|y|^{(N-2)p}}\,dx\,dy \\
&\le O\bigl(\varepsilon^{(N-2)p}\bigr)\left[\int_{B_2\setminus B_1}|x|^{-\frac{2N(N-2)p}{N+\alpha}}\,dx\right]^{\frac{N+\alpha}{N}} = O\bigl(\varepsilon^{(N-2)p}\bigr).
\end{aligned} \tag{7}$$

Combining (4), (6), and (7), we obtain

$$\mathcal{L} \ge O\bigl(\varepsilon^{N+\alpha-(N-2)p}\bigr) - O\bigl(\varepsilon^{\frac{N}{s_1}}\bigr) - O\bigl(\varepsilon^{(N-2)p}\bigr).$$

Noting that $s_1 > \frac{N}{(N-2)p}$,

$$\mathcal{L} \ge O\bigl(\varepsilon^{N+\alpha-(N-2)p}\bigr) - O\bigl(\varepsilon^{\frac{N}{s_1}}\bigr). \tag{8}$$

Case $\mathcal{H}$: It is easy to see that

$$\begin{aligned}
\mathcal{H} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}|u_\varepsilon(x)|^p|u_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy = \int_{B_2}\int_{B_2}\frac{\hat{C}|u_\varepsilon(x)|^p|u_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\ge \int_{B_1}\int_{B_1}\frac{\hat{C}|u_\varepsilon(x)|^p|u_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy = \int_{B_1}\int_{B_1}\frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&= \int_{B_2}\int_{B_2}\frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy - \int_{B_2\setminus B_1}\int_{B_1}\frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\quad - \int_{B_1}\int_{B_2\setminus B_1}\frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy - \int_{B_2\setminus B_1}\int_{B_2\setminus B_1}\frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&= \mathcal{H}_1 - \mathcal{H}_2 - \mathcal{H}_3 - \mathcal{H}_4,
\end{aligned}$$

where

$$\mathcal{H}_1 := \int_{B_2}\int_{B_2} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy,$$
$$\mathcal{H}_2 := \int_{B_2\setminus B_1}\int_{B_1} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy,$$
$$\mathcal{H}_3 := \int_{B_1}\int_{B_2\setminus B_1} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy,$$
$$\mathcal{H}_4 := \int_{B_2\setminus B_1}\int_{B_2\setminus B_1} \frac{\hat{C}|U_\varepsilon(x)|^p|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy.$$

For $\varepsilon < 1$, we have

$$\begin{aligned}
\mathcal{H}_1 &= O\big(\varepsilon^{-\frac{(N-2)(p+q)}{2}}\big)\int_{B_2}\int_{B_2} \frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(1+|\frac{y}{\varepsilon}|^2)^{\frac{(N-2)q}{2}}}\,dx\,dy \\
&= O\big(\varepsilon^{N+\alpha-\frac{(N-2)(p+q)}{2}}\big)\int_{B_{2/\varepsilon}}\int_{B_{2/\varepsilon}} \frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(1+|y|^2)^{\frac{(N-2)q}{2}}}\,dx\,dy \\
&\ge O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big)\int_{B_{\frac{1}{2}}}\bigg[\int_{B_{x,\frac{1}{2}}} \frac{1}{(1+|y|^2)^{\frac{(N-2)q}{2}}|x-y|^{N-\alpha}}\,dy\bigg]\frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}}\,dx \\
&\ge O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big)\int_{B_{\frac{1}{2}}}\bigg[\int_{B_{x,\frac{1}{2}}} \frac{1}{|x-y|^{N-\alpha}}\,dy\bigg]\frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}}\,dx \\
&= O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big)\int_{B_{\frac{1}{2}}} \frac{1}{|y|^{N-\alpha}}\,dy\cdot\int_{B_{\frac{1}{2}}} \frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}}\,dx = O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big).
\end{aligned} \tag{9}$$

By direct computation, we have

$$\mathcal{H}_2 = O\big(\varepsilon^{\frac{(N-2)(p+q)}{2}}\big)\int_{B_1}\int_{B_2\setminus B_1} \frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(\varepsilon^2+|y|^2)^{\frac{(N-2)q}{2}}}\,dx\,dy.$$

By the Hardy–Littlewood–Sobolev inequality with $\frac{1}{r_2}+\frac{1}{s_2} = 1+\frac{\alpha}{N}$ and

$$s_2 \in \bigg(\frac{N}{N+\alpha}, \frac{N}{N+\alpha-(N-2)p}\bigg),$$

we have

$$\begin{aligned}
\mathcal{H}_2 &\le O\big(\varepsilon^{\frac{(N-2)(p+q)}{2}}\big)\bigg(\int_{B_2\setminus B_1} \frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)pr_2}{2}}}\,dx\bigg)^{\frac{1}{r_2}}\bigg(\int_{B_1} \frac{1}{(\varepsilon^2+|y|^2)^{\frac{(N+\alpha)s_2}{2}}}\,dy\bigg)^{\frac{1}{s_2}} \\
&\le O\big(\varepsilon^{\frac{(N-2)(p+q)}{2}-(N+\alpha)+\frac{N}{s_2}}\big)\bigg(\int_0^{\frac{1}{\varepsilon}} \frac{z^{N-1}}{(1+|z|^2)^{\frac{(N+\alpha)s_2}{2}}}\,dz\bigg)^{\frac{1}{s_2}} \\
&\le O\big(\varepsilon^{\frac{N}{s_2}-\frac{(N+\alpha)-(N-2)p}{2}}\big)\bigg(1+\int_1^{\frac{1}{\varepsilon}} |z|^{N-1-(N+\alpha)s_2}\,dz\bigg)^{\frac{1}{s}} = O\big(\varepsilon^{\frac{N}{s_2}-\frac{(N+\alpha)-(N-2)p}{2}}\big).
\end{aligned} \tag{10}$$

By direct computation, we get

$$\mathcal{H}_3 = O\big(\varepsilon^{\frac{(N-2)(p+q)}{2}}\big)\int_{B_2\setminus B_1}\int_{B_1}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(\varepsilon^2+|y|^2)^{\frac{(N-2)q}{2}}}\,dx\,dy.$$

By the Hardy–Littlewood–Sobolev inequality with $\frac{1}{r_3}+\frac{1}{s_3}=1+\frac{\alpha}{N}$ and

$$s_3 > \frac{N}{(N-2)p},$$

we have

$$\begin{aligned}
\mathcal{H}_3 &\le O\big(\varepsilon^{\frac{(N-2)(p+q)}{2}}\big)\bigg(\int_{B_1}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)ps_3}{2}}}\,dx\bigg)^{\frac{1}{s_3}}\bigg(\int_{B_2\setminus B_1}\frac{1}{(\varepsilon^2+|y|^2)^{\frac{(N+\alpha)r_3}{2}}}\,dy\bigg)^{\frac{1}{r_3}}\\
&\le O\big(\varepsilon^{\frac{(N-2)(p+q)}{2}-(N-2)p+\frac{N}{s_3}}\big)\bigg(\int_0^{\frac{1}{\varepsilon}}\frac{z^{N-1}}{(1+|z|^2)^{\frac{(N-2)ps_3}{2}}}\,dz\bigg)^{\frac{1}{s_3}}\\
&\le O\big(\varepsilon^{\frac{N}{s_3}+\frac{N+\alpha-(N-2)p}{2}}\big)\bigg(1+\int_1^{\frac{1}{\varepsilon}}|z|^{N-1-(N-2)ps_3}\,dz\bigg)^{\frac{1}{s_3}}\\
&= O\big(\varepsilon^{\frac{N}{s_3}+\frac{N+\alpha-(N-2)p}{2}}\big).
\end{aligned}\tag{11}$$

We also get

$$\begin{aligned}
\mathcal{H}_4 &= O\big(\varepsilon^{\frac{N+\alpha+(N-2)p}{2}}\big)\int_{B_2\setminus B_1}\int_{B_2\setminus B_1}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)p}{2}}|x-y|^{N-\alpha}(\varepsilon^2+|y|^2)^{\frac{(N-2)q}{2}}}\,dx\,dy\\
&\le O\big(\varepsilon^{\frac{N+\alpha+(N-2)p}{2}}\big)\int_{B_2\setminus B_1}\int_{B_2\setminus B_1}\frac{1}{|x|^{(N-2)p}|x-y|^{N-\alpha}|y|^{(N-2)q}}\,dx\,dy\\
&\le O\big(\varepsilon^{\frac{N+\alpha+(N-2)p}{2}}\big)\bigg[\int_{B_2\setminus B_1}|x|^{-\frac{2pN(N-2)}{N+\alpha}}\,dx\bigg]^{\frac{N+\alpha}{2N}}\cdot\bigg[\int_{B_2\setminus B_1}|x|^{-2N}\,dx\bigg]^{\frac{N+\alpha}{2N}}\\
&\le O\big(\varepsilon^{\frac{N+\alpha+(N-2)p}{2}}\big).
\end{aligned}\tag{12}$$

Combining (9), (10), (11), and (12), we have

$$\mathcal{H}\ge O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big)-O\big(\varepsilon^{\frac{N}{s_2}-\frac{(N+\alpha)-(N-2)p}{2}}\big)-O\big(\varepsilon^{\frac{N}{s_3}+\frac{N+\alpha-(N-2)p}{2}}\big)-O\big(\varepsilon^{\frac{N+\alpha+(N-2)p}{2}}\big).$$

Noting that $s_2<\frac{N}{N+\alpha-(N-2)p}$, we obtain

$$\mathcal{H}\ge O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big).\tag{13}$$

Case $\mathcal{M}$: By the definition of $u_\varepsilon(x)$, we have

$$\begin{aligned}
\mathcal{M} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{C}|u_\varepsilon(x)|^q|u_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy = \int_{B_2}\int_{B_2} \frac{\hat{C}|u_\varepsilon(x)|^q|u_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\geq \int_{B_1}\int_{B_1} \frac{\hat{C}|u_\varepsilon(x)|^q|u_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy = \int_{B_1}\int_{B_1} \frac{\hat{C}|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy - 2\int_{\mathbb{R}^N\setminus B_1}\int_{B_1} \frac{\hat{C}|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N\setminus B_1}\int_{\mathbb{R}^N\setminus B_1} \frac{\hat{C}|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{C}|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy - 2\int_{\mathbb{R}^N\setminus B_1}\int_{B_1} \frac{\hat{C}|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^N\setminus B_1}\int_{\mathbb{R}^N\setminus B_1} \frac{\hat{C}|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&= \mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} - 2\mathcal{M}_1 - \mathcal{M}_2,
\end{aligned}$$

where

$$\mathcal{M}_1 := \int_{\mathbb{R}^N\setminus B_1}\int_{B_1} \frac{\hat{C}|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy$$

and

$$\mathcal{M}_2 := \int_{\mathbb{R}^N\setminus B_1}\int_{\mathbb{R}^N\setminus B_1} \frac{\hat{C}|U_\varepsilon(x)|^q|U_\varepsilon(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy.$$

By direct computation, we have, for $\varepsilon < 1$,

$$\mathcal{M}_1 = O\big(\varepsilon^{(N-2)q}\big)\int_{\mathbb{R}^N\setminus B_1}\int_{B_1} \frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)q}{2}}|x-y|^{N-\alpha}(\varepsilon^2+|y|^2)^{\frac{(N-2)q}{2}}}\,dx\,dy.$$

By the Hardy–Littlewood–Sobolev inequality with $t = r = \frac{2N}{N+\alpha}$ and $q = \frac{N+\alpha}{N-2}$, we have

$$\begin{aligned}
\mathcal{M}_1 &\leq O\big(\varepsilon^{(N-2)q}\big)\left(\int_{\mathbb{R}^N\setminus B_1} \frac{1}{(\varepsilon^2+|x|^2)^N}\,dx\right)^{\frac{N+\alpha}{2N}}\left(\int_{B_1} \frac{1}{(\varepsilon^2+|y|^2)^N}\,dy\right)^{\frac{N+\alpha}{2N}} \\
&\leq O\big(\varepsilon^{\frac{N+\alpha}{2}}\big)\left(\int_1^{+\infty} z^{-1-N}\,dz\right)^{\frac{N+\alpha}{2N}}\left(\int_0^{\frac{1}{\varepsilon}} \frac{z^{N-1}}{(1+|z|^2)^N}\,dz\right)^{\frac{N+\alpha}{2N}} \\
&\leq O\big(\varepsilon^{\frac{N+\alpha}{2}}\big)\cdot\left(1+\int_1^{\frac{1}{\varepsilon}} z^{-1-N}\,dz\right)^{\frac{N+\alpha}{2N}} \\
&\leq O\big(\varepsilon^{\frac{N+\alpha}{2}}\big)\left(1+\frac{1}{-N}r^{-N}\Big|_1^{\frac{1}{\varepsilon}}\right)^{\frac{N+\alpha}{2N}} \leq O\big(\varepsilon^{\frac{N+\alpha}{2}}\big).
\end{aligned} \tag{14}$$

We also get

$$\begin{aligned}\mathcal{M}_2 &= O\big(\varepsilon^{(N-2)q}\big)\int_{\mathbb{R}^N\setminus B_1}\int_{\mathbb{R}^N\setminus B_1}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)q}{2}}|x-y|^{N-\alpha}(\varepsilon^2+|y|^2)^{\frac{(N-2)q}{2}}}\,dx\,dy\\ &\le O\big(\varepsilon^{(N-2)q}\big)\int_{\mathbb{R}^N\setminus B_1}\int_{\mathbb{R}^N\setminus B_1}\frac{1}{|x|^{(N-2)q}|x-y|^{N-\alpha}|y|^{(N-2)q}}\,dx\,dy\\ &\le O\big(\varepsilon^{(N-2)q}\big)\bigg[\int_{\mathbb{R}^N\setminus B_1}|x|^{-2N}\,dx\bigg]^{\frac{N+\alpha}{N}} = O\big(\varepsilon^{N+\alpha}\big).\end{aligned} \tag{15}$$

Combining (14) and (15), we have

$$\mathcal{M} \ge \mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} - O\big(\varepsilon^{\frac{N+\alpha}{2}}\big). \tag{16}$$

From (2), (3), (8), (13), and (16), we have

$$\begin{aligned}\Phi(tu_\varepsilon) &\le \frac{t^2}{2}\big(\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} + O\big(\varepsilon^{N-2}\big)\big) - \frac{t^{2q}}{2q^2}\big(\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} - O\big(\varepsilon^{\frac{N+\alpha}{2}}\big)\big)\\ &\quad + t^2h(\varepsilon) - t^{2p}\big[O\big(\varepsilon^{N+\alpha-(N-2)p}\big) - O\big(\varepsilon^{\frac{N}{s_1}}\big)\big] - O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big)t^{p+q}.\end{aligned}$$

It is easy to see that there exist constants $\bar{\varepsilon} > 0$ and $t_2 > t_1 > 0$ such that, for all $\varepsilon \in (0,\bar{\varepsilon})$ and $t \in [0,t_1] \cup [t_2,\infty)$,

$$\Phi(tu_\varepsilon) < \frac{1}{2}\bigg(1-\frac{1}{q}\bigg)q^{\frac{1}{q-1}}\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}}.$$

In the following we may set $t \in [t_1,t_2]$ and $\varepsilon \in (0,\bar{\varepsilon})$. Then we have

$$\begin{aligned}\Phi(tu_\varepsilon) &\le \frac{t^2}{2}\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} - \frac{t^{2q}}{2q^2}\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} + \frac{t^2}{2}O\big(\varepsilon^{N-2}\big) + \frac{t^{2q}}{2q^2}O\big(\varepsilon^{\frac{N+\alpha}{2}}\big)\\ &\quad + t^2h(\varepsilon) - t^{2p}O\big(\varepsilon^{N+\alpha-(N-2)p}\big) + t^{2p}O\big(\varepsilon^{\frac{N}{s_1}}\big) - t^{p+q}O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big)\\ &\le \frac{t^2}{2}\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} - \frac{t^{2q}}{2q^2}\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} + \frac{t_2^2}{2}O\big(\varepsilon^{N-2}\big) + \frac{t_2^{2q}}{2q^2}O\big(\varepsilon^{\frac{N+\alpha}{2}}\big)\\ &\quad + t_2^2h(\varepsilon) - t_1^{2p}O\big(\varepsilon^{N+\alpha-(N-2)p}\big) + t_2^{2p}O\big(\varepsilon^{\frac{N}{s_1}}\big) - t_1^{p+q}O\big(\varepsilon^{\frac{N+\alpha-(N-2)p}{2}}\big).\end{aligned}$$

By the definition of $h(\varepsilon)$ and (22), for ε small enough, we obtain

$$\Phi(tu_\varepsilon) < \frac{1}{2}\bigg(1-\frac{1}{q}\bigg)q^{\frac{1}{q-1}}\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} \quad \text{for all } t \ge 0,$$

provided one of the following conditions holds:

(1) $N \ge 4$ and $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$;
(2) $N = 3$ and $p \in (1+\frac{\alpha}{N-2}, \frac{N+\alpha}{N-2})$.

The proof is finished. □

The constant $\mathcal{S}_2$ is defined by

$$\mathcal{S}_2 := \inf\bigg\{\frac{|u|_2^2}{[\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\,dx]^{1/p}} : u \in H^1\big(\mathbb{R}^N\big) \setminus \{0\}\bigg\}$$

and is attained by the functions

$$v_\sigma(x)=\frac{\tilde{C}\sigma^{N/2}}{(1+\sigma^2|x|^2)^{N/2}},$$

where $\sigma>0$ (see [6, Theorem 4.3]).

Lemma 2.3 *Suppose that $N\geq 3$ and $p,q\in[\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}]$. There exists a positive constant σ_0 such that if $\sigma>\sigma_0$ then*

$$\Phi(tv_\sigma)<\frac{1}{2}\left(1-\frac{1}{p}\right)p^{\frac{1}{p-1}}\mathcal{S}_2^{\frac{p}{p-1}}\quad \textit{for all } t\geq 0,$$

under $p=\frac{N+\alpha}{N}$ and one of the following conditions:
(1) $N>4+\alpha$ *and* $q\in(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}]$,
(2) $N<4+\alpha$ *and* $q\in(\frac{N+\alpha}{N},\frac{N+\alpha+4}{N})$.

Proof According to the definition of v_σ, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\left|v_\sigma(x)\right|^2dx&=\int_{\mathbb{R}^N}\frac{\tilde{C}^2\sigma^N}{(1+\sigma^2|x|^2)^N}dx\\&=\int_{\mathbb{R}^N}\frac{\tilde{C}^2}{(1+|\sigma x|^2)^N}d(\sigma x)=\int_{\mathbb{R}^N}\frac{\tilde{C}^2}{(1+|x|^2)^N}dx=\mathcal{S}_2^{\frac{N+\alpha}{\alpha}}\end{aligned}\tag{17}$$

and for

$$\begin{aligned}&r\in\left[\frac{N+\alpha}{N},\frac{N+\alpha}{N-2}\right],\\&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}|v_\sigma(x)|^r|v_\sigma(y)|^r}{|x-y|^{N-\alpha}}dx\,dy\\&\quad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}\tilde{C}^{2r}\sigma^{Nr}}{(1+\sigma^2|x|^2)^{\frac{Nr}{2}}|x-y|^{N-\alpha}(1+\sigma^2|y|^2)^{\frac{Nr}{2}}}dx\,dy\\&\quad=\sigma^{Nr-(N+\alpha)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}\tilde{C}^{2r}}{(1+|x|^2)^{\frac{Nr}{2}}|x-y|^{N-\alpha}(1+|y|^2)^{\frac{Nr}{2}}}dx\,dy.\end{aligned}\tag{18}$$

Especially, for $r=p$, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}|v_\sigma(x)|^p|v_\sigma(y)|^p}{|x-y|^{N-\alpha}}dx\,dy=\mathcal{S}_2^{\frac{N+\alpha}{\alpha}}.\tag{19}$$

We also get

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla v_\sigma|^2dx\\&\quad=\int_{\mathbb{R}^N}\frac{\tilde{C}^2N^2\sigma^{N+4}|x|^2}{(1+\sigma^2|x|^2)^{N+2}}dx=\sigma^2\int_{\mathbb{R}^N}\frac{\tilde{C}^2N^2|x|^2}{(1+|x|^2)^{N+2}}dx\end{aligned}\tag{20}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\sigma(x)|^p|v_\sigma(y)|^q}{|x-y|^{N-\alpha}}\,dx\,dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}\tilde{C}^{p+q}\sigma^{\frac{N(p+q)}{2}}}{(1+\sigma^2|x|^2)^{\frac{Np}{2}}|x-y|^{N-\alpha}(1+\sigma^2|y|^2)^{\frac{Nq}{2}}}\,dx\,dy \\
&= \sigma^{\frac{Nq-(N+\alpha)}{2}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\hat{C}\tilde{C}^{p+q}}{(1+|x|^2)^{\frac{Np}{2}}|x-y|^{N-\alpha}(1+|y|^2)^{\frac{Nq}{2}}}\,dx\,dy.
\end{aligned} \tag{21}$$

By the definition of Φ and combining (17)–(21), we obtain

$$\begin{aligned}
\Phi(tv_\sigma) &= \frac{t^2}{2}\int_{\mathbb{R}^N}\big[|\nabla v_\sigma|^2+v_\sigma^2\big]\,dx-\frac{1}{2}\int_{\mathbb{R}^N}\big(I_\alpha*F(tv_\sigma)\big)F(tv_\sigma)\,dx \\
&= \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla v_\sigma|^2\,dx+\frac{t^2}{2}\int_{\mathbb{R}^N}v_\sigma^2\,dx-\frac{t^{2p}}{2p^2}\int_{\mathbb{R}^N}\big(I_\alpha*|v_\sigma|^p\big)|v_\sigma|^p\,dx \\
&\quad -\frac{t^{p+q}}{pq}\int_{\mathbb{R}^N}\big(I_\alpha*|v_\sigma|^p\big)|v_\sigma|^q\,dx-\frac{t^{2q}}{2q^2}\int_{\mathbb{R}^N}\big(I_\alpha*|v_\sigma|^q\big)|v_\sigma|^q\,dx \\
&= \left(\frac{t^2}{2}-\frac{t^{2p}}{2p^2}\right)\mathcal{S}_2^{\frac{N+\alpha}{\alpha}}+c_1\sigma^2t^2-c_2\sigma^{Nq-(N+\alpha)}t^{2q}-c_3\sigma^{\frac{Nq-(N+\alpha)}{2}}t^{p+q},
\end{aligned}$$

where c_1, c_2, and c_3 are positive constants. It is easy to see that there exist constants $\sigma_1>0$ and $t_4>t_3>0$ such that, for all $\sigma\in(0,\sigma_1)$ and $t\in[0,t_3]\cup[t_4,\infty)$,

$$\Phi(tv_\sigma)<\frac{1}{2}\left(1-\frac{1}{p}\right)p^{\frac{1}{p-1}}\mathcal{S}_2^{\frac{N+\alpha}{\alpha}}.$$

In the following we set $t\in[t_3,t_4]$ and $\sigma\in(0,\sigma_1)$. Thus we have

$$\Phi(tv_\sigma)\le\frac{1}{2}\left(1-\frac{1}{p}\right)p^{\frac{1}{p-1}}\mathcal{S}_2^{\frac{N+\alpha}{\alpha}}+c_1\sigma^2t_4^2-c_3\sigma^{\frac{Nq-(N+\alpha)}{2}}t_3^{p+q}.$$

By $q<\frac{N+\alpha+4}{N}$, there exists a positive constant σ_0 such that if $\sigma\in(0,\sigma_0)$ then

$$\Phi(tv_\sigma)<\frac{1}{2}\left(1-\frac{1}{p}\right)p^{\frac{1}{p-1}}\mathcal{S}_2^{\frac{N+\alpha}{\alpha}}\quad\text{for all } t\ge 0.$$

Noting that if $N>\alpha+4$ then $\frac{N+\alpha}{N-2}<\frac{N+\alpha+4}{N}$; if $N<\alpha+4$ then $\frac{N+\alpha}{N-2}>\frac{N+\alpha+4}{N}$, the conclusion follows. □

3 Proof of the main theorem

It is easy to prove that there exist $\beta,\rho>0$ and $v\in H_r^1(\mathbb{R}^N)$ such that

(i) $\inf_{\|u\|=\rho}\Phi(u)>\beta$;

(ii) $\|v\|>\rho$ and $\Phi(v)<0$.

Thus Φ has mountain pass geometry. Define the mountain pass level c by

$$c:=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1}\Phi\big(\gamma(t)\big),$$

where

$$\Gamma = \{\gamma \in C([0,1], H_r^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = v\}.$$

Combining Lemmas 2.2 and 2.3, we have the following conclusions:

(i) If $q = \frac{N+\alpha}{N-2}$, $N \geq 4$, and $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ or $N = 3$ and $p \in (1 + \frac{\alpha}{N-2}, \frac{N+\alpha}{N-2})$, then

$$c \in \left(0, \frac{1}{2}\left(1 - \frac{1}{q}\right) q^{\frac{1}{q-1}} \mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}}\right). \tag{22}$$

(ii) If $p = \frac{N+\alpha}{N}$, $N > 4 + \alpha$, and $q \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ or $N < 4 + \alpha$ and $q \in (\frac{N+\alpha}{N}, \frac{N+\alpha+4}{N})$, then

$$c \in \left(0, \frac{1}{2}\left(1 - \frac{1}{p}\right) p^{\frac{1}{p-1}} \mathcal{S}_2^{\frac{N+\alpha}{\alpha}}\right). \tag{23}$$

(iii) If $q = \frac{N+\alpha}{N-2}$, $p = \frac{N+\alpha}{N}$, and $N > 4 + \alpha$, then

$$c \in \left(0, \min\left\{\frac{1}{2}\left(1 - \frac{1}{q}\right) q^{\frac{1}{q-1}} \mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}}, \frac{1}{2}\left(1 - \frac{1}{p}\right) p^{\frac{1}{p-1}} \mathcal{S}_2^{\frac{N+\alpha}{\alpha}}\right\}\right). \tag{24}$$

From Proposition 2.1 in [8], there exists a Pohozǎev–Palais–Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ in $H_r^1(\mathbb{R}^N)$ such that, as $n \to \infty$,

$$\begin{cases} \Phi(u_n) \to c, \\ \Phi'(u_n) \to 0 \quad \text{strongly in } (H_r^1(\mathbb{R}^N))', \\ \mathcal{P}(u_n) \to 0. \end{cases}$$

For every $n \in \mathbb{N}$,

$$\Phi(u_n) - \frac{1}{N+\alpha}\mathcal{P}(u_n) = \frac{2+\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |u_n|^2 \, dx.$$

As the left-hand side is bounded, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H_r^1(\mathbb{R}^N)$.

By extracting if necessary to a subsequence, we may assume that $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^N)$. It is obvious that u is a solution of problem (P). Thus

$$\Phi(u) = \Phi(u) - \frac{1}{N+\alpha}\mathcal{P}(u) = \frac{2+\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{\alpha}{2(N+\alpha)} \int_{\mathbb{R}^N} |u|^2 \, dx \geq 0.$$

Let $v_n = u_n - u$. By the Brezis–Lieb lemma (see [16, Lemma 1.32]),

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + o(1)$$

and

$$\int_{\mathbb{R}^N} |u_n|^2 \, dx = \int_{\mathbb{R}^N} |v_n|^2 \, dx + \int_{\mathbb{R}^N} |u|^2 \, dx + o(1).$$

According to the situation of p, q, α, and N, we divide the discussion into three cases.

Case (i): $q = \frac{N+\alpha}{N-2}$, $N \geq 4$, and $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ or $N = 3$ and $p \in (1 + \frac{\alpha}{N-2}, \frac{N+\alpha}{N-2})$. From Lemma 2.1, Propositions 2.4 and 2.5 in [14], we see that

$$\int_{\mathbb{R}^N} \big(I_\alpha * |u_n|^p\big)|u_n|^p\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^p\,dx + o(1),$$
$$\int_{\mathbb{R}^N} \big(I_\alpha * |u_n|^q\big)|u_n|^q\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * |v_n|^q\big)|v_n|^q\,dx + \int_{\mathbb{R}^N} \big(I_\alpha * |u|^q\big)|u|^q\,dx + o(1),$$

and

$$\int_{\mathbb{R}^N} \big(I_\alpha * |u_n|^p\big)|u_n|^q\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * |u|^p\big)|u|^q\,dx + o(1).$$

Then we have

$$\begin{aligned}\big\langle \Phi'(u_n), u_n\big\rangle &= \int_{\mathbb{R}^N} \big[|\nabla u_n|^2 + u_n^2\big]\,dx - \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big)f(u_u)\,dx \\ &= \big\langle \Phi'(v), v\big\rangle + \int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + v_n^2\big]\,dx - \frac{1}{q}\int_{\mathbb{R}^N} \big(I_\alpha * |v_n|^q\big)|v_n|^q\,dx + o(1).\end{aligned} \tag{25}$$

From $\langle \Phi'(v), v\rangle = 0$ and $\langle \Phi'(u_n), u_n\rangle \to 0$,

$$\int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + v_n^2\big]\,dx = \frac{1}{q}\int_{\mathbb{R}^N} \big(I_\alpha * |v_n|^q\big)|v_n|^q\,dx + o(1). \tag{26}$$

We also have

$$\begin{aligned}\mathcal{P}(u_n) &= \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} u_n^2\,dx - \frac{N+\alpha}{2}\int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big)F(u_n)\,dx \\ &= \mathcal{P}(u) + \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} v_n^2\,dx \\ &\quad - \frac{N}{2q}\int_{\mathbb{R}^N} \big(I_\alpha * |v_n|^q\big)|v_n|^q\,dx + o(1).\end{aligned}$$

Thus

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N} v_n^2\,dx = \frac{N-2}{2q}\int_{\mathbb{R}^N} \big(I_\alpha * |v_n|^q\big)|v_n|^q\,dx + o(1). \tag{27}$$

Combining (26) and (27), we deduce that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx = \frac{1}{q}\int_{\mathbb{R}^N} \big(I_\alpha * |v_n|^q\big)|v_n|^q\,dx + o(1)$$

and

$$\int_{\mathbb{R}^N} v_n^2\,dx = o(1).$$

We may assume that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \to a, \qquad \frac{1}{q}\int_{\mathbb{R}^N} \big(I_\alpha * |v_n|^q\big)|v_n|^q\,dx \to a,$$

where a is a nonnegative constant.

We claim that $a = 0$. If $a \neq 0$, by the definition of $\mathcal{S}_1$, we have

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \geq \mathcal{S}_1 \left[\int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^q\right)|v_n|^q\,dx \right]^{1/q}.$$

It follows that $a \geq \mathcal{S}_1 (q \cdot a)^{1/q}$, which yields

$$a \geq q^{\frac{1}{q-1}} \mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}}. \tag{28}$$

Similarly to the discussion of (25), we have

$$\begin{aligned}\Phi(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + u_n^2\right] dx - \frac{1}{2}\int_{\mathbb{R}^N} \left(I_\alpha * F(u_n)\right)F(u_n)\,dx \\ &= \Phi(u) + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx - \frac{1}{2q^2}\int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^q\right)|v_n|^q\,dx + o(1).\end{aligned}$$

It follows from $\Phi(u) \geq 0$ and (28) that

$$\begin{aligned}c &= \lim_{n\to\infty} \Phi(u_n) \\ &\geq \lim_{n\to\infty} \left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx - \frac{1}{2q^2}\int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^q\right)|v_n|^q\,dx \right] \\ &= \frac{1}{2}\left(1 - \frac{1}{q}\right) q^{\frac{1}{q-1}} \mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}},\end{aligned}$$

which contradicts (22). Hence $a = 0$. This gives $v_n \to 0$ in $H_r^1(\mathbb{R}^n)$.

Case (ii): $p = \frac{N+\alpha}{N}$, $N > 4 + \alpha$, and $q \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ or $N < 4 + \alpha$ and $q \in (\frac{N+\alpha}{N}, \frac{N+\alpha+4}{N})$.

From Lemma 2.1, Propositions 2.4 and 2.5 in [14], we see that

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^p\,dx = \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p\right)|v_n|^p\,dx + \int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p\,dx + o(1), \\ &\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^q\right)|u_n|^q\,dx = \int_{\mathbb{R}^N} \left(I_\alpha * |u|^q\right)|u|^q\,dx + o(1),\end{aligned}$$

and

$$\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^q\,dx = \int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^q\,dx + o(1).$$

Then we have

$$\begin{aligned}\left\langle \Phi'(u_n), u_n\right\rangle &= \int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + u_n^2\right] dx - \int_{\mathbb{R}^N} \left(I_\alpha * F(u_n)\right) f(u_u)\,dx \\ &= \left\langle \Phi'(v), v\right\rangle + \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + v_n^2\right] dx - \frac{1}{p}\int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p\right)|v_n|^p\,dx + o(1).\end{aligned} \tag{29}$$

From $\langle \Phi'(v), v\rangle = 0$ and $\langle \Phi'(u_n), u_n\rangle \to 0$,

$$\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + v_n^2\right] dx = \frac{1}{p}\int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p\right)|v_n|^p\,dx + o(1). \tag{30}$$

We also have

$$\begin{aligned}\mathcal{P}(u_n) &= \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N}u_n^2\,dx - \frac{N+\alpha}{2}\int_{\mathbb{R}^N}\bigl(I_\alpha * F(u_n)\bigr)F(u_n)\,dx \\ &= \mathcal{P}(u) + \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N}v_n^2\,dx \\ &\quad - \frac{N-2}{2p}\int_{\mathbb{R}^N}\bigl(I_\alpha * |v_n|^p\bigr)|v_n|^p\,dx + o(1).\end{aligned}$$

Thus

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx + \frac{N}{2}\int_{\mathbb{R}^N}v_n^2\,dx = \frac{N}{2p}\int_{\mathbb{R}^N}\bigl(I_\alpha * |v_n|^p\bigr)|v_n|^p\,dx + o(1). \tag{31}$$

Combining (30) and (31), we deduce that

$$\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx = o(1)$$

and

$$\int_{\mathbb{R}^N}v_n^2\,dx = \frac{1}{p}\int_{\mathbb{R}^N}\bigl(I_\alpha * |v_n|^p\bigr)|v_n|^p\,dx + o(1).$$

We may assume that

$$\int_{\mathbb{R}^N}v_n^2\,dx \to b, \qquad \frac{1}{p}\int_{\mathbb{R}^N}\bigl(I_\alpha * |v_n|^p\bigr)|v_n|^p\,dx \to b,$$

where b is a nonnegative constant.

We claim that $b = 0$. If $b \neq 0$, by the definition of $\mathcal{S}_2$, we have

$$\int_{\mathbb{R}^N}|v_n|^2\,dx \geq \mathcal{S}_2\biggl[\int_{\mathbb{R}^N}\bigl(I_\alpha * |v_n|^p\bigr)|v_n|^p\,dx\biggr]^{1/p}.$$

It follows that $b \geq \mathcal{S}_2(p\cdot b)^{1/p}$, which yields

$$b \geq p^{\frac{1}{p-1}}\mathcal{S}_2^{\frac{N+\alpha}{\alpha}}. \tag{32}$$

Similarly to the discussion of (29), we have

$$\begin{aligned}\Phi(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N}\bigl[|\nabla u_n|^2 + u_n^2\bigr]\,dx - \frac{1}{2}\int_{\mathbb{R}^N}\bigl(I_\alpha * F(u_n)\bigr)F(u_n)\,dx \\ &= \Phi(u) + \frac{1}{2}\int_{\mathbb{R}^N}v_n^2\,dx - \frac{1}{2p^2}\int_{\mathbb{R}^N}\bigl(I_\alpha * |v_n|^p\bigr)|v_n|^p\,dx + o(1).\end{aligned}$$

It follows from $\Phi(u) \geq 0$, and (36) or (37) that

$$\begin{aligned} c &= \lim_{n\to\infty} \Phi(u_n) \\ &\geq \lim_{n\to\infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} v_n^2 \, dx - \frac{1}{2p^2} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p\right)|v_n|^p \, dx \right] \\ &= \frac{1}{2}\left(1 - \frac{1}{p}\right) p^{\frac{1}{p-1}} \mathcal{S}_2^{\frac{N+\alpha}{\alpha}}, \end{aligned}$$

which contradicts (23). Hence $b = 0$. This gives $v_n \to 0$ in $H_r^1(\mathbb{R}^n)$.

Case (iii): $q = \frac{N+\alpha}{N-2}$, $p = \frac{N+\alpha}{N}$, and $N > 4 + \alpha$.

From Lemma 2.1, Propositions 2.4 and 2.5 in [14], we see that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^p \, dx = \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p\right)|v_n|^p \, dx + \int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p \, dx + o(1), \\ &\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^q\right)|u_n|^q \, dx = \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^q\right)|v_n|^q \, dx + \int_{\mathbb{R}^N} \left(I_\alpha * |u|^q\right)|u|^p \, dx + o(1), \end{aligned}$$

and

$$\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^p\right)|u_n|^q \, dx = \int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^q \, dx + o(1).$$

Then we have

$$\begin{aligned} \langle \Phi'(u_n), u_n \rangle &= \int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + u_n^2\right] dx - \int_{\mathbb{R}^N} \left(I_\alpha * F(u_n)\right) f(u_u) \, dx \\ &= \langle \Phi'(v), v \rangle + \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + v_n^2\right] dx - \frac{1}{p} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p\right)|v_n|^p \, dx \\ &\quad - \frac{1}{q} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^q\right)|v_n|^q \, dx + o(1). \end{aligned} \tag{33}$$

From $\langle \Phi'(v), v \rangle = 0$ and $\langle \Phi'(u_n), u_n \rangle \to 0$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + v_n^2\right] dx \\ &\quad = \frac{1}{p} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p\right)|v_n|^p \, dx + \frac{1}{q} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^q\right)|v_n|^q \, dx + o(1). \end{aligned} \tag{34}$$

We also have

$$\begin{aligned} \mathcal{P}(u_n) &= \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} u_n^2 \, dx - \frac{N+\alpha}{2} \int_{\mathbb{R}^N} \left(I_\alpha * F(u_n)\right) F(u_n) \, dx \\ &= \mathcal{P}(u) + \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} v_n^2 \, dx \\ &\quad - \frac{N-2}{2p} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p\right)|v_n|^p \, dx - \frac{N}{2q} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^q\right)|v_n|^q \, dx + o(1). \end{aligned}$$

Thus

$$\begin{aligned}&\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx+\frac{N}{2}\int_{\mathbb{R}^N}v_n^2\,dx\\&\quad=\frac{N}{2p}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^p\big)|v_n|^p\,dx+\frac{N-2}{2q}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,dx+o(1).\end{aligned}\tag{35}$$

Combining (34) and (35), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx=\frac{1}{q}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,dx+o(1),\\&\int_{\mathbb{R}^N}v_n^2\,dx=\frac{1}{p}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^p\big)|v_n|^p\,dx+o(1).\end{aligned}$$

We may assume that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx\to a,\qquad\frac{1}{q}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,dx\to a,\\&\int_{\mathbb{R}^N}v_n^2\,dx\to b,\qquad\frac{1}{p}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^p\big)|v_n|^p\,dx\to b,\end{aligned}$$

where a, b are nonnegative constants.

We claim that $a=b=0$. We prove this by taking off any other cases: (1) $a\neq0$, $b=0$; (2) $a=0$, $b\neq0$; (3) $a\neq0$, $b\neq0$. If $a\neq0$, by the definition of $\mathcal{S}_1$, we have

$$\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx\geq\mathcal{S}_1\biggl[\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,dx\biggr]^{1/q}.$$

It follows that $a\geq\mathcal{S}_1(q\cdot a)^{1/q}$, which yields

$$a\geq q^{\frac{1}{q-1}}\mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}}.\tag{36}$$

If $b\neq0$, by the definition of $\mathcal{S}_2$, we have

$$\int_{\mathbb{R}^N}|v_n|^2\,dx\geq\mathcal{S}_2\biggl[\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^p\big)|v_n|^p\,dx\biggr]^{1/p}.$$

It follows that $b\geq\mathcal{S}_2(p\cdot b)^{1/p}$, which yields

$$b\geq p^{\frac{1}{p-1}}\mathcal{S}_2^{\frac{N+\alpha}{\alpha}}.\tag{37}$$

Similarly to the discussion of (33), we have

$$\begin{aligned}\Phi(u_n)&=\frac{1}{2}\int_{\mathbb{R}^N}\big[|\nabla u_n|^2+u_n^2\big]\,dx-\frac{1}{2}\int_{\mathbb{R}^N}\big(I_\alpha*F(u_n)\big)F(u_n)\,dx\\&=\Phi(u)+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}v_n^2\,dx\\&\quad-\frac{1}{2p^2}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^p\big)|v_n|^p\,dx-\frac{1}{2q^2}\int_{\mathbb{R}^N}\big(I_\alpha*|v_n|^q\big)|v_n|^q\,dx+o(1).\end{aligned}$$

It follows from $\Phi(u) \geq 0$, and (36) or (37) that

$$\begin{aligned} c &= \lim_{n\to\infty} \Phi(u_n) \\ &\geq \lim_{n\to\infty} \left[\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} v_n^2 \, dx \right. \\ &\quad \left. - \frac{1}{2p^2} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^p \right) |v_n|^p \, dx - \frac{1}{2q^2} \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^q \right) |v_n|^q \, dx \right] \\ &= \frac{1}{2}\left(1 - \frac{1}{q}\right) q^{\frac{1}{q-1}} \mathcal{S}_1^{\frac{N+\alpha}{2+\alpha}} \text{ or } \frac{1}{2}\left(1 - \frac{1}{p}\right) p^{\frac{1}{p-1}} \mathcal{S}_2^{\frac{N+\alpha}{\alpha}}, \end{aligned}$$

which contradicts (24). Hence $a = b = 0$. This gives $v_n \to 0$ in $H_r^1(\mathbb{R}^n)$.

Combining Cases (i)–(iii), we can assume, going if necessary to a subsequence, $u_n \to u$ in $H_r^1(\mathbb{R}^N)$. Clearly, $\Phi'(u) = 0$ and $\Phi(u) = c$. Thus problem (P) has a nontrivial critical point u.

Then, by the same approaches which appear in [8, Sect. 4], we obtain Theorem 1.1.

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