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Infinitely many solutions for the discrete Schrödinger equations with a nonlocal term

Qilin Xie¹ and Huafeng Xiao^{2*} 

*Correspondence:

huafeng@gzhu.edu.cn

²School of Mathematics and Information Science, Guangzhou University, Guangzhou, China
Full list of author information is available at the end of the article

Abstract

In the present paper, we consider the following discrete Schrödinger equations

$$-\left(a + b \sum_{k \in \mathbb{Z}} |\Delta u_{k-1}|^2\right) \Delta^2 u_{k-1} + V_k u_k = f_k(u_k) \quad k \in \mathbb{Z},$$

where a, b are two positive constants and $V = \{V_k\}$ is a positive potential. $\Delta u_{k-1} = u_k - u_{k-1}$ and $\Delta^2 = \Delta(\Delta)$ is the one-dimensional discrete Laplacian operator. Infinitely many high-energy solutions are obtained by the Symmetric Mountain Pass Theorem when the nonlinearities $\{f_k\}$ satisfy 4-superlinear growth conditions. Moreover, if the nonlinearities are sublinear at infinity, we obtain infinitely many small solutions by the new version of the Symmetric Mountain Pass Theorem of Kajikiya.

Keywords: Solutions; Discrete Schrödinger equations; Kirchhoff type

1 Introduction

In the present paper, we consider the following discrete Schrödinger equations

$$-\left(a + b \sum_{k \in \mathbb{Z}} |\Delta u_{k-1}|^2\right) \Delta^2 u_{k-1} + V_k u_k = f_k(u_k) \quad k \in \mathbb{Z}, \quad (1)$$

where a, b are two positive constants and $V = \{V_k\}$ is a positive potential. $\Delta u_{k-1} = u_k - u_{k-1}$ and $\Delta^2 = \Delta(\Delta)$ is the one-dimensional discrete Laplacian operator.

The discrete Schrödinger equations play a significant role in many areas, such as nonlinear optics [7], biomolecular chains [11] and Bose–Einstein condensates [16]. If $a = 1$ and $b = 0$, problem (1) reduces to the classical discrete Schrödinger equations, which have been extensively studied by many authors in the past several decades. In Ma and Guo [17] and Zhang and Pankov [32], the authors studied the nontrivial solution of discrete Schrödinger equations with a coercive potential by variation methods. In Lin et al. [15], the authors considered a class of discrete nonlinear nonperiodic systems and obtained the existence of the homoclinic solutions. For more related works, we refer to [4–6, 18–20, 25, 28] and their references.

The problem (1) is a discrete case for a class of nonlocal problems and we call this nonlocal term a Kirchhoff type. This class of nonlocal equations is an extension of the classi-

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cal d'Alembert's wave equations because of the effects of the changes in the length of the string during the vibrations. There is a large number of papers concerning the solutions of Kirchhof-type problems in the continuous case, such as [9, 12, 26, 29, 31] and their references. As for the discrete equations with a nonlocal term, Yang and Liu in [30] investigated the existence of the nontrivial solutions via critical groups. In [3], Chakrone et al. considered the multiplicity results for a p -Laplacian discrete problem with Neumann boundary conditions. For more related works, we refer to [1, 2, 8, 13, 14, 22–24, 34] and their references. Recently, Wu and Huang [27] have studied the statistical solutions for nonautonomous discrete Klein–Gordon–Schrödinger-type equations.

Most of the previous works have been concerned with boundary value problems, while little has been done in discrete Schrödinger equations of the Kirchhoff type. We will consider the existence of infinitely many nontrivial solutions for this class of discrete equations. In this paper, the potential V satisfies the coercive condition:

(V_1) $V = \{V_k\}$ is positive and $\lim_{|k| \rightarrow +\infty} V_k = +\infty$.

Set $L := -\Delta^2 + V$, where $Lu_k = -\Delta^2 u_{k-1} + V_k u_k$, then L is a self-adjoint operator and unbounded in l^2 . With the help of the condition (V_1), the authors in [32] proved that the spectrum $\sigma(L)$ is discrete going to $+\infty$ and we assume that λ_i are the eigenvalues of L . If we set that

$$E = \left\{ u \in l^2 : \sum_{k \in \mathbb{Z}} V_k |u_k|^2 < +\infty \right\},$$

then E is a Hilbert space, in which the norm and the inner product are defined by

$$\|u\| := \left\{ \sum_{k \in \mathbb{Z}} (|\Delta u_{k-1}|^2 + V_k |u_k|^2) \right\}^{1/2} \quad \text{and} \quad \langle u, v \rangle_E = \sum_{k \in \mathbb{Z}} (\Delta u_{k-1} \Delta v_{k-1} + V_k u_k v_k),$$

respectively. Denote the standard norms of l^q by $\|\cdot\|_q$ for $q \in [2, +\infty]$. First, we consider the 4-superlinear nonlinearities as follows:

(f_1) $f_k \in C(\mathbb{R}, \mathbb{R})$ and there exist $C > 0$ and $p > 2$ such that

$$|f_k(t)| \leq C(|t| + |t|^{p-1}) \quad \text{for any } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

(f_2) $\lim_{|t| \rightarrow +\infty} \frac{F_k(t)}{t^4} = +\infty$ uniformly in k , where $F_k(t) := \int_0^t f_k(s) ds$.

(f_3) There exists $\alpha > 0$ such that

$$F_k(t) \leq \frac{1}{4} f_k(t) t + \alpha t^2 \quad \text{for any } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

(f'_3) There exists $L > 0$ such that

$$F_k(t) \leq \frac{1}{4} f_k(t) t \quad \text{for any } k \in \mathbb{Z} \text{ and } |t| \geq L.$$

(f_4) $f_k(-t) = -f_k(t)$ for any $k \in \mathbb{Z}$ and $t \in \mathbb{R}$.

If $u = \{u_k\}$ satisfies problem (1) for any $k \in \mathbb{Z}$, we call that u a solution of this problem. Moreover, if $u \neq 0$, we call that u a nontrivial solution. With the above assumptions (V_1)

and (f_1) , the energy functional $I: E \rightarrow \mathbf{R}$ is well defined by

$$I(u) = \frac{1}{2} \sum_{k \in \mathbf{Z}} (a |\Delta u_{k-1}|^2 + V_k u_k^2) + \frac{1}{4} b \left(\sum_{k \in \mathbf{Z}} |\Delta u_{k-1}|^2 \right)^2 - \sum_{k \in \mathbf{Z}} F_k(u_k).$$

Moreover, I is a class of $C^1(E, \mathbf{R})$ and

$$\langle I'(u), v \rangle = (a + b \|\Delta u\|_2^2) \sum_{k \in \mathbf{Z}} \Delta u_{k-1} \Delta v_{k-1} + \sum_{k \in \mathbf{Z}} V_k u_k v_k - \sum_{k \in \mathbf{Z}} f'_k(u_k) v_k,$$

for any u, v in E . Thus, u is a critical point of I in E if and only if u is a solution of problem (1). Now, the first result is stated concerning the infinitely many solutions when the nonlinearities are 4-superlinear growth.

Theorem 1.1 *If (V_1) and (f_1) – (f_4) hold, problem (1) possesses infinitely many nontrivial solutions $\{u^n\}$ in E with high energies, i.e., $I(u^n) \rightarrow +\infty$ as $n \rightarrow \infty$.*

It is evident to check that (f_1) and (f'_3) imply (f_3) . Hence, we have the following result.

Theorem 1.2 *If (V_1) , (f_1) , (f_2) , (f'_3) and (f_4) hold, problem (1) possesses infinitely many nontrivial solutions $\{u^n\}$ in E with high energies, i.e., $I(u^n) \rightarrow +\infty$ as $n \rightarrow \infty$.*

The above two results obtained infinitely many high-energy solutions are strictly dependent on the 4-superlinear growth assumption. The standard Symmetric Mountain Pass Theorem may be invalid without the superlinear assumption. In this case, we try to establish the existence of infinitely many small solutions by the new version of the Symmetric Mountain Pass Theorem of Kajikiya in [10] when the nonlinearities $f_k(t)$ are sublinear at infinity. More precisely, we assume that

$$(f_5) \quad f_k(t) = \kappa \xi_k |t|^{\kappa-2} t, \text{ where } \kappa \in (1, 2) \text{ is a constant and } \xi = \{\xi_k\} \in l^{\frac{2}{2-\kappa}}.$$

Theorem 1.3 *If (V_1) and (f_5) hold, problem (1) possesses infinitely many nontrivial solutions $\{u^n\}$ in E with small energy, i.e., $I(u^n) \rightarrow 0^-$ as $n \rightarrow \infty$.*

In fact, we will prove a more general result than the above theorem.

Theorem 1.4 *If (V_1) , (f_4) and the following assumption hold, (f_6) there exist $\kappa, \delta, \sigma \in (1, 2)$, $p > 2$ and C such that*

$$\xi_k |t|^\kappa \leq f_k(t) t$$

and

$$|f_k(t)| \leq w_k |t|^{\delta-1} + h_k |t|^{\sigma-1} + C |t|^{p-1} \quad \text{for any } k \in \mathbf{Z} \text{ and } t \in \mathbf{R},$$

where $\xi = \{\xi_k\}$, $w = \{w_k\}$, $h = \{h_k\}$ are three positive sequences satisfying $\xi \in l^{\frac{2}{2-\kappa}}$, $w \in l^{\frac{2}{2-\delta}}$ and $h \in l^{\frac{2}{2-\sigma}}$.

Then, problem (1) possesses infinitely many nontrivial solutions $\{u^n\}$ in E with small energies, i.e., $I(u^n) \rightarrow 0^-$ as $n \rightarrow \infty$.

Remark 1.1 It is obvious to check that Theorem 1.4 generalizes Theorem 1.3. For example, let

$$f_k(t) = \kappa \xi_k |t|^{\kappa-2} t + \sigma h_k |t|^{\sigma-2} t + C |t|^{p-2} t \quad \text{for } k \in \mathbf{Z} \text{ and } t \in \mathbf{R},$$

where $\kappa, \sigma \in (1, 2)$, $\xi = \{\xi_k\}$ and $h = \{h_k\}$ are two positive sequences satisfying $\xi \in l^{\frac{2}{2-\kappa}}$ and $h \in l^{\frac{2}{2-\sigma}}$. It is easy to see that f_k satisfy all the assumptions of Theorem 1.4, but f_k do not satisfy (f_5) .

2 The 4-superlinear case

The following Symmetric Mountain Pass Theorem is crucial in proving the existence of infinitely many solutions with 4-superlinear nonlinearities.

Proposition 2.1 (Symmetric Mountain Pass Theorem [21]) *Let X be an infinite-dimensional Banach space, $X = Y \oplus Z$, where Y is a finite-dimensional space. If $J \in C^1(X, \mathbf{R})$ satisfies the (Ce) condition, and*

(J₁) $J(0) = 0, J(-u) = J(u)$ for all $u \in X$;

(J₂) *there exist two constants $\delta, \alpha > 0$ such that $J|_{\partial B_\delta \cap Z} \geq \alpha$;*

(J₃) *for any finite-dimensional subspace $\tilde{X} \subset X$, there exists $R = R(\tilde{X}) > 0$ such that $J(u) \leq 0$ on $\tilde{X} \setminus B_R$,*

then J possesses an unbounded sequence of critical values.

In [21], the Symmetric Mountain Pass Theorem is established under the (PS) condition. Since the Deformation Theorem is still valid under the (Ce) condition, we see that the Symmetric Mountain Pass Theorem also holds under the (Ce) condition. The following embedding lemma, which follows from [17] or [33], plays a significant role in recovering the compactness result.

Lemma 2.1 *If V satisfies the condition (V_1) , then the embedding map from E into l^q is compact for $2 \leq q \leq +\infty$.*

With the help of Lemma 2.1, we have the following compactness result.

Lemma 2.2 *Under the assumption of Theorem 1.1, the functional I satisfies the (Ce) condition.*

Proof Suppose that $\{u^n\}$ is a (Ce) sequence,

$$|I(u^n)| \leq M, \quad (1 + \|u^n\|)I'(u^n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It suffices to prove that $\{u^n\}$ has a converging subsequence in E . We first obtain that $\{u_n\}$ is bounded in E . Otherwise, $\|u^n\| \rightarrow +\infty$ as $n \rightarrow \infty$. Let $v^n = u^n / \|u^n\|$. Moving, if necessary, to a subsequence, we assume $v^n \rightharpoonup v$ in E , by Lemma 2.1, $v^n \rightarrow v$ in l^q and $v_k^n \rightarrow v_k$ for any

$k \in \mathbf{Z}$ as $n \rightarrow \infty$. There are only two cases $v = 0$ or $v \neq 0$. If $v = 0$, by (f_3) , we have

$$\begin{aligned} \frac{1}{\|u^n\|^2}(M+1) &\geq \frac{1}{\|u^n\|^2} \left(I(u^n) - \frac{1}{4} \langle I'(u^n), u^n \rangle \right) \\ &\geq \frac{1}{4} \min\{a, 1\} + \frac{1}{\|u^n\|^2} \sum_{k \in \mathbf{Z}} \left(\frac{1}{4} f_k(u_k^n) u_k^n - F_k(u_k^n) \right) \\ &\geq \frac{1}{4} \min\{a, 1\} - \alpha \sum_{k \in \mathbf{Z}} |v_k^n|^2, \end{aligned}$$

which implies that $\frac{1}{4} \min\{a, 1\} \leq 0$. That is impossible. If $v \neq 0$, then $\Omega_1 := \{k \in \mathbf{Z} | v_k \neq 0\} \neq \emptyset$. For any $k \in \Omega_1$, we have $|u_k^n| \rightarrow +\infty$ as $n \rightarrow \infty$. By (f_2) , one obtains that for any $k \in \Omega_1$,

$$\frac{F_k(u_k^n)}{|u_k^n|^4} |v_k^n|^4 \rightarrow +\infty \quad \text{as } n \rightarrow \infty,$$

combined with Fatou's Lemma, which implies that

$$\sum_{k \in \Omega_1} \frac{F_k(u_k^n)}{|u_k^n|^4} |v_k^n|^4 \rightarrow +\infty \quad \text{as } n \rightarrow \infty. \quad (2)$$

It follows from (f_2) that there exists $L_1 > 0$ such that

$$F_k(t) \geq 0 \quad \text{for any } k \in \mathbf{Z} \text{ and } |t| \geq L_1. \quad (3)$$

By (f_1) , we obtain $|F_k(t)| \leq Ct^2$ for any $k \in \mathbf{Z}$ and $|t| \leq L_1$. Combining with (3), we have

$$F_k(t) \geq -Ct^2 \quad \text{for any } k \in \mathbf{Z} \text{ and } t \in \mathbf{R}.$$

Hence, we obtain

$$\sum_{k \in \mathbf{Z} \setminus \Omega_1} \frac{F_k(u_k^n)}{\|u^n\|^4} \geq -\frac{C}{\|u^n\|^4} \sum_{k \in \mathbf{Z} \setminus \Omega_1} |u_k^n|^2 \geq -C \frac{\|u^n\|_2^2}{\|u^n\|^4} \geq -\frac{C}{\|u^n\|^2},$$

which implies that

$$\liminf_{n \rightarrow \infty} \sum_{k \in \mathbf{Z} \setminus \Omega_1} \frac{F_k(u_k^n)}{\|u^n\|^4} \geq 0. \quad (4)$$

Note that

$$I(u^n) + \sum_{k \in \mathbf{Z}} F_k(u_k^n) \leq \frac{1}{2} \max\{a, 1\} \|u^n\|^2 + \frac{b}{4} \|u^n\|^4.$$

Dividing by $\|u^n\|^4$ on both sides and letting $n \rightarrow \infty$, we obtain

$$\begin{aligned} \frac{b}{4} &\geq \limsup_{n \rightarrow \infty} \sum_{k \in \mathbb{Z}} \frac{F_k(u_k^n)}{\|u^n\|^4} \\ &\geq \limsup_{n \rightarrow \infty} \left(\sum_{k \in \mathbb{Z} \setminus \Omega_1} \frac{F_k(u_k^n)}{\|u^n\|^4} + \sum_{k \in \Omega_1} \frac{F_k(u_k^n)}{\|u^n\|^4} \right) \rightarrow +\infty, \end{aligned}$$

via (2) and (4), which is impossible. In any case, we obtain a contradiction and hence $\{u^n\}$ is bounded in E .

Moving if necessary to a subsequence, we can assume $u^n \rightharpoonup u$ in E . It follows that

$$\begin{aligned} &\langle I'(u^n) - I'(u), u^n - u \rangle \\ &= (a + b \|\Delta u^n\|_2^2) \sum_{k \in \mathbb{Z}} \Delta u_{k-1}^n \Delta(u_{k-1}^n - u_{k-1}) + \sum_{k \in \mathbb{Z}} V_k u_k^n (u_k^n - u_k) \\ &\quad - (a + b \|\Delta u\|_2^2) \sum_{k \in \mathbb{Z}} \Delta u_{k-1} \Delta(u_{k-1}^n - u_{k-1}) - \sum_{k \in \mathbb{Z}} V_k u_k (u_k^n - u_k) \\ &\quad - \sum_{k \in \mathbb{Z}} (f_k(u_k^n) - f_k(u_k)) (u_k^n - u_k) \\ &= (a + b \|\Delta u^n\|_2^2) \sum_{k \in \mathbb{Z}} |\Delta(u_{k-1}^n - u_{k-1})|^2 + \sum_{k \in \mathbb{Z}} V_k |u_k^n - u_k|^2 \\ &\quad - b(\|\Delta u\|_2^2 - \|\Delta u^n\|_2^2) \sum_{k \in \mathbb{Z}} \Delta u_{k-1} \Delta(u_{k-1}^n - u_{k-1}) \\ &\quad - \sum_{k \in \mathbb{Z}} (f_k(u_k^n) - f_k(u_k)) (u_k^n - u_k) \\ &\geq \min\{a, 1\} \|u^n - u\|^2 - b(\|\Delta u\|_2^2 - \|\Delta u^n\|_2^2) \sum_{k \in \mathbb{Z}} \Delta u_{k-1} \Delta(u_{k-1}^n - u_{k-1}) \\ &\quad - \sum_{k \in \mathbb{Z}} (f_k(u_k^n) - f_k(u_k)) (u_k^n - u_k). \end{aligned}$$

One has

$$\begin{aligned} \min\{a, 1\} \|u^n - u\|^2 &\leq \langle I'(u^n) - I'(u), u^n - u \rangle \\ &\quad + \sum_{k \in \mathbb{Z}} (f_k(u_k^n) - f_k(u_k)) (u_k^n - u_k) \\ &\quad + b(\|\Delta u\|_2^2 - \|\Delta u^n\|_2^2) \sum_{k \in \mathbb{Z}} \Delta u_{k-1} \Delta(u_{k-1}^n - u_{k-1}). \end{aligned} \quad (5)$$

By the boundedness of $\{u^n\}$ and $u^n \rightharpoonup u$ in E , it is obvious that

$$\langle I'_n(u^n) - I'(u), u^n - u \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (6)$$

By (f₁), Lemma 2.1 and Lebesgue's dominated convergence theorem

$$\sum_{k \in \mathbb{Z}} (f_k(u_k^n) - f_k(u_k)) (u_k^n - u_k) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (7)$$

Let us consider the functional $P: E \rightarrow \mathbf{R}$,

$$P(w) = \sum_{k \in \mathbf{Z}} \Delta u_{k-1} \Delta w_{k-1}.$$

Since $|P(w)| \leq \|u\| \|w\|$, we can deduce that P is a continuous linear functional on E . By $u^n \rightharpoonup u$ in E , we obtain

$$P(u^n - u) = \sum_{k \in \mathbf{Z}} \Delta u_{k-1} \Delta (u_{k-1}^n - u_{k-1}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By the boundedness of $\{u^n\}$ in E , we have

$$b(\|\Delta u\|_2^2 - \|\Delta u^n\|_2^2) \sum_{k \in \mathbf{Z}} \Delta u_{k-1} \Delta (u_{k-1}^n - u_{k-1}) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (8)$$

It follows from (5)–(8) that $\|u^n - u\| \rightarrow 0$ as $n \rightarrow \infty$. Thus, $u^n \rightarrow u$ strongly in E as $n \rightarrow \infty$. \square

Let $\{e^j\}$ be an orthonormal basis of E and define $X_j = \text{span}\{e_j\}$, $Y_m = \bigoplus_{j=1}^m X_j$ and $Z_m = \overline{\bigoplus_{j=m+1}^{\infty} X_j}$ for any $m \in \mathbf{N}$.

Lemma 2.3 *Under the assumption (V_1) , for any $2 \leq q \leq +\infty$,*

$$\beta_m(q) := \sup_{u \in Z_m, \|u\|=1} \|u\|_q \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (9)$$

Proof It is obvious that $0 < \beta_{m+1}(q) \leq \beta_m(q)$, so that $\beta_m(q) \rightarrow \beta(q) \geq 0$ as $m \rightarrow \infty$. For every $m \in \mathbf{N}$, there exists $u^m \in Z_m$ with $\|u^m\| = 1$ such that

$$\|u^m\|_q > \frac{\beta_m(q)}{2}. \quad (10)$$

For any $w \in E$, $w = \sum_{j=1}^{\infty} c_j e_j$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\langle u^m, w \rangle| &= \left| \left\langle u^m, \sum_{j=1}^{\infty} c_j e_j \right\rangle \right| = \left| \left\langle u^m, \sum_{j=m}^{\infty} c_j e_j \right\rangle \right| \\ &\leq \|u^m\| \left\| \sum_{j=m}^{\infty} c_j e_j \right\| = \left\| \sum_{j=m}^{\infty} c_j e_j \right\| \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$, which implies that $u^m \rightarrow 0$ in E . The compact embedding of $E \hookrightarrow l^q$, $q \in [2, \infty]$, implies that $u^m \rightarrow 0$ in l^q . Let $m \rightarrow \infty$ in (10) and we obtain $\beta_m(q) \rightarrow 0$ as $m \rightarrow \infty$. \square

The proof of Theorem 1.1 We will make use of the Symmetric Mountain Pass Theorem and Proposition 2.1 to prove Theorem 1.1. It is easy to see that (J_1) follows from the condition (f_4) . It follows from (f_1) that

$$F_k(t) \leq c_1 t^2 + c_2 |t|^p \quad \text{for any } k \in \mathbf{Z} \text{ and } t \in \mathbf{R}.$$

By Lemma 2.3, there exists $m \in \mathbb{N}$ large enough such that

$$\|u\|_2^2 \leq \frac{\min\{a, 1\}}{4c_1} \|u\|^2 \quad \text{and} \quad \|u\|_p^p \leq \frac{\min\{a, 1\}}{4c_2} \|u\|^p \quad \text{for any } u \in Z_m.$$

It follows from the above three inequalities that

$$\begin{aligned} I(u) &= \frac{1}{2} \min\{a, 1\} \|u\|^2 - \sum_{k \in \mathbb{Z}} F_k(u_k^n) \\ &\geq \frac{1}{2} \min\{a, 1\} \|u\|^2 - c_1 \|u\|_2^2 - c_2 \|u\|_p^p \\ &\geq \frac{1}{4} \min\{a, 1\} \|u\|^2 (1 - \|u\|^{p-2}). \end{aligned}$$

It follows from $p > 2$ that there exist $\delta_1, \alpha > 0$ such that $I|_{\partial B_{\delta_1} \cap Z_m} \geq \alpha$. Thus, (J_2) holds.

It remains to prove (J_3) . Since all norms are equivalent in a finite-dimensional space, there exists c_4 such that $\|u\|_4^4 \geq c_4 \|u\|^4$ for any $u \in \tilde{E}$.

By (f_2) , for any $M > \frac{b}{4c_4}$, there exists $L_1 > 0$ such that $F_k(t) \geq Mt^4$ for $|t| \geq L_1$. It follows from (f_1) that there exists $C_1 > 0$ such that $F_k(t) \geq -C_1 t^2$ for $|t| \leq L_1$. From the above two inequalities, it follows that for any $k \in \mathbb{Z}$,

$$F_k(t) \geq Mt^4 - C_M t^2 \quad \text{for any } k \in \mathbb{Z} \text{ and } t \in \mathbb{R},$$

where $C_M = C_1 + ML_1^2$. It follows that

$$\begin{aligned} I(u) &\leq \frac{1}{2} \max\{a, 1\} \|u\|^2 + \frac{b}{4} \|u\|^4 - \sum_{k \in \mathbb{Z}} F_k(u_k) \\ &\leq \frac{1}{2} \max\{a, 1\} \|u\|^2 + \frac{b}{4} \|u\|^4 - M \|u\|_4^4 + C_M \|u\|_2^2 \\ &\leq C \|u\|^2 - \left(Mc_4 - \frac{b}{4} \right) \|u\|^4, \end{aligned}$$

for all $u \in \tilde{E}$. Hence, there exists a large $R = R(\tilde{E})$ such that $I(u) \leq 0$ on $\tilde{E} \setminus B_R$. This completes the proof. \square

3 The proof of Theorem 1.4

To deal with the sublinear case in Theorem 1.4, we need the following Symmetric Mountain Pass Theorem of Kajikiya [10]. Let X be a Banach space and A be a subset of X . Denote by Γ the family of closed symmetric subsets A of X ,

$$\Gamma = \{A \in X \setminus \{0\} | A \text{ is closed and symmetric with the origin}\}.$$

We define the \mathbb{Z}_2 -genus of $A \in \Gamma$ by $\gamma(A)$. Set $\Gamma_m = \{A \in \Gamma | \gamma(A) \geq m\}$.

Proposition 3.1 *Let X be an infinite-dimensional Banach space and $J \in C^1(X, \mathbb{R})$ satisfies the following conditions:*

- (a) $J(u)$ is even, bounded from below, $J(0) = 0$ and J satisfies the (PS) condition.
- (b) For any $m \in \mathbb{N}$, there exists $A_m \in \Gamma_m$ such that $\sup_{u \in A_m} J(u) < 0$.

Then, J has a sequence of critical points $\{u^n\}$ such that $J(u^n) \leq 0$, $u^n \neq 0$ and $\lim_{n \rightarrow \infty} u^n = 0$.

Lemma 3.1 *For the finite-dimensional subspace \tilde{E} of E , there exists $\varepsilon > 0$ such that*

$$\text{meas}\{k \in \mathbf{Z} \mid |\xi_k| |u_k|^\kappa \geq \varepsilon \|u\|^\kappa\} \geq 1 \quad \text{for any } u \in \tilde{E} \setminus \{0\}, \quad (11)$$

where κ and ξ_k are given in (f_6) .

Proof Arguing indirectly, if not, for any positive integer n there exist $u^n \in \tilde{E}$ such that

$$\text{meas}\left\{k \in \mathbf{Z} \mid |\xi_k| |u_k^n|^\kappa \geq \frac{1}{n} \|u^n\|^\kappa\right\} = 0.$$

By the above inequality, we can assume that $\|u^n\| = 1$ and

$$\text{meas}\left\{k \in \mathbf{Z} \mid |\xi_k| |u_k^n|^\kappa \geq \frac{1}{n}\right\} = 0.$$

By the compactness of the unit sphere of the finite-dimensional subspace \tilde{E} , there exists a subsequence such that $u^n \rightarrow u^0$ in \tilde{E} and $u^n \rightarrow u^0$ in l^2 . By the Hölder inequality, we have

$$\sum_{k \in \mathbf{Z}} \xi_k |u_k^n - u_k^0|^\kappa \leq \left(\sum_{k \in \mathbf{Z}} |\xi_k|^{\frac{2}{2-\kappa}} \right)^{\frac{2-\kappa}{2}} \left(\sum_{k \in \mathbf{Z}} |u_k^n - u_k^0|^2 \right)^{\frac{\kappa}{2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (12)$$

Since $\|u^0\| = 1$, there exists $d_1 > 0$ such that $\text{meas}\{k \in \mathbf{Z} \mid |\xi_k| |u_k^0|^\kappa \geq d_1\} \geq 1$. Otherwise, one has $\text{meas}\{k \in \mathbf{Z} \mid |\xi_k| |u_k^0|^\kappa \geq \frac{1}{n}\} = 0$, which implies that

$$0 \leq \sum_{k \in \mathbf{Z}} \xi_k |u_k^0|^{\kappa+2} \leq \frac{1}{n} \sum_{k \in \mathbf{Z}} |u_k^0|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, $u^0 = 0$, which is impossible. Set that $\Omega_n = \{k \in \mathbf{Z} \mid |\xi_k| |u_k^n|^\kappa < \frac{1}{n}\}$ and $\Omega_0 = \{k \in \mathbf{Z} \mid |\xi_k| |u_k^0|^\kappa \geq d_1\}$. Thus, $\text{meas}(\Omega_0 \cap \Omega_n) \geq \text{meas}(\Omega_0) - \text{meas}(\Omega_n^c) \geq 1$. Then, for n large, we obtain

$$\begin{aligned} \sum_{k \in \mathbf{Z}} \xi_k |u_k^n - u_k^0|^\kappa &\geq \sum_{\Omega_0 \cap \Omega_n} \xi_k |u_k^n - u_k^0|^\kappa \geq \frac{1}{2^\kappa} \sum_{\Omega_0 \cap \Omega_n} \xi_k |u_k^0|^\kappa - \sum_{\Omega_0 \cap \Omega_n} \xi_k |u_k^n|^\kappa \\ &\geq \left(\frac{d_1}{2^\kappa} - \frac{1}{n} \right) \text{meas}(\Omega_0 \cap \Omega_n) \geq \left(\frac{d_1}{2^\kappa} - \frac{1}{n} \right), \end{aligned}$$

which contradicts (12). \square

The proof of Theorem 1.4 We consider the truncated functional

$$\varphi(u) = \frac{1}{2} \sum_{k \in \mathbf{Z}} (a |\Delta u_{k-1}|^2 + V_k u_k^2) + \frac{1}{4} b \left(\sum_{k \in \mathbf{Z}} |\Delta u_{k-1}|^2 \right)^2 - h(\|u\|) \sum_{k \in \mathbf{Z}} F_k(u_k),$$

for all $u \in E$, where $h \in C^1(\mathbf{R}^+, \mathbf{R}^+)$ such that $0 \leq h \leq 1$, $h(t) = 1$ for $0 \leq t \leq 1$ and $h(t) = 0$ for $t \geq 2$. To obtain Theorem 1.4, it suffices to show that φ possesses a sequence of critical points $\{u^n\}$ such that $u^n \rightarrow 0$ as $n \rightarrow \infty$.

By (f_4) , we have $\varphi(-u) = \varphi(u)$. For any $u \in E$ with $\|u\| = 2$, one has

$$\varphi(u) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (a|\Delta u_{k-1}|^2 + V_k u_k^2) + \frac{1}{4} b \left(\sum_{k \in \mathbb{Z}} |\Delta u_{k-1}|^2 \right)^2 \geq \frac{1}{2} \min\{a, 1\} \|u\|^2,$$

which implies that $\varphi(u)$ is bounded from below and coercive. By condition (f_6) and Lemma 2.1, we can prove that φ satisfies the (PS) condition. For any Y_m , by Lemma 3.1, there exists $\varepsilon_m > 0$ such that

$$\text{meas}\{k \in \mathbb{Z} | \xi_k |u_k|^\kappa \geq \varepsilon_m \|u\|^\kappa\} \geq 1 \quad \text{for any } u \in Y_m \setminus \{0\}.$$

Let $\Omega_u = \{k \in \mathbb{Z} | \xi_k |u_k|^\kappa \geq \varepsilon_m \|u\|^\kappa\}$. For any $u \in Y_m \setminus \{0\}$, with $\|u\| \leq \min\{1, (\frac{\varepsilon_m}{2\kappa(a+b+1)})^{1/(2-\kappa)}\}$, we deduce that

$$\begin{aligned} \varphi(u) &\leq \frac{1}{2} \sum_{k \in \mathbb{Z}} (a|\Delta u_{k-1}|^2 + V_k u_k^2) + \frac{1}{4} b \left(\sum_{k \in \mathbb{Z}} |\Delta u_{k-1}|^2 \right)^2 - \sum_{k \in \mathbb{Z}} F_k(u_k) \\ &\leq \frac{1}{2} \max\{a, 1\} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{1}{\kappa} \sum_{k \in \mathbb{Z}} \xi_k |u_k|^\kappa \\ &\leq \frac{1}{2} \max\{a, 1\} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{1}{\kappa} \sum_{k \in \Omega_u} \xi_k |u_k|^\kappa \\ &\leq (a + b + 1) \|u\|^2 - \frac{1}{\kappa} \varepsilon_m \|u\|^\kappa \text{meas}(\Omega_u) \\ &\leq -(a + b + 1) \|u\|^2. \end{aligned}$$

Then, choosing $0 < d_m \leq \min\{1, (\frac{\varepsilon_m}{2\kappa(a+b+1)})^{1/(2-\kappa)}\}$, the above inequality implies that

$$\{u \in Y_m | \|u\| = d_m\} \subset \{u \in E | \varphi(u) \leq -(a + b + 1)d_m^2\}.$$

Setting that $A_m = \{u \in E | \varphi(u) \leq -(a + b + 1)d_m^2\}$, we have

$$\gamma(A_m) \geq \gamma(\{u \in Y_m | \|u\| = d_m\}) \geq m.$$

Hence, $A_m \in \Gamma_m$ and $\sup_{u \in A_m} I(u) \leq -(a + b + 1)d_m^2 < 0$. This completes the proof. \square

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Availability of data and materials

The data and material in this paper are original.

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to this work. All authors read and approved the final manuscript.

Author details

¹School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou, China. ²School of Mathematics and Information Science, Guangzhou University, Guangzhou, China.

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