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Three positive solutions for a nonlinear partial discrete Dirichlet problem with (p,q)-Laplacian operator

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Abstract

In this paper, we prove the existence of three solutions to a partial difference equation with (p,q)-Laplacian operator by using critical point theory. Furthermore, based on the strong maximum principle, we prove that the three solutions are positive under appropriate nonlinearity assumptions. Finally, we also give an example to illustrate our main results.

Keywords: Boundary value problem; Three positive solutions; Partial difference equation; (p,q)-Laplacian; Critical point theory

1 Introduction

Let $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of integers and real numbers, respectively. Define $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$ for $a \le b$.

We consider the following partial discrete Dirichlet problem $(\triangle^\lambda)$:

$$\begin{aligned} &-\bigl[\Delta_1\bigl(\phi_p\bigl(\Delta_1 w(k-1,l)\bigr)\bigr) + \Delta_2\bigl(\phi_p\bigl(\Delta_2 w(k,l-1)\bigr)\bigr)\bigr] + h(k,l)\phi_q\bigl(w(k,l)\bigr) \\ &\quad = \lambda f\bigl((k,l), w(k,l)\bigr), \quad (k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d), \end{aligned}$$

with boundary conditions

$$\begin{aligned} &w(k,0) = w(k,d+1) = 0, \quad k \in \mathbb{Z}(0,c+1), \\ &w(0,l) = w(c+1,l) = 0, \quad l \in \mathbb{Z}(0,d+1), \end{aligned} \tag{1.1}$$

where c and d are given positive integers, λ is a positive real parameter, Δ_1 and Δ_2 are the forward difference operators defined by $\Delta_1 w(k,l) = w(k+1,l) - w(k,l)$ and $\Delta_2 w(k,l) = w(k,l+1) - w(k,l)$, $\Delta_1^2 w(k,l) = \Delta_1(\Delta_1 w(k,l))$, $\Delta_2^2 w(k,l) = \Delta_2(\Delta_2 w(k,l))$, $\phi_r(u) = |u|^{r-2}u$ for $u \in \mathbb{R}$, $1 < q \le p < +\infty$, $h(k,l) \ge 0$ for all $(k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d)$, and $f((k,l),\cdot) \in C(\mathbb{R},\mathbb{R})$ for all $(k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d)$.

Difference equations have gained extensive uses in various domains, like biomathematics, as shown [1–3]. Admittedly, for the boundary value problem of difference equations, there are the following important research tools: fixed point methods, upper and lower solution techniques, and invariant sets of descending flow [4–9]. In 2003, Yu and Guo [10] firstly studied a class of difference equation by critical point theory. Since then, by means of critical point theory, numerous scholars have committed to the research on difference equations and obtained many results, such as the results on periodic solutions [10–12], homoclinic solutions [13–21], and boundary value problems [22–33].

Candito and D'Aguì [29] in 2010 investigated the following discrete Neumann problem:

$$\begin{cases} -\triangle(\phi_p(\triangle w(l-1))) + q(l)\phi_p(w(l)) = \lambda f(l,w(l)), \quad l \in \mathbb{Z}(1,M), \\ \triangle w(0) = \triangle w(M) = 0, \end{cases} \tag{1.2}$$

and several sufficient conditions concerning the existence of three solutions of (1.2) were acquired.

Subsequently, they [30] in 2011 considered the discrete Dirichlet problem

$$\begin{cases} \Delta^2(w(l-1)) = \lambda f(w(l)) + \mu g(l,w(l)), \quad l \in \mathbb{Z}(1,M), \\ w(0) = w(M+1) = 0 \end{cases} \tag{1.3}$$

and proved that it has at least three positive solutions.

Heidarkhani and Moghadam [31] in 2014 considered the discrete Dirichlet problem

$$\begin{cases} -\triangle(\phi_p(\triangle w(l-1))) + q(l)\phi_p(w(l)) = \lambda f(l,w(l)) + \mu g(l,w(l)), \quad l \in \mathbb{Z}(1,M), \\ w(0) = w(M+1) = 0 \end{cases} \tag{1.4}$$

and proved that it has at least three solutions.

Mugnai and Papageorgiou [34] in 2014 studied the discrete Dirichlet problem

$$\begin{cases} -\Delta(\phi_p(\Delta w)) - \mu\Delta(\phi_q(\Delta w)) = f(l,w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $1 < q \leq 2 \leq p < \infty$, $\mu \geq 0$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory's conditions, and proved the existence of multiple solutions of (1.5).

Nastasi et al. [32] in 2017 focused on the discrete Dirichlet problem

$$\begin{cases} -\Delta(\phi_p(\Delta w(l-1))) - \Delta(\phi_q(\Delta w(l-1))) + \alpha(l)\phi_p(w(l)) + \beta(l)\phi_q(w(l)) \\ \quad = \lambda g(l,w(l)), \quad l \in \mathbb{Z}(1,M), \\ w(0) = w(M+1) = 0, \end{cases} \tag{1.6}$$

where $\phi_r(u) = |u|^{r-2}u$ for $u \in \mathbb{R}$, and proved that it has at least two positive solutions.

Xiong and Zhou [33] in 2021 considered the discrete Dirichlet problem

$$\begin{cases} \triangle[p(l)\phi_p(\triangle w(l-1))] + \lambda f(l,w(l)) = 0, \quad l \in \mathbb{Z}(1,M), \\ w(0) = w(M+1) = 0 \end{cases} \tag{1.7}$$

and obtained the existence of three solutions of (1.7) under appropriate nonlinearity assumptions.

It should be noted that the above-mentioned difference equations have only one variable. However, difference equations involving two or more variables have rarely been studied and are called partial difference equations. In recent years, the partial difference equations have been extensively employed in various domains. However, it should be mentioned that the boundary value problem of the partial difference equation is a challenging problem attracting many mathematical researchers [35, 36].

Heidarkhani and Imbesi [35] in 2015 considered the partial discrete Dirichlet problem

$$\Delta_1^2 w(k-1,l) + \Delta_2^2 w(k,l-1) + \lambda f\big((k,l), w(k,l)\big) = 0, \quad (k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d), \tag{1.8}$$

with boundary conditions (1.1) and proved the existence of at least three solutions of (1.8).

Lately, Du and Zhou [36] in 2020 studied the partial discrete Dirichlet problem (s^λ):

$$\begin{aligned} &\Delta_1\big(\phi_p\big(\Delta_1 w(k-1,l)\big)\big) + \Delta_2\big(\phi_p\big(\Delta_2 w(k,l-1)\big)\big) + \lambda f\big((k,l), w(k,l)\big) = 0, \\ &(k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d), \end{aligned}$$

with boundary conditions (1.1) and proved the existence of multiple solutions of (s^λ).

Compared with the results of the partial difference equations with p-Laplacian, those with (p,q)-Laplacian have rarely been studied. Thus in this paper, we demonstrate the existence of three solutions to a partial difference equation with (p,q)-Laplacian operator by using different methods. Furthermore, based on the strong maximum principle, we prove that the three solutions are positive under appropriate nonlinearity assumptions.

The main tool of this paper is as follows.

(Λ) Let $(X, \|\cdot\|)$ be a real finite-dimensional Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux-differentiable functionals with coercive Φ and such that

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Lemma 1.1 (Theorem 4.1 of [37]) *Assume that* (Λ) *holds and there exist* $r > 0$ *and* $\bar{x} \in X$ *with* $r < \Phi(\bar{x})$ *such that*

(a_1) $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$,

(a_2) *For each* $\lambda \in \Lambda_r := (\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)})$, *the function* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

The rest of this paper is organized as follows. In Sect. 2, we establish the variational framework associated with ($\triangle^\lambda$). In Sect. 3, we present our main results. Finally, in Sect. 4, we present an example illustrating our main results.

2 Preliminaries

In this section, we establish the variational framework associated with ($\triangle^\lambda$). We consider the cd-dimensional Banach space

$V = \{w : \mathbb{Z}(0,c+1) \times \mathbb{Z}(0,d+1) \to \mathbb{R} : w(k,0) = w(k,d+1) = 0, k \in \mathbb{Z}(0,c+1)$ and $w(0,l) = w(c+1,l) = 0, l \in \mathbb{Z}(0,d+1)\}$, endowed with the norm

$$\|w\| = \left(\sum_{l=1}^{d}\sum_{k=1}^{c+1}\bigl|\Delta_1 w(k-1,l)\bigr|^p + \sum_{k=1}^{c}\sum_{l=1}^{d+1}\bigl|\Delta_2 w(k,l-1)\bigr|^p\right)^{\frac{1}{p}}, \quad w \in V.$$

We also define the other norm $\|w\|_\infty = \max\{|w(k,l)| : (k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d)\}$ in V.

Define

$$\Phi(w) = \Phi_1(w) + \Phi_2(w)$$

and

$$\Psi(w) = \sum_{l=1}^{d}\sum_{k=1}^{c} F\bigl((k,l), w(k,l)\bigr)$$

for $w \in V$, where $\Phi_1(w) = \frac{\|w\|^p}{p}$, $\Phi_2(w) = \frac{\sum_{l=1}^{d}\sum_{k=1}^{c} h(k,l)|w(k,l)|^q}{q}$, and $F((k,l),w) = \int_0^w f((k,l),\tau)\,d\tau$ for $((k,l),w) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d) \times \mathbb{R}$.

Let

$$I_\lambda(w) = \Phi(w) - \lambda\Psi(w)$$

for $w \in V$. Obviously, $\Phi, \Psi \in C^1(V,\mathbb{R})$, that is, Φ_1, Φ_2, and Ψ are continuously Fréchet differentiable in V, and

$$\begin{aligned}
\Phi_1'(w)(s) &= \lim_{t\to 0}\frac{\Phi_1(w+ts) - \Phi_1(w)}{t} \\
&= \sum_{l=1}^{d}\sum_{k=1}^{c+1}\phi_p\bigl(\Delta_1 w(k-1,l)\bigr)\Delta_1 s(k-1,l) \\
&\quad + \sum_{k=1}^{c}\sum_{l=1}^{d+1}\phi_p\bigl(\Delta_2 w(k,l-1)\bigr)\Delta_2 s(k,l-1) \\
&= -\sum_{l=1}^{d}\sum_{k=1}^{c}\Delta_1\phi_p\bigl(\Delta_1 w(k-1,l)\bigr)s(k,l) \\
&\quad - \sum_{k=1}^{c}\sum_{l=1}^{d}\Delta_2\phi_p\bigl(\Delta_2 w(k,l-1)\bigr)s(k,l), \\
\Phi_2'(w)(s) &= \sum_{l=1}^{d}\sum_{k=1}^{c} h(k,l)\phi_q\bigl(w(k,l)\bigr)s(k,l),
\end{aligned}$$

and

$$\Psi'(w)(s) = \lim_{t\to 0}\frac{\Psi(w+ts) - \Psi(w)}{t} = \sum_{l=1}^{d}\sum_{k=1}^{c} f\bigl((k,l), w(k,l)\bigr)s(k,l)$$

for $w, s \in V$.

Therefore for all $w, s \in V$,

$$\begin{aligned}(\Phi - \lambda\Psi)'(w)(s) = -\sum_{l=1}^{d}\sum_{k=1}^{c}\big[&\Delta_1\phi_p\big(\Delta_1 w(k-1,l)\big) + \Delta_2\phi_p\big(\Delta_2 w(k,l-1)\big) \\ &- h(k,l)\phi_q\big(w(k,l)\big) + \lambda f\big((k,l), w(k,l)\big)\big]s(k,l).\end{aligned} \tag{2.1}$$

Obviously, w is a critical point of the functional $\Phi - \lambda\Psi$ in V if and only if it is a solution of problem (Δ^λ). Therefore we reduce the existence of solutions of (Δ^λ) to the existence of the critical points of $\Phi - \lambda\Psi$ on V.

Lemma 2.1 (Lemma 2.1 of [38]) *Suppose that there exists $w : \mathbb{Z}(0, c+1) \times \mathbb{Z}(0, d+1) \to \mathbb{R}$ such that*

$$\begin{aligned}&w(k,l) > 0 \quad \text{or} \\ &-\Delta_1\big(\phi_p\big(\Delta_1 w(k-1,l)\big)\big) - \Delta_2\big(\phi_p\big(\Delta_2 w(k,l-1)\big)\big) + h(k,l)\phi_q\big(w(k,l)\big) \geq 0\end{aligned} \tag{2.2}$$

for all $(k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d)$, $w(k,0) = w(k,d+1) = 0$ for $k \in \mathbb{Z}(0,c+1)$, and $w(0,l) = w(c+1,l) = 0$ for $l \in \mathbb{Z}(0,d+1)$.

Then either $w(k,l) > 0$ for all $(k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d)$, or $w \equiv 0$.

Lemma 2.2 (Proposition 1 of [38]) *We have the following inequality*:

$$\begin{aligned}\|w\|_\infty \leq \max\Bigg\{&\left(\frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}}\right)^{1/q}\left(\frac{\|w\|^p}{p} + \frac{\sum_{l=1}^{d}\sum_{k=1}^{c} h(k,l)|w(k,l)|^q}{q}\right)^{1/q}, \\ &\left(\frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}}\right)^{1/p}\left(\frac{\|w\|^p}{p} + \frac{\sum_{l=1}^{d}\sum_{k=1}^{c} h(k,l)|w(k,l)|^q}{q}\right)^{1/p}\Bigg\},\end{aligned}$$

where $h_* = \min\{h(k,l) : (k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d)\}$.

3 Main results

Now we state the following theorem.

Theorem 3.1 *Let $f((k,l), w)$ be a continuous function with respect to w for all $(k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d)$. Suppose that there exist two positive constants c_1 and d_1 such that*

$$\left(\frac{2c+2d}{p}\right)d_1^p + \frac{\tilde{h}}{q}d_1^q > r = \frac{4^p + h_*(c+d+2)^{p-1}}{p(c+d+2)^{p-1}}\min\big\{c_1^q, c_1^p\big\}$$

and the following conditions are satisfied:

(H_1) *$f((k,l),\xi) > 0$ for all $(k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d)$ and $\xi \in [-c_1, c_1]$,*

(H_2) $\frac{\sum_{l=1}^{d}\sum_{k=1}^{c} F((k,l),d_1)}{(\frac{2c+2d}{p})d_1^p + \frac{\tilde{h}}{q}d_1^q} > \frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}} \times \frac{\sum_{l=1}^{d}\sum_{k=1}^{c}\max_{|\eta|\leq c_1} F((k,l),\eta)}{\min\{c_1^q, c_1^p\}}$,

(H_3) $\limsup_{|\xi|\to+\infty}\frac{F((k,l),\xi)}{|\xi|^p} < \frac{4^p \sup_{\Phi(w)\leq r}\sum_{l=1}^{d}\sum_{k=1}^{c} F((k,l),w(k,l))}{cd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}$.

Then for each $\lambda \in \Lambda$, problem (Δ^λ) possesses at least three nontrivial solutions, where

$$\Lambda := \left(\frac{(\frac{2c+2d}{p})d_1^p + \frac{\tilde{h}}{q}d_1^q}{\sum_{l=1}^{d}\sum_{k=1}^{c} F((k,l),d_1)}, \frac{(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\sup_{\Phi(w)\leq r}\sum_{l=1}^{d}\sum_{k=1}^{c} F((k,l),w(k,l))}\right).$$

Proof Let $\Phi(w)$, $\Psi(w)$, and $I_\lambda(w)$ for $w \in V$ be as defined in Sect. 2. Take $X = V$. Obviously, Φ and Ψ are two continuously Gâteaux-differentiable functionals.

First, we prove that $\Phi(w)$ is coercive, that is, $\lim_{\|w\|\to+\infty} \Phi(w) = +\infty$:

$$\begin{aligned}\Phi(w) &= \Phi_1(w) + \Phi_2(w)\\ &= \frac{\|w\|^p}{p} + \frac{\sum_{l=1}^{d}\sum_{k=1}^{c} h(k,l)|w(k,l)|^q}{q}\\ &\geq \frac{\|w\|^p}{p}.\end{aligned} \tag{3.1}$$

By the definition of Φ and Ψ we have

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Thus condition (Λ) holds.

Assume that $w \in V$ and

$$\Phi(w) = \frac{\|w\|^p}{p} + \frac{\sum_{l=1}^{d}\sum_{k=1}^{c} h(k,l)|w(k,l)|^q}{q} \leq r.$$

If $r = \frac{4^p+h_*(c+d+2)^{p-1}}{p(c+d+2)^{p-1}} c_1^q$, then $c_1 \geq 1$, and by Lemma 2.2 we have

$$\begin{aligned}\|w\|_\infty &\leq \max\left\{\left(\frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}}\right)^{1/q} r^{1/q},\right.\\ &\qquad \left.\left(\frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}}\right)^{1/p} r^{1/p}\right\}\\ &= \max\left\{c_1, c_1^{q/p}\right\} = c_1.\end{aligned}$$

If $r = \frac{4^p+h_*(c+d+2)^{p-1}}{p(c+d+2)^{p-1}} c_1^p$, then $0 < c_1 < 1$, and by Lemma 2.2 we have

$$\begin{aligned}\|w\|_\infty &\leq \max\left\{\left(\frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}}\right)^{1/q} r^{1/q},\right.\\ &\qquad \left.\left(\frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}}\right)^{1/p} r^{1/p}\right\}\\ &= \max\left\{c_1^{p/q}, c_1\right\} = c_1.\end{aligned}$$

Therefore $\{w \in V : \Phi(w) \leq r\} \subseteq \{w \in V : \|w\|_\infty \leq c_1\}$, and

$$\begin{aligned}\frac{\sup_{\Phi(w)\leq r}\Psi(w)}{r} &\leq \frac{\sup_{\|w\|_\infty\leq c_1}\sum_{l=1}^{d}\sum_{k=1}^{c} F((k,l),w(k,l))}{r}\\ &\leq \frac{\sum_{l=1}^{d}\sum_{k=1}^{c}\max_{|\eta|\leq c_1} F((k,l),\eta)}{r}\\ &= \frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}} \times \frac{\sum_{l=1}^{d}\sum_{k=1}^{c}\max_{|\eta|\leq c_1} F((k,l),\eta)}{\min\{c_1^q, c_1^p\}}.\end{aligned} \tag{3.2}$$

Define the sequence $\{\bar{w}\}$ in V by

$$\bar{w}(k,l) = \begin{cases} d_1 & \text{if } (k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d), \\ 0 & \text{if } k = 0 \text{ and } l \in \mathbb{Z}(0,d+1) \text{ or } k = c+1 \text{ and } l \in \mathbb{Z}(0,d+1), \\ 0 & \text{if } l = 0 \text{ and } k \in \mathbb{Z}(0,c+1) \text{ or } l = d+1 \text{ and } k \in \mathbb{Z}(0,c+1). \end{cases}$$

By the assumed conditions we have

$$\begin{aligned} &\Phi(\bar{w}) = \left(\frac{2c+2d}{p}\right) d_1^p + \frac{\tilde{h}}{q} d_1^q > r, \\ &\frac{\Psi(\bar{w})}{\Phi(\bar{w})} = \frac{\sum_{l=1}^d \sum_{k=1}^c F((k,l),d_1)}{(\frac{2c+2d}{p}) d_1^p + \frac{\tilde{h}}{q} d_1^q}. \end{aligned} \tag{3.3}$$

According to (H_2), (3.2), and (3.3), we have

$$\frac{\sup_{\Phi(w)\le r} \Psi(w)}{r} < \frac{\Psi(\bar{w})}{\Phi(\bar{w})},$$

and thus condition (a_1) of Lemma 1.1 holds.

Next, we prove that the functional $\Phi - \lambda\Psi$ is coercive. By (H_3) there exists e such that

$$\limsup_{|\xi|\to+\infty} \frac{F((k,l),\xi)}{|\xi|^p} < e < \frac{4^p \sup_{\Phi(w)\le r} \sum_{l=1}^d \sum_{k=1}^c F((k,l),w(k,l))}{cd(4^p + h_*(c+d+2)^{p-1}) \min\{c_1^q, c_1^p\}}$$
$$\text{for } (k,l) \in \mathbb{Z}(1,c) \times \mathbb{Z}(1,d).$$

Therefore there is a positive constant s such that

$$F\bigl((k,l),\xi\bigr) \le e|\xi|^p + s.$$

According to [36, Proposition 1], we have

$$\|w\|_\infty^p \le \frac{(c+d+2)^{p-1}}{4^p} \|w\|^p. \tag{3.4}$$

Since $\lambda < \frac{(4^p + h_*(c+d+2)^{p-1}) \min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1} \sup_{\Phi(w)\le r} \sum_{l=1}^d \sum_{k=1}^c F((k,l),w(k,l))}$ for all $w \in V$, by (3.4) we have

$$\begin{aligned} \lambda \sum_{l=1}^d \sum_{k=1}^c F\bigl((k,l),w(k,l)\bigr) &\le \lambda \sum_{l=1}^d \sum_{k=1}^c \bigl(e\bigl|w(k,l)\bigr|^p + s\bigr) \\ &\le \lambda \sum_{l=1}^d \sum_{k=1}^c \bigl(e\|w\|_\infty^p + s\bigr) \\ &\le \frac{ecd\lambda(c+d+2)^{p-1}}{4^p} \|w\|^p + \lambda scd \end{aligned} \tag{3.5}$$

$$< \frac{ecd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p4^p \sup_{\Phi(w)\le r}\sum_{l=1}^{d}\sum_{k=1}^{c}F((k,l),w(k,l))}\|w\|^p + \frac{scd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\sup_{\Phi(w)\le r}\sum_{l=1}^{d}\sum_{k=1}^{c}F((k,l),w(k,l))},$$

where $\|w\|_\infty = \max\{|w(k,l)| : (k,l) \in \mathbb{Z}(1,c)\times\mathbb{Z}(1,d)\}$.

Combining (3.1) with (3.5), we have

$$\Phi(w) - \lambda\Psi(w) \ge \frac{\|w\|^p}{p} - \frac{ecd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p4^p \sup_{\Phi(w)\le r}\sum_{l=1}^{d}\sum_{k=1}^{c}F((k,l),w(k,l))}\|w\|^p - \frac{scd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\sup_{\Phi(w)\le r}\sum_{l=1}^{d}\sum_{k=1}^{c}F((k,l),w(k,l))}, \tag{3.6}$$

that is,

$$\Phi(w) - \lambda\Psi(w) \ge \left[\frac{1}{p} - \frac{ecd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p4^p \sup_{\Phi(w)\le r}\sum_{l=1}^{d}\sum_{k=1}^{c}F((k,l),w(k,l))}\right]\|w\|^p - \frac{scd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\sup_{\Phi(w)\le r}\sum_{l=1}^{d}\sum_{k=1}^{c}F((k,l),w(k,l))}. \tag{3.7}$$

Thus we obtain $\lim_{\|w\|\to+\infty}\Phi(w) - \lambda\Psi(w) = +\infty$, that is, I_λ is coercive. Therefore condition (a_2) of Lemma 1.1 is verified.

Thus we have proved that all assumptions of Lemma 1.1 are satisfied, so that the functional $\Phi(w) - \lambda\Psi(w)$ possesses at least three distinct critical points. Since $w = 0$ is not a solution of problem (Δ^λ), it possesses at least three nontrivial solutions. Therefore the proof of Theorem 3.1 is completed. □

From Theorem 3.1 we have the following:

Corollary 3.2 *Let $f((k,l),w)$ be a continuous function with respect to w for every $(k,l) \in \mathbb{Z}(1,c)\times\mathbb{Z}(1,d)$. Suppose that there exist two positive constants c_1 and d_1 such that*

$$\left(\frac{2c+2d}{p}\right)d_1^p + \frac{\tilde{h}}{q}d_1^q > r = \frac{4^p + h_*(c+d+2)^{p-1}}{p(c+d+2)^{p-1}}\min\{c_1^q, c_1^p\}.$$

Suppose that the following conditions are satisfied:

$(\tilde{H}_1)$ *$f((k,l),\xi) > 0$ for all $(k,l) \in \mathbb{Z}(1,c)\times\mathbb{Z}(1,d)$ and $\xi \in [0,c_1]$,*

$(\tilde{H}_2)$ $\frac{\sum_{l=1}^{d}\sum_{k=1}^{c}F((k,l),d_1)}{(\frac{2c+2d}{p})d_1^p + \frac{\tilde{h}}{q}d_1^q} > \frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}} \times \frac{\sum_{l=1}^{d}\sum_{k=1}^{c}\max_{\eta\in[0,c_1]}F((k,l),\eta)}{\min\{c_1^q, c_1^p\}}$,

$(\tilde{H}_3)$ $\limsup_{\xi\to+\infty}\frac{F((k,l),\xi)}{\xi^p} < \frac{4^p\sup_{\Phi(w)\le r}\sum_{l=1}^{d}\sum_{k=1}^{c}F((k,l),w(k,l))}{cd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}$.

Then for every $\lambda \in \Lambda$, problem (Δ^λ) possesses at least three positive solutions.

Proof We consider the auxiliary problem (Δ^{λ^+}):

$$-\left[\Delta_1\left(\phi_p\left(\Delta_1 w(k-1,l)\right)\right) + \Delta_2\left(\phi_p\left(\Delta_2 w(k,l-1)\right)\right)\right] + h(k,l)\phi_q\left(w(k,l)\right) = \lambda f\left((k,l), w^+(k,l)\right), \quad (k,l) \in \mathbb{Z}(1,c)\times\mathbb{Z}(1,d),$$

with boundary conditions (1.1).

Let

$$F^+\big((k,l),w\big)=\int_0^w f\big((k,l),\tau^+\big)\,d\tau,\quad (k,l)\in\mathbb{Z}(1,c)\times\mathbb{Z}(1,d),$$

where $\tau^+=\max\{0,\tau\}$.

$$f^+\big((k,l),w\big)=\begin{cases} f((k,l),w) & \text{if } w>0,\\ f((k,l),0) & \text{if } w\le 0,\end{cases}$$

for $(k,l)\in\mathbb{Z}(1,c)\times\mathbb{Z}(1,d)$. Then condition (H_1) of Theorem 3.1 holds since

$$\limsup_{w\to+\infty}\frac{F^+((k,l),w)}{|w|^p}=\limsup_{w\to+\infty}\frac{F((k,l),w)}{w^p}$$

and

$$\lim_{w\to-\infty}\frac{F^+((k,l),w)}{|w|^p}=\lim_{w\to-\infty}\frac{f((k,l),0)w}{|w|^p}=-\lim_{w\to-\infty}\frac{f((k,l),0)}{|w|^{p-1}}=0.$$

Note that

$$\sum_{l=1}^{d}\sum_{k=1}^{c}\max_{|\eta|\le c_1}F^+\big((k,l),\eta\big)=\sum_{l=1}^{d}\sum_{k=1}^{c}\max_{\eta\in[0,c_1]}F\big((k,l),\eta\big)>0.$$

Therefore we have

$$\limsup_{|w|\to+\infty}\frac{F^+((k,l),w)}{|w|^p}<\frac{4^p\sup_{\Phi(w)\le r}\sum_{l=1}^{d}\sum_{k=1}^{c}F^+((k,l),w(k,l))}{cd(4^p+h_*(c+d+2)^{p-1})\min\{c_1^q,c_1^p\}}.$$

So all the conditions of Theorem 3.1 are satisfied. As a result, problem $(\triangle^{\lambda^+})$ possesses at least three nontrivial solutions. Suppose $w=w(k,l)$ is a nontrivial solution. Then for any $(k,l)\in\mathbb{Z}(1,c)\times\mathbb{Z}(1,d)$, there exists $w(k,l)>0$, or

$$\begin{aligned}&-\big[\Delta_1\big(\phi_p\big(\Delta_1 w(k-1,l)\big)\big)+\Delta_2\big(\phi_p\big(\Delta_2 w(k,l-1)\big)\big)\big]+h(k,l)\phi_q\big(w(k,l)\big)\\&\quad=\lambda f\big((k,l),w^+(k,l)\big)\\&\quad=\lambda f\big((k,l),0\big)\\&\quad>0.\end{aligned}\tag{3.8}$$

From Lemma 2.1 we can conclude that $w>0$ for $(k,l)\in\mathbb{Z}(1,c)\times\mathbb{Z}(1,d)$. Problem $(\triangle^{\lambda^+})$ possesses at least three positive solutions. Since problem $(\triangle^{\lambda^+})$ shares the same solutions with problem $(\triangle^{\lambda})$, the latter possesses at least three positive solutions. Therefore the proof of Corollary 3.2 is completed. □

As a particular case of problem $(\triangle^{\lambda})$, we consider the following problem $(\triangle^{\nu g})$:

$$\begin{aligned}&-\big[\Delta_1\big(\phi_p\big(\Delta_1 w(k-1,l)\big)\big)+\Delta_2\big(\phi_p\big(\Delta_2 w(k,l-1)\big)\big)\big]+h(k,l)\phi_q\big(w(k,l)\big)\\&\quad=\lambda\nu(k,l)g\big(w(k,l)\big),\quad (k,l)\in\mathbb{Z}(1,c)\times\mathbb{Z}(1,d),\end{aligned}$$

with boundary conditions (1.1), where $\nu > 0 : \mathbb{Z}(1,c) \times \mathbb{Z}(1,d) \to \mathbb{R}$, and $g : [0,+\infty) \to \mathbb{R}$ is a continuous function.

Define

$$V = \sum_{l=1}^{d}\sum_{k=1}^{c} \nu(k,l), \quad G(t) = \int_0^t g(s)\,ds, \quad M = \max\bigl\{\nu(k,l),(k,l) \in \mathbb{Z}(1,c)\times\mathbb{Z}(1,d)\bigr\} > 0.$$

Corollary 3.3 *Suppose that there exist two positive constants c_1 and d_1 such that*

$$\left(\frac{2c+2d}{p}\right)d_1^p + \frac{\tilde{h}}{q}d_1^q > r = \frac{4^p + h_*(c+d+2)^{p-1}}{p(c+d+2)^{p-1}}\min\bigl\{c_1^q, c_1^p\bigr\}$$

and

$$\sum_{l=1}^{d}\sum_{k=1}^{c}\max_{\eta\in[0,c_1]} F\bigl((k,l),\eta\bigr) = VG(c_1).$$

Suppose that the following conditions are satisfied:

(H_1') $g(\xi) > 0$ *for each* $\xi \in [0,c_1]$,

(H_2') $\frac{G(d_1)}{(\frac{2c+2d}{p})d_1^p + \frac{\tilde{h}}{q}d_1^q} > \frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}} \times \frac{G(c_1)}{\min\{c_1^q, c_1^p\}}$,

(H_3') $\limsup_{\xi\to+\infty}\frac{G(\xi)}{\xi^p} < \frac{4^p V \sup_{\Phi(w)\le r} G(w)}{Mcd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}$.

Then for any $\lambda \in \bar{\Lambda}_1$, problem $(\triangle^{\nu g})$ possesses at least three positive solutions, where

$$\bar{\Lambda}_1 := \left(\frac{(\frac{2c+2d}{p})d_1^p + \frac{\tilde{h}}{q}d_1^q}{G(d_1)}, \frac{(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\sup_{\Phi(w)\le r} G(w)}\right).$$

4 An example

We give an example illustrating our main results.

Example 4.1 Consider problem $(\triangle^{\nu g})$ where $c = d = q = 2$, $p = 3$,

$$h(k,l) = \frac{k}{l}, \quad (k,l) \in \mathbb{Z}(1,2)\times\mathbb{Z}(1,2),$$

$$\nu(k,l) = k + l, \quad (k,l) \in \mathbb{Z}(1,2)\times\mathbb{Z}(1,2),$$

and

$$g(w) = \begin{cases} 10w^9 + \frac{1}{2}\cos w, & 0 \le w \le \frac{5}{2}\pi, \\ \frac{10\times 5^9}{2^9}\pi^9 \sin w, & w > \frac{5}{2}\pi. \end{cases}$$

We can infer that $\tilde{h} = \sum_{l=1}^2\sum_{k=1}^2 h(k,l) = 4\frac{1}{2}$, $h_* = \min\{h(k,l) : (k,l) \in \mathbb{Z}(1,2)\times\mathbb{Z}(1,2)\} = \frac{1}{2}$, $V = \sum_{l=1}^2\sum_{k=1}^2 \nu(k,l) = 12$, $M = 4$, and

$$G(w) = \begin{cases} w^{10} + \frac{1}{2}\sin w, & 0 \le w \le \frac{5}{2}\pi, \\ -\frac{10\times 5^9}{2^9}\pi^9\cos w + \frac{5^{10}}{2^{10}}\pi^{10} + \frac{1}{2}, & w > \frac{5}{2}\pi. \end{cases}$$

Let $c_1 = 2$ and $d_1 = 3$. The we have that $h(k,l) = \frac{k}{l}$, and then $\tilde{h} = 4\frac{1}{2}$, $h_* = 2$. Letting $c = d = p = q = c_1 = 2$, and $d_1 = 3$, we have

$$\left(\frac{2c+2d}{p}\right)d_1^p + \frac{\tilde{h}}{q}d_1^q = 92.25 > \frac{4^p + h_*(c+d+2)^{p-1}}{p(c+d+2)^{p-1}} \min\left\{c_1^q, c_1^p\right\} \approx 3.037.$$

Obviously, $g(w) > 0$ for all $w \in [0,2]$, so that condition (H_1') of Corollary 3.3 is satisfied.

Then we obtain

$$\frac{p(c+d+2)^{p-1}}{4^p + h_*(c+d+2)^{p-1}} \times \frac{G(c_1)}{\min\{c_1^q, c_1^p\}} \approx \frac{1024.455}{3.037} \approx 337.324 \tag{4.1}$$

and

$$\frac{G(d_1)}{(\frac{2c+2d}{p})d_1^p + \frac{\tilde{h}}{q}d_1^q} \approx \frac{59{,}049.0705}{92.25} \approx 640.098. \tag{4.2}$$

Therefore, by (4.1) and (4.2) condition (H_2') of Corollary 3.3 is satisfied.

We continue by checking condition (H_3') of Corollary 3.3. Obviously, $V > 0$, $M > 0$, $\sup_{\Phi(w)\le r} G(w) > 0$, and $cd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\} > 0$,

$$\limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = \limsup_{\xi\to+\infty} \frac{-\frac{10\times 5^9}{2^9}\pi^9\cos\xi + \frac{5^{10}}{2^{10}}\pi^{10} + \frac{1}{2}}{\xi^3} = 0, \tag{4.3}$$

and

$$\frac{4^p V \sup_{\Phi(w)\le r} G(w)}{Mcd(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}} > 0. \tag{4.4}$$

Combining (4.3) with (4.4), we obtain that condition (H_3') of Corollary 3.3 is satisfied.

To sum up, all the conditions of Corollary 3.3 can be satisfied.

Since $\sup_{\Phi(w)\le r} G(w) \le \max_{|w|\le c_1} G(w)$, we have

$$\frac{(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\sup_{\Phi(w)\le r} G(w)} \ge \frac{(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\max_{|w|\le c_1} G(w)}.$$

We approach

$$\left(\frac{(\frac{2c+2d}{p})d_1^p + \frac{\tilde{h}}{q}d_1^q}{G(d_1)}, \frac{(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\sup_{\Phi(w)\le r} G(w)}\right)$$
$$\text{with } \left(\frac{(\frac{2c+2d}{p})d_1^p + \frac{\tilde{h}}{q}d_1^q}{G(d_1)}, \frac{(4^p + h_*(c+d+2)^{p-1})\min\{c_1^q, c_1^p\}}{p(c+d+2)^{p-1}\max_{|w|\le c_1} G(w)}\right).$$

Then for every $\lambda \in (0.002, 0.003)$, problem $(\triangle^{vg})$ possesses at least three positive solutions.

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