Du *Boundary Value Problems* (2022) 2022:6
https://doi.org/10.1186/s13661-022-01587-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

On three-dimensional Hall-magnetohydrodynamic equations with partial dissipation

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Abstract

In this paper, we address the Hall-MHD equations with partial dissipation. Applying some important inequalities (such as the logarithmic Sobolev inequality using BMO space, bilinear estimates in BMO space, Young's inequality, cancellation property, interpolation inequality) and delicate energy estimates, we establish an improved blow-up criterion for the strong solution. Moreover, we also obtain the existence of the strong solution for small initial data, the smallness conditions of which are given by the suitable Sobolev norms.

MSC: 35L60; 35K55; 35Q80

Keywords: Hall-magnetohydrodynamics; Blow-up criterion; BMO norms; Small initial data; Partial dissipation

1 Introduction

The incompressible Hall-magnetohydrodynamic equations with full dissipation in three dimensions read as:

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla(p+\pi) = \kappa_1 u_{x_1x_1} + \kappa_2 u_{x_2x_2} + \kappa_3 u_{x_3x_3} + (B\cdot\nabla)B, \\ \operatorname{div} u = 0, \\ B_t + (u\cdot\nabla)B = (B\cdot\nabla)u + \Delta B - \nabla\times((\nabla\times B)\times B), \\ \operatorname{div} B = 0, \\ u(0,x) = u_0, \qquad B(0,x) = B_0. \end{cases} \tag{1}$$

Here $u(t,x)$, $B(t,x)$ denote velocity field and magnetic field, respectively; κ_1, κ_2, κ_3 are the kinematic viscosity, $(t,x)\in\mathbb{R}^+\times\mathbb{R}^3$.

Compared to usual MHD system and the Boussinesq equations, Hall-MHD equations involve $\nabla\times((\nabla\times B)\times B)$, it is Hall term and plays a crucial position in magnetic reconnection due to Ohm's law. Magnetic reconnection corresponds to changes in the topology of magnetic field lines, which are ubiquitously observed in space. The Hall term becomes important when large magnetic shear appears because it has second-order derivatives, and it restores the influence of the electric current in the Lorentz force occurring

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in Ohm's law, which was neglected in usual MHD. Therefore, Hall-MHD is very important for such problems as magnetic reconnection in neutron stars, geo-dynamo, space plasmas, and star formation. The paper [1] introduces the physical background to Hall-magnetohydrodynamics, and papers [7, 8, 13, 15–18, 24] present the recent progress of the Hall-MHD system.

The nonlinear Jordan–Moore–Gibson–Thompson equation with memory read as

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - \beta\Delta u_t - \int_0^t h(t-s)\Delta u(s)\,ds = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right),$$

where $u = u(x,t)$ denotes the scalar acoustic velocity. The Jordan–Moore–Gibson–Thompson equation is one of the nonlinear sound equations that describe the propagation of sound waves in gases and liquids. Recent works on the Jordan–Moore–Gibson–Thompson equation can be found in [4, 12]. The Hall-MHD Eqs. (1) describe the magnetic properties for a conductive fluid moving in a magnetic field, in which magnetic reconnection happens in the case of large magnetic shear. In the Hall-MHD Eqs. (1), $u = u(x,t)$, $B = B(x,t)$ are non-dimensional quantities corresponding to the fluid velocity field, the magnetic field.

Many results on usual MHD system have been obtained in [10, 11, 14, 17, 21–23, 26–28, 30–33]. However, the Hall-MHD system had few results until recently. The paper [7] got the local existence and global small solutions for the Hall-magnetohydrodynamics. Some results on the Boussinesq and MHD equations with partial viscosity were obtained in [5, 6, 15, 24]. Two new blow-up criteria for the system (1) with $\kappa_1 = \kappa_2 = \kappa_3 = 1$ were obtained by Chae and Lee in [8]. Fei and Xiang [19] got a blow-up criterion and small existence to (1) with $\kappa_1 = \kappa_2 = 1$, $\kappa_3 = 0$.

The paper [20] established regularity criterion for the Hall-MHD equations without viscosity and full dissipation, the papers [2, 3] obtained regularity criterion for the Hall-MHD equations with full viscosity and full dissipation in different spaces. In this paper, we investigate the Hall-magnetohydrodynamic system with full viscosity and partial dissipation. Inspired by [8, 13, 19, 33], we find a new blow-up criterion for strong solution, which imposes the condition is $(u, \nabla B) \in L^2(0,T;\mathrm{BMO})$. Additionally, we also get the existence of the strong solution for small initial data.

The first aim of this paper is to get blow-up criterion for the strong solution to (1) with $\kappa_1, \kappa_2 > 0$, $\kappa_3 = 0$.

Theorem 1.1 *Assume that* $\kappa_1 > 0$, $\kappa_2 > 0$, $\kappa_3 = 0$, $(u_0, B_0) \in H^3(\mathbb{R}^3)$, *and* $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$, *let* $T_0 < \infty$ *be the first blow-up time to the problem* (1), *then*

$$\limsup_{t\nearrow T_0}\left(\|u(t)\|_{H^3}^2 + \|B(t)\|_{H^3}^2\right) = \infty,$$

is equivalent to

$$\int_0^{T_0}\left(\|u(t)\|_{\mathrm{BMO}}^2 + \|\nabla B(t)\|_{\mathrm{BMO}}^2\right)dt = \infty.$$

Remark 1.1 Compared to previous results, the blow-up condition $\int_0^{T_0}(\|u\|_{\mathrm{BMO}}^2 + \|\nabla B\|_{\mathrm{BMO}}^2)\,dt < \infty$ instead of $\int_0^{T^*}(\|\nabla u\|_{L^p}^q + \|\Delta B\|_{L^\beta}^\gamma)\,dt < \infty$ with $p, \beta \in (3,\infty]$ in [19].

Noticing the fact $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \hookrightarrow \mathrm{BMO}(\mathbb{R}^3)$, $p > 3$, thus the above blow-up criterion is meaningful.

Remark 1.2 The similar blow-up criterion can also be established for the system (1) with cases when $\kappa_1 = 0, \kappa_2, \kappa_3 > 0$ and $\kappa_1 > 0, \kappa_2 = 0, \kappa_3 > 0$.

Based on the Theorem 1.1, we can obtain the small initial data solutions to (1) with $\kappa_1, \kappa_2 > 0, \kappa_3 = 0$.

Theorem 1.2 *Suppose the conditions in Theorem 1.1 hold, there exists a universal positive constant ε^*, then (1) has a solution $(u,B) \in L^\infty(0,\infty; H^3(\mathbb{R}^3))$, provided that $\|u_0\|_{H^2} + \|B_0\|_{H^2} < \varepsilon^*$.*

Remark 1.3 Compared to [19], the smallness condition $\|(u_0,B_0)\|_{H^2}$ instead of $\|(u_0,B_0)\|_{H^3}$ in [19] is sufficiently small.

2 Notations and preliminaries

Through the paper, ∂_k and u_k represent the k th components of ∇ and u, and the following simplified notation will be adopted throughout the paper:

$$\int_{\mathbb{R}^3} f\,dx := \int\int\int_{\mathbb{R}^3} f(t,x)\,dx_1\,dx_2\,dx_3; \qquad \|\cdot\|_k := \|\cdot\|_{L^k};$$
$$f_0 := f(0,x); \qquad \nabla_p := (\partial_1,\partial_2,0); \qquad u_p := (u_1,u_2,0).$$

Next, some lemmas are given.

Lemma 2.1 (See [9]) *Let $f, g, h, \nabla_p f, \nabla_p g, \partial_3 h \in L^2(\mathbb{R}^3)$, then*

$$\int_{\mathbb{R}^3} fgh\,dx \le C\|f\|_2^{\frac{1}{2}}\|\nabla_p f\|_2^{\frac{1}{2}}\|g\|_2^{\frac{1}{2}}\|\nabla_p g\|_2^{\frac{1}{2}}\|h\|_2^{\frac{1}{2}}\|\partial_3 h\|_2^{\frac{1}{2}}.$$

Lemma 2.2 (See [29]) *Suppose $\nabla g \in W^{1,q}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, then*

$$\|\nabla g\|_{L^\infty} \le C\big[\|\nabla g\|_{\mathrm{BMO}} \ln^{\frac{1}{2}}\big(e + \|\nabla g\|_{W^{1,q}} + \|g\|_{L^\infty}\big) + 1\big],$$

here $q > 3$.

Lemma 2.3 (See [25, Lemma 1]) *The bilinear estimates in BMO space, let $h_1, h_2 \in \mathrm{BMO} \cap H^{|\zeta|+|\eta|}$. Then*

$$\big\|\partial^\zeta h_1 \cdot \partial^\eta h_2\big\|_2 \le C\big(\|h_1\|_{\mathrm{BMO}}\big\|(-\Delta)^{\frac{|\zeta|+|\eta|}{2}} h_2\big\|_2 + \|h_2\|_{\mathrm{BMO}}\big\|(-\Delta)^{\frac{|\zeta|+|\eta|}{2}} h_1\big\|_2\big),$$

where $\zeta = (\zeta_1,\zeta_2,\zeta_3)$, $\eta = (\eta_1,\eta_2,\eta_3)$, and $|\zeta|, |\eta| \ge 1$.

3 Proof of Theorem 1.1

We adopt the following notations: $\nabla^\varrho := \partial^{|\varrho|}/\partial_1^{\varrho_1}\partial_2^{\varrho_2}\partial_3^{\varrho_3}$, where $\varrho = (\varrho_1,\varrho_2,\varrho_3) \in (\mathbb{N} \cup \{0\})^3$ with $|\varrho| = \varrho_1 + \varrho_2 + \varrho_3 \le 3$, $\kappa := \min\{\kappa_1,\kappa_2\}$ and $\kappa_0 := \min\{\kappa, 1\}$.

Operating ∇^ϱ on $(1)_1$ and $(1)_3$, and multiplying them by $\nabla^\varrho u$ and $\nabla^\varrho B$, respectively, and then integrating by parts, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|u(t)\|_{H^3}^2+\|B(t)\|_{H^3}^2\big)+\kappa_1\|\partial_1 u\|_{H^3}^2+\kappa_2\|\partial_2 u\|_{H^3}^2+\|\nabla B\|_{H^3}^2\\
&\quad=-\sum_{0\le|\varrho|\le 3}\int_{\mathbb{R}^3}\nabla^\varrho\big[\nabla\times\big((\nabla\times B)\times B\big)\big]\cdot\nabla^\varrho B\,dx\\
&\qquad-\sum_{0\le|\varrho|\le 3}\int_{\mathbb{R}^3}\nabla^\varrho(u\cdot\nabla B)\cdot\nabla^\varrho B\,dx-\sum_{0\le|\varrho|\le 3}\int_{\mathbb{R}^3}\nabla^\varrho(u\cdot\nabla u)\cdot\nabla^\varrho u\,dx\\
&\qquad+\sum_{0\le|\varrho|\le 3}\int_{\mathbb{R}^3}\nabla^\varrho(B\cdot\nabla u)\cdot\nabla^\varrho B\,dx+\sum_{0\le|\varrho|\le 3}\int_{\mathbb{R}^3}\nabla^\varrho(B\cdot\nabla B)\cdot\nabla^\varrho u\,dx\\
&\quad:=H_1+H_2+H_3+H_4+H_5.
\end{aligned}\tag{2}$$

Using Lemma 2.2, we have

$$\|\nabla B\|_{L^\infty}\le C\big[\|\nabla B\|_{\mathrm{BMO}}\ln^{\frac{1}{2}}\big(e+\|\nabla B\|_{W^{1,\frac{7}{2}}}+\|B\|_{L^\infty}\big)+1\big].$$

Noticing the fact that $H^2(\mathbb{R}^3)\hookrightarrow W^{1,\frac{7}{2}}(\mathbb{R}^3)$, and $H^3(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, we get

$$\|\nabla B\|_{L^\infty}\le C\big[\|\nabla B\|_{\mathrm{BMO}}\ln^{\frac{1}{2}}\big(e+\|B\|_{H^3}\big)+1\big].$$

By the above inequality, cancellation property and Young's inequality, one obtains

$$\begin{aligned}
|H_1|&\le C\|B\|_{H^3}\|\nabla B\|_\infty\|\nabla B\|_{H^3}\\
&\le\frac{1}{8}\|\nabla B\|_{H^3}^2+C\|\nabla B\|_\infty^2\|B\|_{H^3}^2\\
&\le\frac{1}{8}\|\nabla B\|_{H^3}^2+C\big(\ln\big(e+\|B\|_{H^3}\big)\|\nabla B\|_{\mathrm{BMO}}^2+1\big)\|B\|_{H^3}^2.
\end{aligned}\tag{3}$$

We apply cancellation property and Lemma 2.3 to deduce that

$$\begin{aligned}
|H_2|&=\bigg|\sum_{0\le|\varrho|\le 3}\int_{\mathbb{R}^3}\big[\nabla^\varrho(u\cdot\nabla B)-(u\cdot\nabla)\nabla^\varrho B\big]\cdot\nabla^\varrho B\,dx\bigg|\\
&\le C\big(\|B\|_{H^3}\|\nabla B\|_{H^3}\|u\|_{\mathrm{BMO}}+\|B\|_{H^3}\|u\|_{H^3}\|\nabla B\|_{\mathrm{BMO}}\big)\\
&\le\frac{1}{8}\|\nabla B\|_{H^3}^2+C\|u\|_{\mathrm{BMO}}^2\|B\|_{H^3}^2+C\|\nabla B\|_{\mathrm{BMO}}\|u\|_{H^3}\|B\|_{H^3}.
\end{aligned}\tag{4}$$

For H_3, when $|\varrho|=0$, the H_3 have cancelled. When $|\varrho|=1$, by $\operatorname{div} u=0$, H_3 can be rewritten as follows

$$\begin{aligned}
H_{31}&=-\int_{\mathbb{R}^3}(\nabla u\cdot\nabla)u\cdot\nabla u\,dx\\
&=-\int_{\mathbb{R}^3}(\nabla_p u\cdot\nabla)u\nabla_p u\,dx-\int_{\mathbb{R}^3}(\partial_3 u_p\cdot\nabla_p)u\partial_3 u\,dx+\int_{\mathbb{R}^3}(\nabla_p\cdot u_p)\partial_3 u\partial_3 u\,dx.
\end{aligned}$$

Thus using the Höledr inequality and Lemma 2.3, one obtains

$$
\begin{aligned}
|H_{31}| &\leq C\|\nabla u \nabla_p u\|_2 \|\nabla u\|_2 \\
&\leq C\|\nabla_p \nabla u\|_2 \|u\|_{\mathrm{BMO}} \|\nabla u\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}} \|\nabla_p u\|_{H^1} \|u\|_{H^1} \\
&\leq \frac{3\kappa}{36} \|\nabla_p u\|_{H^1}^2 + C\|u\|_{\mathrm{BMO}}^2 \|u\|_{H^1}^2.
\end{aligned} \tag{5}
$$

When $|\varrho| = 2$, one can write H_3 as

$$
\begin{aligned}
H_{32} &= -\int_{\mathbb{R}^3} \left(\nabla^2 u \cdot \nabla\right) u \nabla^2 u \, dx - 2\int_{\mathbb{R}^3} (\nabla u \cdot \nabla) \nabla u \nabla^2 u \, dx \\
&= H_{321} + H_{322}
\end{aligned}
$$

H_{321}, H_{322} can be further decomposed into three parts, respectively.

$$
\begin{aligned}
H_{321} &= -\int_{\mathbb{R}^3} (\nabla \nabla_p u \cdot \nabla) u \nabla \nabla_p u \, dx - \int_{\mathbb{R}^3} \left(\partial_3^2 u_p \cdot \nabla_p\right) u \partial_3^2 u \, dx \\
&\quad + \int_{\mathbb{R}^3} (\partial_3 \nabla_p \cdot u_p) \partial_3 u \partial_3^2 u \, dx \\
&= H_{3211} + H_{3212} + H_{3213}. \\
H_{322} &= -2\int_{\mathbb{R}^3} (\nabla_p u \cdot \nabla) \nabla u \nabla \nabla_p u \, dx - 2\int_{\mathbb{R}^3} (\partial_3 u_p \cdot \nabla_p) \nabla u \partial_3^2 u \, dx \\
&\quad + 2\int_{\mathbb{R}^3} (\nabla_p \cdot u_p) \partial_3 \nabla u \partial_3^2 u \, dx \\
&= H_{3221} + H_{3222} + H_{3223}.
\end{aligned}
$$

By the Höledr inequality and Lemma 2.3, we have

$$
\begin{aligned}
|H_{3211}| &\leq C\|\nabla_p u \nabla u\|_2 \left\|\nabla^2 u\right\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}} \left\|\nabla_p \nabla^2 u\right\|_2 \left\|\nabla^2 u\right\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}} \|\nabla_p u\|_{H^2} \|u\|_{H^2} \\
&\leq \frac{\kappa}{36} \|\nabla_p u\|_{H^2}^2 + C\|u\|_{\mathrm{BMO}}^2 \|u\|_{H^2}^2,
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
|H_{3212}| &\leq C\left\|\nabla_p u \partial_3^2 u\right\|_2 \left\|\partial_3^2 u\right\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}} \left\|\nabla_p \nabla^2 u\right\|_2 \left\|\nabla^2 u\right\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}} \|\nabla_p u\|_{H^2} \|u\|_{H^2} \\
&\leq \frac{\kappa}{36} \|\nabla_p u\|_{H^2}^2 + C\|u\|_{\mathrm{BMO}}^2 \|u\|_{H^2}^2.
\end{aligned} \tag{7}
$$

Similarly to H_{3211} and H_{3212}, we have

$$
|H_{3213}, H_{3221}, H_{3222}, H_{3223}| \leq \frac{\kappa}{36} \|\nabla_p u\|_{H^2}^2 + C\|u\|_{\mathrm{BMO}}^2 \|u\|_{H^2}^2. \tag{8}
$$

Collecting (7), (8), and (9), we have

$$|H_{32}| \leq \frac{6\kappa}{36}\|\nabla_h u\|_{H^2}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^2}^2. \tag{9}$$

When $|\varrho| = 3$, we rewrite H_3 as follows

$$\begin{aligned}
H_{33} &= -\int_{\mathbb{R}^3}\big(\nabla^3 u\cdot\nabla\big)u\cdot\nabla^3 u\,dx - 3\int_{\mathbb{R}^3}\big(\nabla^2 u\cdot\nabla\big)\nabla u\cdot\nabla^3 u\,dx \\
&\quad - 3\int_{\mathbb{R}^3}(\nabla u\cdot\nabla)\nabla^2 u\cdot\nabla^3 u\,dx \\
&= H_{331} + H_{332} + H_{3233}.
\end{aligned}$$

Since

$$\begin{aligned}
H_{331} &= -\int_{\mathbb{R}^3}\big(\nabla^2\nabla_p u\cdot\nabla\big)u\nabla^2\nabla_p u\,dx - \int_{\mathbb{R}^3}\big(\partial_3^3 u_p\cdot\nabla_p\big)u\partial_3^3 u\,dx \\
&\quad + \int_{\mathbb{R}^3}\big(\partial_3^2\nabla_p\cdot u_p\big)\partial_3 u\partial_3^3 u\,dx \\
&= H_{3311} + H_{3312} + H_{3313},
\end{aligned}$$

and

$$\begin{aligned}
H_{332} &= -3\int_{\mathbb{R}^3}\big(\nabla^2 u\cdot\nabla\big)\nabla_p u\nabla^2\nabla_p u\,dx - 3\int_{\mathbb{R}^3}\big(\partial_3^2 u_p\cdot\nabla_p\big)\partial_3 u\partial_3^3 u\,dx \\
&\quad + 3\int_{\mathbb{R}^3}(\partial_3\nabla_p\cdot u_p)\partial_3^2 u\partial_3^3 u\,dx \\
&= H_{3321} + H_{3322} + H_{3323}.
\end{aligned}$$

Applying the Höledr inequality, Lemma 2.3, and Young's inequality, one has

$$\begin{aligned}
|H_{3311}| &\leq C\big\|\nabla_p\nabla^2 u\nabla u\big\|_2\big\|\nabla_p\nabla^2 u\big\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}}\big\|\nabla^3\nabla_p u\big\|_2\big\|\nabla_p\nabla^2 u\big\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}}\|\nabla_p u\|_{H^3}\|u\|_{H^3} \\
&\leq \frac{\kappa}{36}\|\nabla_p u\|_{H^3}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
|H_{3312}| &\leq C\big\|\partial_3^3 u_p\nabla_p u\big\|_2\big\|\partial_3^3 u\big\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}}\big\|\nabla_p\nabla^3 u\big\|_2\big\|\nabla^3 u\big\|_2 \\
&\leq C\|u\|_{\mathrm{BMO}}\|\nabla_p u\|_{H^3}\|u\|_{H^3} \\
&\leq \frac{\kappa}{36}\|\nabla_p u\|_{H^3}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2.
\end{aligned} \tag{11}$$

Similarly to (11) and (12), one obtains

$$\begin{aligned}|H_{3321}| &\le C\big\|\nabla_p\nabla^2 u\big\|_2\big\|\nabla^2 u\nabla_p\nabla u\big\|_2\\ &\le C\big\|\nabla^3\nabla_p u\big\|_2\big\|\nabla^2\nabla_p u\big\|_2\|u\|_{\mathrm{BMO}}\\ &\le \frac{\kappa}{36}\|\nabla_p u\|_{H^3}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2,\end{aligned} \tag{12}$$

and

$$\begin{aligned}|H_{3322}| &\le C\big\|\partial_3^2 u_p\nabla_p\partial_3 u\big\|_2\big\|\partial_3^3 u\big\|_2\\ &\le C\|u\|_{\mathrm{BMO}}\big\|\nabla_p\nabla^3 u\big\|_2\big\|\nabla^3 u\big\|_2\\ &\le \frac{\kappa}{36}\|\nabla_p u\|_{H^3}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2.\end{aligned} \tag{13}$$

One can estimate H_{3313}, H_{3323} as H_{3312}, H_{3322} to get

$$|H_{3313}, H_{3323}| \le \frac{\kappa}{36}\|\nabla_p u\|_{H^3}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2. \tag{14}$$

Clearly, H_{333} can be similarly estimated as H_{331}, so we have

$$|H_{333}| \le \frac{3\kappa}{36}\|\nabla_p u\|_{H^3}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2. \tag{15}$$

Putting (10)–(15) together, we obtain

$$|H_{33}| \le \frac{9\kappa}{36}\|\nabla_p u\|_{H^3}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2. \tag{16}$$

Combining (5), (9), and (16), we get

$$|H_3| \le \frac{18\kappa}{36}\|\nabla_p u\|_{H^3}^2 + C\|u\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2. \tag{17}$$

Applying cancellation property and integration by parts, one can deduce that

$$\begin{aligned}H_4 + H_5 &= \sum_{0\le|\varrho|\le 3}\int_{\mathbb{R}^3}\big[\nabla^\varrho(Bu) - B\nabla^\varrho u\big]\cdot\nabla\nabla^\varrho B + \big[\nabla^\varrho(B\cdot DB) - (B\cdot D)\nabla^\varrho B\big]\cdot\nabla^\varrho u\,dx\\ &= H_{41} + H_{42}.\end{aligned}$$

By the Höledr inequality, Lemma 2.3, and Young's inequality, we get

$$\begin{aligned}|H_{41}| &\le C\|\nabla B\|_{H^3}\big(\|u\|_{H^3}\|B\|_{\mathrm{BMO}} + \|B\|_{H^3}\|u\|_{\mathrm{BMO}}\big)\\ &\le \frac{1}{8}\|\nabla B\|_{H^3}^2 + C\big(\|B\|_{H^3}^2 + \|u\|_{H^3}^2\big)\big(\|u\|_{\mathrm{BMO}}^2 + \|B\|_{\mathrm{BMO}}^2\big),\end{aligned} \tag{18}$$

and

$$\begin{aligned}|H_{42}| &\le C\|B\|_{\mathrm{BMO}}\|\nabla B\|_{H^3}\|u\|_{H^3}\\ &\le \frac{1}{8}\|\nabla B\|_{H^3}^2 + C\|B\|_{\mathrm{BMO}}^2\|u\|_{H^3}^2.\end{aligned} \tag{19}$$

Collecting (18) and (19), we have

$$|H_4 + H_5| \le \frac{2}{8}\|\nabla B\|_{H^3}^2 + C\big(\|B\|_{\mathrm{BMO}}^2 + \|u\|_{\mathrm{BMO}}^2\big)\big(\|u\|_{H^3}^2 + \|B\|_{H^3}^2\big). \tag{20}$$

Combining (2)–(4), (17), and (20), we get

$$\begin{aligned}&\frac{d}{dt}\big(\|u\|_{H^3} + \|B\|_{H^3}\big) + \kappa\|\nabla_p u\|_{H^3} + \|\nabla B\|_{H^3}\\ &\quad\le C\big(1 + \|B\|_{\mathrm{BMO}}^2 + \|u\|_{\mathrm{BMO}}^2 + \|\nabla B\|_{\mathrm{BMO}}^2\ln\big(e + \|B\|_{H^3}\big)\big)\big(\|u\|_{H^3}^2 + \|B\|_{H^3}^2\big).\end{aligned} \tag{21}$$

Setting $R(t) := e + \|u\|_{H^3} + \|B\|_{H^3}$, from (21), one obtains

$$\frac{d}{dt}R(t) \le C\big(\|\nabla B\|_{\mathrm{BMO}}^2 + \|u\|_{\mathrm{BMO}}^2 + C\big)R(t)\ln R(t).$$

Applying the Gronwall inequality, one gets

$$\sup_{0\le t\le T} R(t) \le \big(\|u_0\|_{H^3}^2 + \|B_0\|_{H^3}^2 + e\big)\exp\bigg(C\exp\bigg(\int_0^T \|u\|_{\mathrm{BMO}}^2 + \|\nabla B\|_{\mathrm{BMO}}^2\,dt\bigg)\bigg),$$

which implies the blow-up criterion in Theorem 1.1 holds.

4 Proof of Theorem 1.2

Operating ∇ to $(1)_1$, $(1)_3$, taking the scalar product of them with ∇u, ∇B, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\nabla u\|_2^2 + \|\nabla B\|_2^2\big) + \kappa_1\|\partial_1\nabla u\|_2^2 + \kappa_2\|\partial_2\nabla u\|_2^2 + \|\Delta B\|_2^2\\ &\quad= -\int_{\mathbb{R}^3}\nabla\big[\nabla\times\big((\nabla\times B)\times B\big)\big]\cdot\nabla B\,dx - \int_{\mathbb{R}^3}\nabla(u\cdot\nabla B)\cdot\nabla B\,dx\\ &\qquad - \int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u\,dx + \int_{\mathbb{R}^3}\nabla(B\cdot\nabla B)\cdot\nabla u\,dx + \int_{\mathbb{R}^3}\nabla(B\cdot\nabla u)\cdot\nabla B\,dx\\ &\quad:= K_1 + K_2 + K_3 + K_4 + K_5.\end{aligned} \tag{22}$$

Firstly, applying the Hölder inequality, commutator estimate and interpolation, one gets

$$\begin{aligned}|K_1| &\le C\big\|\nabla\big((\nabla\times B)\times B\big) - \nabla(\nabla\times B)\times B\big\|_{\frac{6}{5}}\big\|\nabla^2 B\big\|_6\\ &\le C\|\nabla B\|_2\|\nabla B\|_3\big\|\nabla^2 B\big\|_6\\ &\le C\|\nabla B\|_2\big\|\nabla^3 B\big\|_2^2,\end{aligned} \tag{23}$$

here we use the fact that $\|\nabla B\|_3 \le C\|B\|_2^{\frac{1}{2}}\|\nabla^3 B\|_2^{\frac{1}{2}}$, $\|\nabla^2 B\|_6 \le C\|\nabla^3 B\|_2$ due to the Gagliardo-Nirenberg-Sobolev inequality. By the Hölder inequality, one obtains

$$\begin{aligned}|K_2| &\le C\|\nabla u\|_2\|\nabla B\|_4^2\\ &\le C\|\nabla u\|_2\big\|\nabla^2 B\big\|_2^2.\end{aligned} \tag{24}$$

Reviewing H_{31} in Sect. 3, we know $K_3 = H_{31}$. Hence, applying Lemma 2.1, one obtains

$$
\begin{aligned}
|K_3| &\le C\|\nabla u\|_2^{\frac{1}{2}}\|\nabla u\|_2^{\frac{1}{2}}\|\nabla_p u\|_2^{\frac{1}{2}}\|\nabla\nabla_p u\|_2^{\frac{1}{2}}\|\nabla\nabla_p u\|_2^{\frac{1}{2}}\|\nabla\nabla_p u\|_2^{\frac{1}{2}} \\
&\le C\|\nabla_p \nabla u\|_2^2\|\nabla u\|_2.
\end{aligned} \tag{25}
$$

$K_4 + K_5$ can be written into two parts:

$$
K_4 + K_5 = \int_{\mathbb{R}^3}(\nabla B\cdot\nabla u)\cdot\nabla B\,dx + \int_{\mathbb{R}^3}(\nabla B\cdot\nabla B)\cdot\nabla u\,dx.
$$

By the Höledr inequality, we obtain

$$
\begin{aligned}
|K_4 + K_5| &\le C\|\nabla u\|_2\|\nabla B\|_4^2 \\
&\le C\|\nabla u\|_2\left\|\nabla^2 B\right\|_2^2.
\end{aligned} \tag{26}
$$

Combining (22)–(26), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla u(t)\right\|_2^2 + \left\|\nabla B(t)\right\|_2^2\right) + \kappa\|\nabla_p\nabla u\|_2^2 + \|\Delta B\|_2^2 \\
&\quad\le C\left(\|\nabla B\|_2 + \|\nabla u\|_2\right)\left(\|\Delta B\|_2^2 + \|\nabla\nabla_p u\|_2^2\right) + C\|\nabla B\|_2\left\|\nabla^3 B\right\|_2^2.
\end{aligned} \tag{27}
$$

Similarly to derivation of (22), one gets

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\Delta u(t)\right\|_2^2 + \left\|\Delta B(t)\right\|_2^2\right) + \kappa_1\|\partial_1\Delta u\|_2^2 + \kappa_2\|\partial_2\Delta u\|_2^2 + \left\|\nabla^3 B\right\|_2^2 \\
&\quad= -\int_{\mathbb{R}^3} D^2\left[\nabla\times\left((\nabla\times B)\times B\right)\right]\cdot D^2 B\,dx - \int_{\mathbb{R}^3} D^2(u\cdot\nabla B)\cdot D^2 B\,dx \\
&\qquad - \int_{\mathbb{R}^3} D^2(u\cdot\nabla u)\cdot D^2 u\,dx + \int_{\mathbb{R}^3} D^2(B\cdot\nabla u)\cdot D^2 B\,dx + \int_{\mathbb{R}^3} D^2(B\cdot\nabla B)\cdot D^2 u\,dx \\
&\quad:= E_1 + E_2 + E_3 + E_4 + E_5.
\end{aligned} \tag{28}
$$

We apply cancellation property, the Hölder inequality, commutator estimate to estimate E_1 as follows

$$
\begin{aligned}
|E_1| &\le C\left\|D^2\left[(\nabla\times B)\times B\right] - D^2(\nabla\times B)\times B\right\|_2\left\|\nabla^3 B\right\|_2 \\
&\le C\|\nabla B\|_3\|\Delta B\|_6\left\|\nabla^3 B\right\|_2 \\
&\le C\|\Delta B\|_2\left\|\nabla^3 B\right\|_2^2.
\end{aligned} \tag{29}
$$

E_2 can be split into two terms:

$$
\begin{aligned}
E_2 &= -\int_{\mathbb{R}^3}\left(D^2 u\cdot\nabla\right)B\cdot D^2 B\,dx - 2\int_{\mathbb{R}^3}(Du\cdot\nabla)\nabla B\cdot D^2 B\,dx \\
&= E_{21} + E_{22}.
\end{aligned}
$$

Noticing the fact that $\|\nabla B\|_3 \le C\|B\|_2^{\frac{1}{2}}\|\nabla^3 B\|_2^{\frac{1}{2}}$, $\|\nabla^2 B\|_6 \le C\|\nabla^3 B\|_2$ due to the Gagliardo-Nirenberg-Sobolev inequality, we have

$$|E_{21}| \le C\|\nabla^2 u\|_2\|\nabla B\|_3\|\nabla^2 B\|_6 \le C\|\nabla^2 u\|_2\|\nabla^3 B\|_2^2, \tag{30}$$

and

$$|E_{22}| \le C\|\nabla u\|_3\|\nabla^2 B\|_3^2 \le C\|\nabla^2 u\|_2^{\frac{3}{4}}\|\nabla^3 B\|_2^{\frac{5}{3}} \le C\|\nabla^2 u\|_2\|\nabla^3 B\|_2^2. \tag{31}$$

Collecting (30) and (31), we have

$$|E_2| \le C\|\nabla^2 u\|_2\|\nabla^3 B\|_2^2. \tag{32}$$

Obviously, $E_3 = H_{32}$, hence we get

$$\begin{aligned} E_3 &= H_{32} \\ &= H_{321} + H_{322} \\ &= H_{3211} + H_{3212} + H_{3213} + H_{3221} + H_{3222} + H_{3223}. \end{aligned}$$

One can use the Hölder inequality to deduce that

$$\begin{aligned} |H_{3211}| &= \left|\int_{\mathbb{R}^3} (\nabla_p \nabla u \cdot \nabla) u \cdot \nabla_p \nabla u dx\right| \\ &\le C\|\nabla_p \nabla u\|_2^{\frac{1}{2}}\|\nabla u\|_2^{\frac{1}{2}}\|\nabla_p \nabla u\|_2^{\frac{1}{2}}\|\nabla_p^2 \nabla u\|_2^{\frac{1}{2}}\|\nabla_p \nabla u\|_2^{\frac{1}{2}}\|\partial_3 \nabla_p \nabla u\|_{L^2}^{\frac{1}{2}} \\ &\le C\|\nabla_p \nabla^2 u\|_2\|\nabla_p \nabla u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{1}{2}} \\ &\le C\|\nabla_p \nabla^2 u\|_2\|\nabla_p \nabla u\|_2\|\nabla_p \nabla u\|_2^{\frac{1}{2}}\|\nabla u\|_2^{\frac{1}{2}} \\ &\le C\|\nabla_p \nabla^2 u\|_2^2\|\nabla^2 u\|_2. \end{aligned} \tag{33}$$

$$\begin{aligned} |H_{3212}| &= \left|\int_{\mathbb{R}^3} \left(\partial_3^2 u_p \cdot \nabla_p\right) u \cdot \partial_3^2 u \, dx\right| \\ &\le C\|\partial_3^2 u\|_2^{\frac{1}{2}}\|\nabla_p u\|_2^{\frac{1}{2}}\|\partial_3^2 u\|_2^{\frac{1}{2}}\|\nabla_p \partial_3^2 u\|_2^{\frac{1}{2}}\|\nabla_p \partial_3 u\|_2^{\frac{1}{2}}\|\nabla_p \partial_3^2 u\|_2^{\frac{1}{2}} \\ &\le C\|\nabla_p \nabla^2 u\|_2\|\nabla^2 u\|_2\|\nabla_p \nabla u\|_2^{\frac{1}{2}}\|\nabla_p u\|_2^{\frac{1}{2}} \\ &\le C\|\nabla_p \nabla^2 u\|_2^2\|\nabla^2 u\|_2. \end{aligned} \tag{34}$$

$$\begin{aligned} |H_{3213}| &= \left|\int_{\mathbb{R}^3} (\partial_3 \nabla_p \cdot u_p) \partial_3 u \cdot \partial_3^2 u \, dx\right| \\ &\le C\|\partial_3 \nabla_p \cdot u_p\|_2^{\frac{1}{2}}\|\partial_3 u\|_2^{\frac{1}{2}}\|\partial_3^2 u\|_2^{\frac{1}{2}}\|\nabla_p \partial_3^2 u\|_2^{\frac{1}{2}}\|\nabla_p \partial_3 u\|_2^{\frac{1}{2}}\|\nabla_p \partial_3^2 u\|_2^{\frac{1}{2}} \\ &\le C\|\nabla_p \nabla^2 u\|_2\|\nabla^2 u\|_2^{\frac{1}{2}}\|\nabla_p \nabla u\|_2\|\nabla u\|_2^{\frac{1}{2}} \\ &\le C\|\nabla_p \nabla^2 u\|_2^{\frac{3}{2}}\|\nabla^2 u\|_2^{\frac{3}{4}} \\ &\le C\|\nabla_p \nabla^2 u\|_2^2\|\nabla^2 u\|_2. \end{aligned} \tag{35}$$

Similarly to the above calculation, one gets

$$\begin{aligned}
|H_{3221}| &= \left|\int_{\mathbb{R}^3} (\nabla_p u \cdot \nabla)\nabla u \cdot \nabla_p \nabla u \, dx\right| \\
&\le C\|\nabla_p u\|_2^{\frac{1}{2}} \left\|\nabla^2 u\right\|_2^{\frac{1}{2}} \|\nabla_p \nabla u\|_2^{\frac{1}{2}} \left\|\nabla_p^2 u\right\|_2^{\frac{1}{2}} \left\|\nabla_p \nabla^2 u\right\|_2^{\frac{1}{2}} \|\nabla_p \partial_3 \nabla u\|_2^{\frac{1}{2}} \\
&\le C\left\|\nabla_p \nabla^2 u\right\|_2 \left\|\nabla^2 u\right\|_2 \|\nabla_p u\|_2^{\frac{1}{2}} \|\nabla_h \nabla u\|_2^{\frac{1}{2}} \|\nabla u\|_2^{\frac{1}{2}} \\
&\le C\left\|\nabla_p \nabla^2 u\right\|_2^2 \left\|\nabla^2 u\right\|_2.
\end{aligned} \tag{36}$$

In similar manner as H_{3213} and H_{3212}, one gets

$$|H_{3222}, H_{3223}| \le C\left\|\nabla_p \nabla^2 u\right\|_2^2 \left\|\nabla^2 u\right\|_2. \tag{37}$$

Combining (33)–(37), we have

$$|E_3| \le C\left\|\nabla_p \nabla^2 u\right\|_2^2 \left\|\nabla^2 u\right\|_2. \tag{38}$$

One can split $E_4 + E_5$ into four terms:

$$\begin{aligned}
E_4 + E_5 &= \int_{\mathbb{R}^3} \left(\nabla^2 B \cdot \nabla\right) B \cdot \nabla^2 u \, dx + 2\int_{\mathbb{R}^3} (\nabla B \cdot \nabla)\nabla B \cdot \nabla^2 u \, dx \\
&\quad + \int_{\mathbb{R}^3} \left(\nabla^2 B \cdot \nabla u\right) \cdot \nabla^2 B \, dx + 2\int_{\mathbb{R}^3} (\nabla B \cdot \nabla)\nabla u \cdot \nabla^2 B \, dx \\
&= E_{41} + E_{42} + E_{43} + E_{44}.
\end{aligned}$$

Similarly to E_{21} and E_{22}, one has

$$|E_{41}, E_{42}, E_{43}, E_{44}| \le C\left\|\nabla^2 u\right\|_2 \left\|\nabla^3 B\right\|_2^2.$$

Hence, one gets

$$|E_4 + E_5| \le C\left\|\nabla^2 u\right\|_2 \left\|\nabla^3 B\right\|_2^2. \tag{39}$$

Combining (28), (29), (32), (38), and (39), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\Delta u(t)\right\|_2^2 + \left\|\Delta B(t)\right\|_2^2\right) + \kappa\|\nabla_p \Delta \, u\|_2^2 + \left\|\nabla^3 B\right\|_2^2 \\
&\quad \le C\left(\|\Delta u\|_2 + \|\Delta B\|_2\right)\left(\|\nabla_p \Delta u\|_2^2 + \left\|\nabla^3 B\right\|_2^2\right).
\end{aligned} \tag{40}$$

Adding (27) to (40), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla B\|_2^2 + \|\nabla u\|_2^2 + \|\Delta B\|_2^2 + \|\Delta u\|_2^2\right) \\
&\qquad + \kappa\|\nabla_p \nabla u\|_2^2 + \|\Delta B\|_2^2 + \kappa\|\nabla_p \Delta u\|_2^2 + \left\|\nabla^3 B\right\|_2^2 \\
&\quad \le C\left(\|\nabla u\|_2 + \|\nabla B\|_2 + \left\|\nabla^2 u\right\|_2 + \left\|\nabla^2 B\right\|_2\right) \\
&\qquad \times \left(\|\nabla_p \nabla u\|_2^2 + \|\Delta B\|_2^2 + \|\nabla_p \Delta u\|_2^2 + \left\|\nabla^3 B\right\|_2^2\right).
\end{aligned}$$

Therefore, one gets

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\nabla B\|_2^2+\|\nabla u\|_2^2+\|\Delta B\|_2^2+\|\Delta u\|_2^2\big)\\
&\quad+\big[\kappa_0-C\big(\|\nabla u\|_2+\|\nabla B\|_2+\big\|\nabla^2 u\big\|_2+\big\|\nabla^2 B\big\|_2\big)\big]\\
&\quad\times\big(\|\nabla_p\nabla u\|_2^2+\|\Delta B\|_2^2+\|\nabla_p\Delta u\|_2^2+\big\|\nabla^3 B\big\|_2^2\big)\le 0.
\end{aligned}$$

where $\kappa_0=\min\{\kappa,1\}$. Choose ε^* sufficiently small such that

$$C\big(\|\nabla u_0\|_2+\|\nabla B_0\|_2+\|\Delta u_0\|_2+\|\Delta B_0\|_2\big)\le\frac{\kappa_0}{2}.$$

Then one obtains:

$$(B,u)\in L^\infty\big(0,T;H^2\big),(\nabla_p u,\nabla B)\in L^2\big(0,T;H^2\big),\quad\forall T\in(0,T_0),$$

noticing

$$H^2\big(\mathbb{R}^3\big)\hookrightarrow\mathrm{BMO}\big(\mathbb{R}^3\big),$$

yields for any $T\in(0,T_0)$

$$(\nabla B,u)\in\big(0,T;\mathrm{BMO}\big(\mathbb{R}^3\big)\big).$$

By Theorem 1.1, applying continuation argument, we obtain the result of Theorem 1.2.

Acknowledgements
The author is indebted to the referee and the associate editor for their detailed comments and valuable suggestions, which greatly improved the manuscript. The author is also grateful to Prof. Lili Du for useful direction on this paper. This research was supported by High-level Talent Sailing Project of Yibin University (2021QH07).

Funding
This research was supported by High-level Talent Sailing Project of Yibin University (2021QH07).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that they have no competing interests.

Authors' contributions
BD prepared the manuscript initially and performed all the steps of the proofs in this research. The main idea of this paper was proposed by BD. All authors read and approved the final manuscript.

Publisher's Note

Received: 18 November 2021 Accepted: 4 January 2022 Published online: 24 January 2022

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