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Regularity and uniqueness for the 3D compressible magnetohydrodynamic equations

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Abstract

In this paper, some new L^p gradient estimates are justified for the three-dimensional compressible magnetohydrodynamic equations in the whole space \mathbb{R}^3 . The key to derive the estimate $\|\nabla \mathbf{u}\|_{L^3}$ is the “div-curl” decomposition technique. For regular initial data with small energy, we prove the existence of global solutions belonging to a new class of functions in which the uniqueness can be shown to hold.

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1 Introduction

We are concerned with the Cauchy problem of three-dimensional compressible magnetohydrodynamic (MHD) equations which read as (cf. [2, 16])

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) = -\nabla \times (\nu \nabla \times \mathbf{B}), \\ \operatorname{div} \mathbf{B} = 0, \end{cases} \quad (1.1)$$

where $t > 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, the unknown functions ρ , $\mathbf{u} = (u^1, u^2, u^3)$, $\mathbf{B} = (B^1, B^2, B^3)$ are the fluid density, velocity, and magnetic field, respectively. The pressure $P = P(\rho)$ satisfies the condition

$$P(\rho) = A\rho^\gamma \quad \text{with } A > 0, \gamma > 1, \quad (1.2)$$

where γ is the adiabatic exponent and A is a physical constant. The positive constant ν is the resistivity coefficient, and the viscosity coefficients μ and λ satisfy

$$\mu > 0, \lambda + \frac{2}{3}\mu \geq 0. \quad (1.3)$$

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Let us consider the Cauchy problem of (1.1)–(1.3) with the far-field behavior

$$(\rho, \mathbf{u}, \mathbf{B})(x, t) \rightarrow (1, 0, 0) \quad \text{as } |x| \rightarrow \infty, t > 0, \quad (1.4)$$

and the initial conditions

$$(\rho, \mathbf{u}, \mathbf{B})(x, 0) = (\rho_0, \mathbf{u}_0, \mathbf{B}_0)(x) \quad \text{with } x \in \mathbb{R}^3. \quad (1.5)$$

Without losing generality, let us assume that the far-field state of density at infinity is equivalent to 1.

This set of equations (1.1) describes the interaction between fluid flow and magnetic field, which has been studied by many literature works [3, 4, 6, 7, 9, 22, 23]. For the local strong solutions to the compressible MHD flows, Vol’pert and Khudiaevev in [20] got the local strong solutions under the conditions of large initial data and positive initial density. Later, Fan and Yu in [10] extended Vol’pert and Khudiaevev’s results as the initial density may contain vacuum. In [18], Lu and Huang investigated the 2D full compressible MHD equations with zero heat-conduction and obtained a local strong solution as the initial density and initial magnetic field decay not too slow at infinity. For the global solutions, Fan and Li in [8] investigated the 3D compressible non-isentropic MHD flows with zero resistivity and got the global strong solutions which do not need the positivity of initial density. In [12], Hu and Wang got the existence of a global variational weak solution to the three-dimensional full magnetohydrodynamic flows. Later, in [13], they also got the global existence and large-time behavior of solutions to the 3D equations of compressible MHD flows. In [22], Zhang, Jiang, and Xie obtained the global existence of weak solutions with cylindrical symmetry to the initial boundary value problems of MHD equations in plasma physics.

The global existence and uniqueness of strong solutions to Cauchy problem (1.1) are much subtle and remain open. Therefore, the main purpose of this paper is to investigate the global existence and uniqueness of solutions to (1.1)–(1.3). Lv, Shi, and Xu in [19] studied the Cauchy problem of 2D or 3D compressible MHD flows with vacuum as far-field density and obtained the global existence and uniqueness of strong solutions under the conditions of small total energy. Li, Xu, and Zhang [17] proved the global well-posedness of classical solution to problem (1.1)–(1.5) provided that the initial energy is suitably small. However, the results obtained in [17, 19] also required the following compatibility conditions:

$$-\mu \Delta \mathbf{u}_0 - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}_0 + \nabla P(\rho_0) - (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = \rho_0^{1/2} g \quad \text{for some } g \in L^2. \quad (1.6)$$

Roughly speaking, when the density contains vacuum, condition (1.6) implies

$$\lim_{t \rightarrow 0^+} (\sqrt{\rho} \dot{\mathbf{u}})(x, t) \in L^2, \quad (1.7)$$

where $\dot{\mathbf{f}} = \mathbf{f}_t + \mathbf{u} \cdot \nabla \mathbf{f}$ denotes the material derivative. However, if the initial density is strictly positive, then (1.7) can be written as

$$\lim_{t \rightarrow 0^+} \dot{\mathbf{u}}(x, t) \in L^2. \quad (1.8)$$

Obviously, in order to ensure that (1.6)/ (1.7) or (1.8) hold, the initial velocity has to at least possess H^2 -regularity:

$$\mathbf{u}_0 \in H^2. \quad (1.9)$$

Thus, it follows from (1.6)/ (1.8) and the standard L^2 -method that for any $0 < T < \infty$ we have

$$\sup_{t \in [0, T]} \|\dot{\mathbf{u}}\|_{L^2} + \int_0^T \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 \leq C(T). \quad (1.10)$$

Clearly, one can use (1.10) to derive the higher-order estimates of the solutions in the previous articles. Similar to the compressible Navier–Stokes equations (see, for example, [11, 15]), we introduce the effective viscous flux F , the vorticity ω :

$$F \triangleq (2\mu + \lambda) \operatorname{div} \mathbf{u} - (P(\rho) - P(1)) - \frac{1}{2} |\mathbf{B}|^2, \quad \omega \triangleq \nabla \times \mathbf{u}. \quad (1.11)$$

Then it is easily derived from (1.1) that

$$\Delta F = \operatorname{div}(\rho \dot{\mathbf{u}}) - \operatorname{div} \operatorname{div}(\mathbf{B} \otimes \mathbf{B}) \quad \text{and} \quad \mu \Delta \omega = \nabla \times (\rho \dot{\mathbf{u}} - \operatorname{div}(\mathbf{B} \otimes \mathbf{B})). \quad (1.12)$$

To sum up, can the condition of (1.9) be weakened so far when exploring the global solutions of the MHD equations? When the magnetic field in (1.1) was replaced by the temperature, Xu and Zhang in [21] obtained a global “intermediate weak” solution in the nonvacuum case with lower regularity. Thus, our main aim in this paper is to prove the following theorem of solutions with lower regularity than that in (1.9).

Theorem 1.1 *For any given number $p \in [9/2, 6)$, suppose that the initial data $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ satisfy*

$$\inf \rho_0(x) > 0, \quad \rho_0 - 1 \in H^1 \cap W^{1,p}, \quad \mathbf{u}_0 \in H^1 \cap W^{1,3}, \quad \mathbf{B}_0 \in H^1. \quad (1.13)$$

Then there exists a positive constant $\varepsilon > 0$, depending on $\mu, \lambda, \nu, A, \gamma, \inf \rho_0, \sup \rho_0, \|\nabla \mathbf{u}_0\|_{L^2}$ and $\|\nabla \mathbf{B}_0\|_{L^2}$, such that if

$$E_0 \triangleq \|(\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0)\|_{L^2}^2 \leq \varepsilon, \quad (1.14)$$

then Cauchy problem (1.1)–(1.5) has a global solution $(\rho, \mathbf{u}, \mathbf{B})$ in $\mathbb{R}^3 \times (0, \infty)$ satisfying, for any $0 < T < \infty$,

$$\left\{ \begin{array}{l} \rho - 1 \in C([0, T]; H^1 \cap W^{1,p}), \quad \inf \rho(x, t) > 0, \\ (\mathbf{u}, \mathbf{B}) \in C([0, T]; L^2 \cap L^r) \quad 2 \leq r < 6, \\ \mathbf{u} \in L^\infty(0, T; H^1 \cap W^{1,3}) \cap L^2(0, T; H^2) \cap L^q(0, T; W^{1,\infty}), \\ \mathbf{B} \in L^\infty([0, T]; H^1) \cap L^2(0, T; H^2), \\ t^{1/2} \dot{\mathbf{u}} \in L^\infty(0, T; L^2), \quad t^{1/2} \nabla \dot{\mathbf{u}} \in L^2(0, T; L^2), \\ (t^{1/2} \mathbf{B}_t, t^{1/2} \nabla^2 \mathbf{B}) \in L^\infty([0, T]; L^2), \quad (t^{1/2} \nabla \mathbf{B}_t, t^{1/2} \nabla^3 \mathbf{B}) \in L^2([0, T]; L^2), \end{array} \right. \quad (1.15)$$

with $1 < q < (4p)/(5p - 6)$. Moreover, the uniqueness of the solutions belonging to the class of functions (1.15) holds.

Remark 1.1 In comparison with (1.9), the regularity of initial velocity $\mathbf{u}_0 \in H^1 \cap W^{1,3}$ is much weaker. The conclusion obtained in Theorem 1.1 becomes a new and interesting one.

Remark 1.2 The existence result obtained in Theorem 1.1 excluded the state in which the initial density is vacuum. Indeed, it is worth mentioning that the estimates stated in Sect. 2, especially in Lemmas 2.2, 2.3, and 2.5, are independent of the lower bound of density; in other words, the initial density can contain a vacuum (i.e. $\rho_0 \geq 0$). However, the next important estimate $\|\nabla \mathbf{u}\|_{L^3}$, which is essentially and technically needed for the analysis of uniqueness, requires the initial density to exclude the vacuum state. In fact, when we attempt to obtain the estimate $\|\nabla \mathbf{u}\|_{L^3}$ directly, there are usually two ways, one is to use the L^p -estimate of equation (1.1)₂, namely,

$$\begin{aligned} \|\nabla \mathbf{u}\|_{L^3} &\leq C\|\rho \dot{\mathbf{u}}\|_{W^{-1,3}} + C\|\mathbf{B} \cdot \nabla \mathbf{B} - \nabla |\mathbf{B}|^2\|_{W^{-1,3}} + C\|\nabla(P(\rho) - P(1))\|_{W^{-1,3}} \\ &\leq C\|\rho \dot{\mathbf{u}}\|_{W^{-1,3}} + C\|\nabla \mathbf{B}\|_{L^2}^2 + C(\bar{\rho})\|\rho - 1\|_{L^2}^{2/3}. \end{aligned}$$

Unfortunately, the first term on the right-hand side cannot be bounded by Lemmas 2.1–2.5. So, another way is to use the “div-curl” decomposition technique, that is,

$$\|\nabla \mathbf{u}\|_{L^3} \leq C(\|\operatorname{div} \mathbf{u}\|_{L^3} + \|\operatorname{curl} \mathbf{u}\|_{L^3}).$$

To do this, we first operate “div” and “curl” to both sides of (1.1)₂, and then multiply $|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}$ and $|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}$, respectively, and integrate them over \mathbb{R}^3 , which yields

$$\begin{aligned} &\frac{1}{3} \frac{d}{dt} \int \rho (|\operatorname{div} \mathbf{u}|^3 + |\operatorname{curl} \mathbf{u}|^3) dx \\ &\quad + (2\mu + \lambda) \int (|\operatorname{div} \mathbf{u}| |\nabla \operatorname{div} \mathbf{u}|^2 + |\operatorname{div} \mathbf{u}| |\nabla |\operatorname{div} \mathbf{u}||^2) dx \\ &\quad + \mu \int (|\operatorname{curl} \mathbf{u}| |\nabla \operatorname{curl} \mathbf{u}|^2 + |\operatorname{curl} \mathbf{u}| |\nabla |\operatorname{curl} \mathbf{u}||^2) dx \\ &= - \int (\mathbf{u}_t \cdot \nabla \rho) (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx - \int (\nabla \rho \times \mathbf{u}_t) (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx + \mathcal{R}, \end{aligned}$$

where the symbol \mathcal{R} denotes the terms which can be absorbed/bounded by the left-hand side and the estimates obtained in Lemmas 2.1–2.5. Due to the lower regularity of initial velocity, $\mathbf{u}_0 \in H^1 \cap W^{1,3}$, but $\mathbf{u}_0 \notin H^2$, thus the estimate $\mathbf{u}_t \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ cannot be obtained. In order to deal with the right two terms of the inequality, we use the following equality to substitute the expression of \mathbf{u}_t into the integrals:

$$\mathbf{u}_t = \rho^{-1} \left(\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 - \nabla P \right) - \mathbf{u} \cdot \nabla \mathbf{u}.$$

Here, we need $\inf \rho(x, t) > 0$, in other words, we need $\inf \rho_0 > 0$. The consequence of doing so is that some new terms appear, but these terms can be controlled by the estimates

obtained in Lemmas 2.2–2.5. To sum up, we need the initial density to exclude the vacuum state, and then the estimate $\|\nabla \mathbf{u}\|_{L^3}$ can be well controlled, which is essentially and technically needed for the analysis of uniqueness.

The rest of the paper is organized as follows: in the next section, Sect. 2, based on the lower-order estimates achieved in [17], we obtain the necessary a priori estimates on strong solutions. Then finally the main result, Theorem 1.1, is proved in Sect. 3.

2 Preliminaries

In this section, we establish some necessary a priori bounds for smooth solutions to Cauchy problem (1.1)–(1.5). Before stating our main results, we first let $(\rho, \mathbf{u}, \mathbf{B})$ be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3 \times [0, T]$ for some $0 < T < \infty$. We can check that the equations of (1.1) can be written as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla P(\rho) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2, \\ \mathbf{B}_t + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{B} \operatorname{div} \mathbf{u} = \nu \Delta \mathbf{B}, \\ \operatorname{div} \mathbf{B} = 0. \end{cases} \quad (2.1)$$

We start with the following estimates, which have been achieved in [17, Proposition 3.1], thus we do not explain in detail.

Lemma 2.1 *For given constants $M > 0$, $\bar{\rho} > 2$, assume that $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ satisfies*

$$0 \leq \inf \rho_0 \leq \sup \rho_0 \leq \bar{\rho}, \quad \|(\nabla \mathbf{u}_0, \nabla \mathbf{B}_0)\|_{L^2} \leq M. \quad (2.2)$$

There exist positive constants K and ε , depending on $\mu, \lambda, \nu, A, \gamma, \bar{\rho}$, and M , such that if

$$E_0 \triangleq \int \left(\frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + G(\rho_0) + \frac{1}{2} |\mathbf{B}_0|^2 \right) dx \leq \varepsilon, \quad (2.3)$$

where $G(\cdot)$ is the potential energy density given by

$$G(\rho) \triangleq \rho \int_1^\rho \frac{P(s) - P(1)}{s^2} ds,$$

then the following estimates hold:

$$0 \leq \rho(x, t) \leq 2\bar{\rho}, \quad \forall (x, t) \in \mathbb{R}^3 \times [0, T], \quad (2.4)$$

$$\sup_{0 \leq t \leq T} \|\mathbf{B}\|_{L^3}^3 + \int_0^T \|\mathbf{B}\|_{L^9}^3 dx \leq E_0^{1/9}, \quad (2.5)$$

$$\sup_{0 \leq t \leq T} (\|\rho - 1\|_{L^2}^2 + \|\sqrt{\rho} \mathbf{u}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2}^2) + \int_0^T (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) dt \leq KE_0, \quad (2.6)$$

$$\sup_{0 \leq t \leq T} (\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2) dt \leq K. \quad (2.7)$$

Remark 2.1 Estimates (2.4)–(2.7) are independent of time T and the lower bound of density. Furthermore, if the initial density possesses a positive lower bound, then it follows from the expression of $G(\cdot)$ that

$$E_0 \triangleq \|(\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0)\|_{L^2}^2. \quad (2.8)$$

In the following, we will use the convention that C or C_i ($i = 1, 2, \dots$) denotes a generic positive constant depending on $\mu, \lambda, \nu, \gamma, A, \bar{\rho}$, the initial data, and T .

First, we will prove the following refined t -weighted estimates of the material derivative and the gradient of magnetic.

Lemma 2.2 *Under the conditions of (2.2) and (2.3), there exists a positive constant C depending on T such that*

$$\begin{aligned} & \sup_{0 \leq t \leq T} t(\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2) \\ & + \int_0^T t(\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{H^2}^2) dt \leq C(T). \end{aligned} \quad (2.9)$$

Proof In order to prove (2.9), we first need to apply $t\dot{u}^j[\partial_t + \operatorname{div}(\mathbf{u} \cdot)]$ to the both sides of the j th equation of (2.1)₂, then integrate by parts over \mathbb{R}^3 , and add the results together. We get by some calculations that

$$\begin{aligned} \frac{1}{2} \left(t \int \rho |\dot{\mathbf{u}}|^2 dx \right)_t &= \frac{1}{2} \int \rho |\dot{\mathbf{u}}|^2 dx + \mu \int t\dot{u}^j [\Delta u_t^j + \operatorname{div}(\mathbf{u} \Delta u^j)] dx \\ &+ (\lambda + \mu) \int t\dot{u}^j [\partial_j \partial_t (\operatorname{div} \mathbf{u}) + \operatorname{div}(\mathbf{u} \partial_j (\operatorname{div} \mathbf{u}))] dx \\ &- \int t\dot{u}^j [\partial_j P_t + \operatorname{div}(\mathbf{u} \partial_j P)] dx \\ &- \frac{1}{2} \int t\dot{u}^j [\partial_t \partial_j (|\mathbf{B}|^2) + \operatorname{div}(\mathbf{u} \partial_j (|\mathbf{B}|^2))] dx \\ &+ \int t\dot{u}^j [\partial_t (B^i \partial_i B^j) + \operatorname{div}(\mathbf{u} B^i \partial_i B^j)] dx \\ &= \frac{1}{2} \int \rho |\dot{\mathbf{u}}|^2 dx + \sum_{i=1}^5 I_i. \end{aligned} \quad (2.10)$$

Now, we estimate the right-hand side of terms of (2.10). Due to the Cauchy–Schwarz inequality, we get by integration by parts that

$$\begin{aligned} I_1 &= -\mu \int t(\partial_k \dot{u}^j \partial_k u_t^j + \partial_k \dot{u}^j u^k \partial_l \partial_l u^j) dx \\ &= -\mu \int t(|\nabla \dot{\mathbf{u}}|^2 - \partial_k \dot{u}^j \partial_k u^l \partial_l u^j + \partial_k \dot{u}^j \partial_l u^l \partial_k u^j - \partial_k \dot{u}^j \partial_l u^k \partial_l u^j) dx \\ &\leq -\frac{7\mu}{8} (t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2) + Ct \|\nabla \mathbf{u}\|_{L^4}^4 \end{aligned} \quad (2.11)$$

and

$$I_2 \leq -(\mu + \lambda)(t \|\operatorname{div} \dot{\mathbf{u}}\|_{L^2}^2) + \frac{\mu}{8}(t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2) + Ct \|\nabla \mathbf{u}\|_{L^4}^4. \quad (2.12)$$

In order to estimate I_3 , we notice that

$$(P(\rho) - P(1))_t + \mathbf{u} \cdot \nabla (P(\rho) - P(1)) + \gamma P(\rho) \operatorname{div} \mathbf{u} = 0, \quad (2.13)$$

which, together with (2.4), yields

$$\begin{aligned} I_3 &= \int t(P_t \operatorname{div} \dot{\mathbf{u}} + \partial_i \dot{u}^j u^i \partial_j P) dx \\ &= \int t(\operatorname{div} \dot{\mathbf{u}}(-\gamma P(\rho) \operatorname{div} \mathbf{u} - \mathbf{u} \cdot \nabla P(\rho)) - P(\rho) \partial_j (\partial_i \dot{u}^j u^i)) dx \\ &= \int tP(\rho)[- \gamma (\operatorname{div} \dot{\mathbf{u}})(\operatorname{div} \mathbf{u}) + \partial_j \dot{u}^j \partial_i u^i - \partial_i \dot{u}^j \partial_j u^i] dx \\ &\leq \frac{\mu}{8}(t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2) + Ct \|\nabla \mathbf{u}\|_{L^2}^2. \end{aligned} \quad (2.14)$$

Next, for I_4 , we obtain after using the integration by parts that

$$\begin{aligned} I_4 &= \int t(\partial_j \dot{u}^j B^i B_t^i + \partial_i \dot{u}^j u^i B^k \partial_j B^k) dx \\ &\leq Ct \|\nabla \dot{\mathbf{u}}\|_{L^2} (\|\mathbf{B}_t\|_{L^6} \|\mathbf{B}\|_{L^3} + \|\mathbf{u}\|_{L^6} \|\mathbf{B}\|_{L^\infty} \|\nabla \mathbf{B}\|_{L^3}) \\ &\leq \frac{\mu}{8}(t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2) + Ct \|\nabla \mathbf{B}_t\|_{L^2}^2 \|\mathbf{B}\|_{L^3}^2 + Ct \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{B}\|_{L^2}^2 \|\nabla^2 \mathbf{B}\|_{L^2}^2 \\ &\leq \frac{\mu}{8}(t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2) + Ct \|\nabla \mathbf{B}_t\|_{L^2}^2 \|\mathbf{B}\|_{L^3}^2 + Ct (\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4) \|\nabla^2 \mathbf{B}\|_{L^2}^2, \end{aligned} \quad (2.15)$$

where we have used the Gagliardo–Nirenberg inequality

$$\|\mathbf{v}\|_{L^p} \leq C \|\mathbf{v}\|_{L^2}^{\frac{6-p}{2p}} \|\nabla \mathbf{v}\|_{L^2}^{\frac{3p-6}{2p}}, \quad \forall \mathbf{v} \in H^1 \text{ and } 2 \leq p \leq 6. \quad (2.16)$$

By using (2.16) and integrating by parts, we obtain

$$\begin{aligned} I_5 &= \int t(\dot{u}^j \partial_i B^j B_t^i + \dot{u}^j \partial_i B_t^j B^i - \partial_k \dot{u}^j u^k B^i \partial_i B^j) dx \\ &\leq Ct \|\nabla \dot{\mathbf{u}}\|_{L^2} \|\nabla \mathbf{B}_t\|_{L^2} \|\mathbf{B}\|_{L^3} + Ct \|\nabla \dot{\mathbf{u}}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\mathbf{B}\|_{L^\infty} \|\nabla \mathbf{B}\|_{L^3} \\ &\leq \frac{\mu}{8}(t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2) + Ct \|\nabla \mathbf{B}_t\|_{L^2}^2 \|\mathbf{B}\|_{L^3}^2 + Ct (\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4) \|\nabla^2 \mathbf{B}\|_{L^2}^2. \end{aligned} \quad (2.17)$$

Substituting (2.11)–(2.17) into (2.10), we have

$$\begin{aligned} \frac{d}{dt}(t \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^2) + t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 &\leq Ct \|\nabla \mathbf{B}_t\|_{L^2}^2 \|\mathbf{B}\|_{L^3}^2 + C(\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^2 + t \|\nabla \mathbf{u}\|_{L^2}^2) \\ &\quad \times Ct \|\nabla \mathbf{u}\|_{L^4}^4 + Ct (\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4) \|\nabla^2 \mathbf{B}\|_{L^2}^2. \end{aligned} \quad (2.18)$$

What is left is to estimate the term $\|\nabla \mathbf{B}_t\|_{L^2}$. To this end, noticing that

$$\mathbf{B}_{tt} - \nu \Delta \mathbf{B}_t = (\mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \operatorname{div} \mathbf{u})_t$$

and using the fact that $\mathbf{u}_t = \dot{\mathbf{u}} - \mathbf{u} \cdot \nabla \mathbf{u}$, we obtain after direct computations that

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} (t \|\mathbf{B}_t\|_{L^2}^2) + \nu t \|\nabla \mathbf{B}_t\|_{L^2}^2 \\
 &= \frac{1}{2} \|\mathbf{B}_t\|_{L^2}^2 + \int t (\mathbf{B}_t \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B}_t - \mathbf{B}_t \operatorname{div} \mathbf{u}) \cdot \mathbf{B}_t \, dx \\
 & \quad + \int t (\mathbf{B} \cdot \nabla \dot{\mathbf{u}} - \dot{\mathbf{u}} \cdot \nabla \mathbf{B} - \mathbf{B} \operatorname{div} \dot{\mathbf{u}}) \cdot \mathbf{B}_t \, dx \\
 & \quad + \int t (-\mathbf{B} \cdot \nabla (\mathbf{u} \cdot \nabla \mathbf{u}) + (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \nabla \mathbf{B} + \mathbf{B} \operatorname{div} (\mathbf{u} \cdot \nabla \mathbf{u})) \cdot \mathbf{B}_t \, dx \\
 &= \frac{1}{2} \|\mathbf{B}_t\|_{L^2}^2 + \sum_{i=1}^3 J_i.
 \end{aligned} \tag{2.19}$$

Now, we estimate J_i as follows. By using (2.16) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
 J_1 &\leq Ct \|\mathbf{B}_t\|_{L^3} \|\mathbf{B}_t\|_{L^6} \|\nabla \mathbf{u}\|_{L^2} + Ct \|\mathbf{u}\|_{L^6} \|\nabla \mathbf{B}_t\|_{L^2} \|\mathbf{B}_t\|_{L^3} \\
 &\leq Ct \|\nabla \mathbf{B}_t\|_{L^2}^{\frac{3}{2}} \|\mathbf{B}_t\|_{L^2}^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{L^2} \\
 &\leq \frac{\nu}{8} (t \|\nabla \mathbf{B}_t\|_{L^2}^2) + Ct \|\mathbf{B}_t\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2}^4
 \end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
 J_2 &= \int t (\mathbf{B} \cdot \nabla \dot{\mathbf{u}} - \dot{\mathbf{u}} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \operatorname{div} \dot{\mathbf{u}}) \cdot \mathbf{B}_t \, dx \\
 &\leq Ct \|\nabla \dot{\mathbf{u}}\|_{L^2} \|\mathbf{B}\|_{L^3} \|\nabla \mathbf{B}_t\|_{L^2} \\
 &\leq \frac{\nu}{8} (t \|\nabla \mathbf{B}_t\|_{L^2}^2) + Ct \|\mathbf{B}\|_{L^3}^2 \|\nabla \dot{\mathbf{u}}\|_{L^2}^2
 \end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
 J_3 &= \int t (B^i u^j \partial_j u^k \partial_i B_t^k + u^k \partial_k u^i \partial_i B^j B_t^j \\
 & \quad - \partial_j B^i u^k \partial_k u^j B_t^i - B^i u^k \partial_k u^j \partial_j B_t^i) \, dx \\
 &\leq \|\nabla \mathbf{B}_t\|_{L^2} \|\nabla \mathbf{B}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^6} \\
 &\leq \frac{\nu}{8} (t \|\nabla \mathbf{B}_t\|_{L^2}^2) + Ct \|\nabla \mathbf{B}\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^6}^2.
 \end{aligned} \tag{2.22}$$

Putting (2.20)–(2.22) into (2.19), we obtain

$$\begin{aligned}
 & \frac{d}{dt} (t \|\mathbf{B}_t\|_{L^2}^2) + t \|\nabla \mathbf{B}_t\|_{L^2}^2 \\
 &\leq Ct \|\mathbf{B}\|_{L^3}^2 \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + C \|\mathbf{B}_t\|_{L^2}^2 + Ct \|\mathbf{B}_t\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2}^4 \\
 & \quad + Ct \|\nabla \mathbf{B}\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^6}^2.
 \end{aligned} \tag{2.23}$$

Thus, integrating the resulting equations (2.18) and (2.23) over $(0, T)$ and using (2.3), (2.6), and (2.7), we deduce after adding them together that

$$\begin{aligned} & \sup_{0 \leq t \leq T} (t \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^2 + t \|\mathbf{B}_t\|_{L^2}^2) + \int_0^T t (\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2) dt \\ & \leq C + C_1 \int_0^T t \|\nabla \mathbf{u}\|_{L^4}^4 dt + C \int_0^T t \|\nabla \mathbf{u}\|_{L^6}^2 dt. \end{aligned} \quad (2.24)$$

We are now in a position of estimating the last two terms on the right-hand side of (2.24). Indeed, it follows from (1.2), (2.4) and the L^p -estimates that

$$\|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C (\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}), \quad \forall p \in [2, 6], \quad (2.25)$$

so that, using (1.11), (2.4)–(2.6), (2.16), (2.25), and the standard L^p -estimates, we find

$$\begin{aligned} \|\nabla \mathbf{u}\|_{L^6} & \leq C (\|\operatorname{div} \mathbf{u}\|_{L^6} + \|\omega\|_{L^6}) \\ & \leq C (\|F\|_{L^6} + \|P(\rho) - P(1)\|_{L^6} + \|\mathbf{B}\|^2_{L^6} + \|\omega\|_{L^6}) \\ & \leq C (\|\nabla F\|_{L^2} + \|\nabla \omega\|_{L^2} + \|\nabla \mathbf{B}\|_{L^6} + 1) \\ & \leq C (1 + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2} + \|\nabla^2 \mathbf{B}\|_{L^2}), \end{aligned} \quad (2.26)$$

which, together with (2.7), gives

$$\int_0^T t \|\nabla \mathbf{u}\|_{L^6}^2 dt \leq C \int_0^T t (1 + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2) dt \leq C(T). \quad (2.27)$$

On the other hand, it follows from (2.1)₃ that

$$\|\nabla^2 \mathbf{B}\|_{L^2} \leq C (\|\mathbf{B}_t\|_{L^2} + \|\nabla \mathbf{B}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2).$$

It follows from (2.7), (2.24), and (2.27) that

$$\begin{aligned} \sup_{0 \leq t \leq T} (t \|\nabla^2 \mathbf{B}\|_{L^2}^2) & \leq C \sup_{0 \leq t \leq T} t (\|\mathbf{B}_t\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2}^4) \\ & \leq C + C \int_0^T t \|\nabla \mathbf{u}\|_{L^4}^4 dt. \end{aligned} \quad (2.28)$$

In view of (2.7), (2.6), and (2.26), we have

$$\begin{aligned} \int_0^T t \|\nabla \mathbf{u}\|_{L^4}^4 dt & \leq C \int_0^T t \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^6}^3 dt \\ & \leq C + C \int_0^T t (\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^3 + \|\nabla^2 \mathbf{B}\|_{L^2}^3) dt \\ & \leq C + C \sup_{0 \leq t \leq T} \sqrt{t} (\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2} + \|\nabla^2 \mathbf{B}\|_{L^2}) \\ & \leq C + \delta \sup_{0 \leq t \leq T} (t \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^2 + t \|\nabla^2 \mathbf{B}\|_{L^2}^2), \end{aligned} \quad (2.29)$$

where $\delta > 0$ is an undetermined number.

Based upon (2.1)₃, it is easy to get that

$$\begin{aligned}\|\nabla \mathbf{B}\|_{H^2} &\leq C + C \|\nabla(\mathbf{B}_t + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{B} \operatorname{div} \mathbf{u})\|_{L^2} \\ &\leq C + C(\|\nabla \mathbf{B}_t\|_{L^2} + \|\nabla \mathbf{u}\|_{L^6} \|\nabla \mathbf{B}\|_{L^3} + \|\mathbf{u}\|_{L^6} \|\nabla^2 \mathbf{B}\|_{L^3} + \|\mathbf{B}\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2}) \\ &\leq C + C(\|\nabla \mathbf{B}_t\|_{L^2} + \|\nabla^2 \mathbf{u}\|_{L^2} \|\nabla \mathbf{B}\|_{L^2}^{1/2} \|\nabla^2 \mathbf{B}\|_{L^2}^{1/2}) \\ &\quad + C(\|\nabla \mathbf{u}\|_{L^2} \|\nabla^2 \mathbf{B}\|_{L^2}^{1/2} \|\nabla \mathbf{B}\|_{H^2}^{1/2} + \|\mathbf{B}\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2}),\end{aligned}$$

thus

$$\|\nabla \mathbf{B}\|_{H^2} \leq C + C(\|\nabla \mathbf{B}_t\|_{L^2} + \|\nabla^2 \mathbf{u}\|_{L^2} \|\nabla^2 \mathbf{B}\|_{L^2}^{1/2} + \|\nabla^2 \mathbf{B}\|_{L^2} + \|\nabla^2 \mathbf{u}\|_{L^2}),$$

from which, together with (2.7), (2.26)–(2.28), we have

$$\begin{aligned}\int_0^T t \|\nabla \mathbf{B}\|_{H^2}^2 dt &\leq C + C \int_0^T t \|\nabla \mathbf{B}_t\|_{L^2}^2 dt + C \sup_{0 \leq t \leq T} (t \|\nabla^2 \mathbf{u}\|_{L^2}^2) \int_0^T \|\nabla^2 \mathbf{B}\|_{L^2}^2 dt \\ &\quad + C \int_0^T t (\|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2) \\ &\leq C(T).\end{aligned}$$

Therefore, putting (2.27)–(2.29) into (2.24) and choosing a suitably small number $\delta > 0$, we immediately obtain (2.9). \square

Lemma 2.3 *Under the conditions of (2.2) and (2.3), there exists a positive constant C depending on T such that*

$$\int_0^T (\|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}^q + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p}^q + \|\operatorname{div} \mathbf{u}\|_{L^\infty}^q + \|\operatorname{curl} \mathbf{u}\|_{L^\infty}^q) dt \leq C(T), \quad (2.30)$$

where $\operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u}$, and (p, q) satisfies

$$3 < p < 6 \quad \text{and} \quad 1 < q < \frac{4p}{5p-6} < \frac{4}{3}. \quad (2.31)$$

Proof It follows from (2.4) and (2.16) that

$$\begin{aligned}\|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}^q + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p}^q &\leq C(\|\mathbf{B}\|_{L^\infty}^q \|\nabla \mathbf{B}\|_{L^p}^q + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^{\frac{q(6-p)}{2p}} \|\nabla \dot{\mathbf{u}}\|_{L^2}^{\frac{q(3p-6)}{2p}}) \\ &\leq C(\|\nabla \mathbf{B}\|_{L^2}^{\frac{3q}{p}} \|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{q(2p-3)}{p}} + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^{\frac{q(6-p)}{2p}} \|\nabla \dot{\mathbf{u}}\|_{L^2}^{\frac{q(3p-6)}{2p}}),\end{aligned}$$

so that, by virtue of Lemma 2.2 and (2.7), we obtain

$$\begin{aligned}\int_0^T (\|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}^q + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p}^q) dt &\leq C \int_0^T (\|\nabla^2 \mathbf{B}\|_{L^2}^{\frac{q(2p-3)}{p}} + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^{\frac{q(6-p)}{2p}} \|\nabla \dot{\mathbf{u}}\|_{L^2}^{\frac{q(3p-6)}{2p}}) dt\end{aligned}$$

$$\begin{aligned}
&\leq C \sup_{0 \leq t \leq T} (t \|\nabla^2 \mathbf{B}\|_{L^2}^2)^{\frac{q(2p-3)}{2p}} \int_0^T t^{-\frac{q(2p-3)}{2p}} dt \\
&\quad + C \sup_{0 \leq t \leq T} (t \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2}^2)^{\frac{q(6-p)}{4p}} \int_0^T t^{-\frac{q}{2}} (t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2)^{\frac{q(3p-6)}{4p}} dt \\
&\leq C \int_0^T t^{-\frac{q(2p-3)}{2p}} dt + C \left(\int_0^T t^{-\frac{2pq}{4p-3pq+6q}} dt \right)^{\frac{4p-3pq+6q}{4p}} \left(\int_0^T t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 dt \right)^{\frac{q(3p-6)}{4p}} \\
&\leq C(T),
\end{aligned} \tag{2.32}$$

since $3 < p < 6$ and $1 < q < \frac{4p}{5p-6} < 2$ yield that

$$0 < \frac{q(2p-3)}{2p} < 1, \quad 0 < \frac{2pq}{4p-3pq+6q} < 1, \quad 0 < \frac{q(3p-6)}{4p} < 1.$$

Due to (2.4)–(2.7), (2.25), and the Sobolev embedding inequality, we have that

$$\begin{aligned}
&\|\operatorname{div} \mathbf{u}\|_{L^\infty} + \|\omega\|_{L^\infty} \\
&\leq C(\|F\|_{L^\infty} + \|P(\rho) - P(1)\|_{L^\infty} + \|\mathbf{B}\|^2_{L^\infty} + \|\omega\|_{L^\infty}) \\
&\leq C(1 + \|F\|_{L^\infty} + \|\omega\|_{L^\infty} + \|\nabla \mathbf{B}\|_{L^2} \|\nabla^2 \mathbf{B}\|_{L^2} + \|\nabla F\|_{L^p} + \|\nabla \omega\|_{L^p}) \\
&\leq C(1 + \|\nabla \mathbf{u}\|_{L^2} + \|\mathbf{B}\|_{L^2}^{1/2} \|\nabla \mathbf{B}\|_{L^2}^{1/2} + \|\nabla^2 \mathbf{B}\|_{L^2} + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}) \\
&\leq C(1 + \|\nabla^2 \mathbf{B}\|_{L^2} + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}).
\end{aligned} \tag{2.33}$$

On the other hand, since $1 < q < \frac{4}{3}$, we obtain

$$\int_0^T \|\nabla^2 \mathbf{B}\|_{L^2}^q dt \leq (t \|\nabla^2 \mathbf{B}\|_{L^2}^2)^{\frac{q}{2}} \int_0^T t^{-\frac{q}{2}} dt \leq C(T),$$

which, combined with (2.32) and (2.33), leads to (2.30). \square

Next, we need to estimate $\|\nabla \mathbf{u}\|_{L^p}$ with $1 < p < \infty$ and $p = \infty$, respectively. Clearly, since it holds that

$$-\Delta \mathbf{u} = \nabla \operatorname{div} \mathbf{u} - \nabla \times \operatorname{curl} \mathbf{u},$$

thus, for $1 < p < \infty$, we have

$$\|\nabla \mathbf{u}\|_{L^p} \leq C(\|\operatorname{div} \mathbf{u}\|_{L^p} + \|\operatorname{curl} \mathbf{u}\|_{L^p}).$$

However, for $p = \infty$, the above inequality cannot work. Thus, the following Beale, Kato, and Majda type inequality (cf. [1, 14]) will be used later to estimate $\|\nabla \mathbf{u}\|_{L^\infty}$.

Lemma 2.4 *For any $k \in \mathbb{Z}^+$ and $p \in (1, +\infty)$, let $D^{k,p} \triangleq \{\mathbf{v} \in L^1_{\text{loc}} | \partial^k \mathbf{v} \in L^p\}$ and $D^1 \triangleq D^{1,2}$ be the homogeneous Sobolev spaces. Then, for any $\mathbf{v} \in D^1 \cap D^{2,p}$ with $p \in (3, +\infty)$, there exists a positive constant $C(p) > 0$ such that, for all $\nabla \mathbf{v} \in L^2 \cap D^{1,p}$,*

$$\|\nabla \mathbf{v}\|_{L^\infty} \leq C(1 + \|\nabla \mathbf{v}\|_{L^2}) + C(\|\operatorname{div} \mathbf{v}\|_{L^\infty} + \|\nabla \times \mathbf{v}\|_{L^\infty}) \ln(e + \|\nabla^2 \mathbf{v}\|_{L^p}). \tag{2.34}$$

With the help of Lemmas 2.3 and 2.4, we can now derive the L^2 - L^p -estimates ($3 < p < 6$) of the gradient of density.

Lemma 2.5 *Under the conditions of (1.13) and (2.3), it holds that*

$$\sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^2 \cap L^p} + \|\rho_t\|_{L^2}) + \int_0^T (\|\nabla^2 \mathbf{u}\|_{L^p}^q + \|\nabla \mathbf{u}\|_{L^\infty}^q) dt \leq C(T), \quad (2.35)$$

where $3 < p < 6$ and $q > 1$ are the same ones as in (2.31).

Proof To prove (2.35), operating ∇ to both sides of (2.1)₁, then multiplying it by $|\nabla \rho|^{p-2} \nabla \rho$ with $p \in [2, 6]$ and integrating by parts over \mathbb{R}^3 , we know from (2.4) that

$$\frac{d}{dt} \|\nabla \rho\|_{L^p} \leq C(\|\nabla \mathbf{u}\|_{L^\infty} \|\nabla \rho\|_{L^p} + \|\nabla^2 \mathbf{u}\|_{L^p}). \quad (2.36)$$

Recalling that $\mathcal{L} \triangleq -\mu \Delta - (\mu + \lambda) \nabla \operatorname{div}$ is a strong elliptic operator (cf. [5]), we deduce from (2.1)₂ and (2.4) that, for any $p \in (3, 6)$,

$$\begin{aligned} \|\nabla^2 \mathbf{u}\|_{L^p} &\leq C(\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\nabla P\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p} + \|\nabla |\mathbf{B}|^2\|_{L^p}) \\ &\leq C(\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\nabla \rho\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}), \end{aligned} \quad (2.37)$$

which, combined with (2.36), yields

$$\frac{d}{dt} \|\nabla \rho\|_{L^p} \leq C(\|\nabla \mathbf{u}\|_{L^\infty} + 1) \|\nabla \rho\|_{L^p} + C(\|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}). \quad (2.38)$$

Using Lemma 2.4 and (2.37), we obtain that, for any $p \in (3, 6)$,

$$\begin{aligned} \|\nabla \mathbf{u}\|_{L^\infty} &\leq C + C(\|\operatorname{div} \mathbf{u}\|_{L^\infty} + \|\nabla \times \mathbf{u}\|_{L^\infty}) \ln(e + \|\nabla \rho\|_{L^p}) \\ &\quad + C(\|\operatorname{div} \mathbf{u}\|_{L^\infty} + \|\nabla \times \mathbf{u}\|_{L^\infty}) \ln(e + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}). \end{aligned} \quad (2.39)$$

Substituting (2.39) into (2.38), we obtain

$$\frac{d}{dt} (e + \|\nabla \rho\|_{L^p}) \leq CA(t) \ln(e + \|\nabla \rho\|_{L^p}), \quad (2.40)$$

where

$$\begin{aligned} A(t) &\triangleq C + C(\|\operatorname{div} \mathbf{u}\|_{L^\infty} + \|\nabla \times \mathbf{u}\|_{L^\infty} + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}) \\ &\quad + C(\|\operatorname{div} \mathbf{u}\|_{L^\infty} + \|\nabla \times \mathbf{u}\|_{L^\infty}) \ln(e + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p} + \|\mathbf{B} \cdot \nabla \mathbf{B}\|_{L^p}). \end{aligned}$$

It follows from Lemma 2.3 and the relation $\ln(e + y) \leq (e + y)^\delta$ for any $y \geq 0$ and $\delta > 0$ that

$$\int_0^T A(t) dt \leq C(T),$$

which, together with (2.40), yields

$$\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^p} \leq C(T), \quad \forall p \in (3, 6). \quad (2.41)$$

In terms of (2.7), (2.30), (2.37), (2.41) and the Sobolev embedding inequality, we obtain that for any p, q being as the ones in (2.31),

$$\begin{aligned} & \int_0^T (\|\nabla^2 \mathbf{u}\|_{L^p}^q + \|\nabla \mathbf{u}\|_{L^\infty}^q) dt \\ & \leq C \int_0^T (1 + \|\nabla^2 \mathbf{u}\|_{L^p}^q) dt \\ & \leq C \int_0^T (1 + \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^p}^q + \|\nabla \rho\|_{L^p}^q + \|\mathbf{B} \cdot \nabla^2 \mathbf{B}\|_{L^p}^q) dt \\ & \leq C(T). \end{aligned} \quad (2.42)$$

Taking $p = 2$ in (2.36), we have

$$\begin{aligned} \frac{d}{dt} \|\nabla \rho\|_{L^2} & \leq C(\|\nabla \mathbf{u}\|_{L^\infty} \|\nabla \rho\|_{L^2} + \|\nabla^2 \mathbf{u}\|_{L^2}) \\ & \leq C(\|\nabla \mathbf{u}\|_{L^\infty}^q + 1) \|\nabla \rho\|_{L^2} + C(1 + \|\nabla^2 \mathbf{u}\|_{L^2}^q), \quad \forall p > 3, \end{aligned}$$

thus, it follows from Gronwall's inequality that

$$\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^2} \leq C(T). \quad (2.43)$$

On the other hand, we know from (2.1)₁ that

$$\begin{aligned} \|\rho_t(t)\|_{L^2} & \leq C(\|\nabla \mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{L^6} + \|\nabla \rho\|_{L^3}) \\ & \leq C(1 + \|\nabla \rho\|_{L^3}) \\ & \leq C(1 + \|\nabla \rho\|_{L^2} + \|\nabla \rho\|_{L^p}) \\ & \leq C(T), \quad \forall p \in (3, 6), \end{aligned}$$

which, together with (2.41)–(2.43), gives (2.35). \square

Remark 2.2 It is worth mentioning that the estimates stated in Lemmas 2.2, 2.3, and 2.5 are independent of the lower bound of density.

The next lemma is to exclude the presence of vacuum, which plays an important role in the treatment of $\|\nabla \mathbf{u}\|_{L^3}$.

Lemma 2.6 Suppose that if ρ_0 satisfies $\inf_{x \in \mathbb{R}^3} \rho_0(x) \geq \underline{\rho} > 0$, then there exists a positive constant $c(\underline{\rho}, T)$, depending on $\underline{\rho}$ and T , such that

$$\rho(x, t) \geq c(\underline{\rho}, T), \quad \forall x \in \mathbb{R}^3, t \in [0, T], \quad (2.44)$$

and moreover,

$$\sup_{0 \leq t \leq T} (t \|\nabla^2 \mathbf{u}\|_{L^2}^2 + t \|\mathbf{u}_t\|_{L^2}^2) + \int_0^T (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\mathbf{u}_t\|_{L^2}^2 + t \|\nabla \mathbf{u}_t\|_{L^2}^2) dt \leq C(T). \quad (2.45)$$

Proof We infer from (2.30) that

$$\rho(x, t) \geq \inf_{x \in \mathbb{R}^3} \rho_0(x) \exp \left\{ - \int_0^t \|\operatorname{div} \mathbf{u}\|_{L^\infty} ds \right\} \geq c(\underline{\rho}, T),$$

which leads to the desired estimate (2.44).

Next, due to (2.1)₁ and the L^2 -theory of elliptic system, we infer from (2.4) and (2.5) that

$$\begin{aligned} \|\nabla^2 \mathbf{u}\|_{L^2} &\leq C(\|\dot{\mathbf{u}}\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\mathbf{B}\|_{L^3} \|\nabla^2 \mathbf{B}\|_{L^2}) \\ &\leq C(\|\dot{\mathbf{u}}\|_{L^2} + \|\nabla \rho\|_{L^2} + \|\nabla^2 \mathbf{B}\|_{L^2}). \end{aligned}$$

Hence it follows from (2.4), (2.7), (2.9), (2.35), and (2.45) that

$$\begin{aligned} &\sup_{0 \leq t \leq T} (t \|\nabla^2 \mathbf{u}\|_{L^2}^2) + \int_0^T \|\nabla^2 \mathbf{u}\|_{L^2}^2 dt \\ &\leq C \sup_{0 \leq t \leq T} (t \|\dot{\mathbf{u}}\|_{L^2}^2 + t \|\nabla \rho\|_{L^2}^2 + t \|\nabla^2 \mathbf{B}\|_{L^2}^2) \\ &\quad + C \int_0^T (\|\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2) dt \\ &\leq C(T). \end{aligned} \tag{2.46}$$

Recalling the definition of the material derivative, we have

$$\|\mathbf{u}_t\|_{L^2}^2 \leq \|\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2}^2 \leq C(\|\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{u}\|_{H^1}^2),$$

so that it follows from (2.4), (2.7), (2.9), (2.44), and (2.46) that

$$\sup_{0 \leq t \leq T} (t \|\mathbf{u}_t\|_{L^2}^2) + \int_0^T \|\mathbf{u}_t\|_{L^2}^2 dt \leq C(T), \tag{2.47}$$

and analogously,

$$\begin{aligned} \int_0^T t \|\nabla \mathbf{u}_t\|_{L^2}^2 dt &\leq \int_0^T t \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 dt + \int_0^T t \|\nabla(\mathbf{u} \cdot \nabla \mathbf{u})\|_{L^2}^2 dt \\ &\leq C + C \int_0^T t \|\nabla^2 \mathbf{u}\|_{L^2}^4 dt + C \int_0^T t \|\mathbf{u}\|_{L^\infty}^2 \|\nabla^2 \mathbf{u}\|_{L^2}^2 dt \\ &\leq C + C \int_0^T t \|\nabla^2 \mathbf{u}\|_{L^2}^4 dt \\ &\leq C. \end{aligned} \tag{2.48}$$

The combination of (2.46)–(2.48) leads to the desired estimate (2.45). \square

The following technical lemma is concerned with the estimate of $\|\nabla \mathbf{u}\|_{L^3}$, which plays an essential role in the entire analysis.

Lemma 2.7 *Assume that the conditions of Theorem 1.1 hold. Then*

$$\sup_{0 \leq t \leq T} \|\nabla \mathbf{u}\|_{L^3} \leq C(T). \tag{2.49}$$

Proof In terms of the standard L^p -estimate, to bound $\|\nabla \mathbf{u}\|_{L^p}$ with $1 < p < \infty$, it suffices to show that both $\|\operatorname{div} \mathbf{u}\|_{L^p}$ and $\|\operatorname{curl} \mathbf{u}\|_{L^p}$ are bounded. To do this, we first operate div and curl to both sides of (2.1)₂ to get that

$$\begin{aligned} & \rho(\operatorname{div} \mathbf{u})_t + \rho \mathbf{u} \cdot \nabla(\operatorname{div} \mathbf{u}) - (2\mu + \lambda)\Delta(\operatorname{div} \mathbf{u}) \\ &= -(\nabla \rho) \cdot \mathbf{u}_t - \partial_j(\rho u^j) \partial_i u^i - \Delta P + \partial_j B^i \partial_i B^j - \frac{1}{2} \Delta |\mathbf{B}|^2 \end{aligned} \quad (2.50)$$

and

$$\begin{aligned} & \rho(\operatorname{curl} \mathbf{u})_t + \rho \mathbf{u} \cdot \nabla(\operatorname{curl} \mathbf{u}) - \mu \Delta(\operatorname{curl} \mathbf{u}) \\ &= -(\nabla \rho) \times \mathbf{u}_t - \nabla(\rho \mathbf{u}^i) \times (\partial_i \mathbf{u}) + (\nabla B^i) \times (\partial_i \mathbf{B}) + \mathbf{B} \cdot \nabla(\operatorname{curl} \mathbf{B}). \end{aligned} \quad (2.51)$$

We shall divide the proofs into three steps.

Step I. Estimation of $\|\operatorname{div} \mathbf{u}\|_{L^3}$.

Multiplying (2.50) by $|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}$ and integrating by parts over \mathbb{R}^3 , we obtain

$$\begin{aligned} & \frac{1}{3} \frac{d}{dt} \int \rho |\operatorname{div} \mathbf{u}|^3 dx + (2\mu + \lambda) \int (|\operatorname{div} \mathbf{u}| |\nabla \operatorname{div} \mathbf{u}|^2 + |\operatorname{div} \mathbf{u}| |\nabla |\operatorname{div} \mathbf{u}||^2) dx \\ &= - \int (\mathbf{u}_t \cdot \nabla \rho) (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx - \int \partial_j(\rho u^j) \partial_i u^i (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \\ &+ \int \nabla P \cdot \nabla (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx + \int \partial_j B^i \partial_i B^j (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \\ &+ \int \nabla |\mathbf{B}|^2 \cdot \nabla (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \triangleq \sum_{i=1}^5 J_i. \end{aligned} \quad (2.52)$$

By virtue of (2.1)₂, we know that

$$\begin{aligned} J_1 &= - \int \rho^{-1} (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} - \nabla P - \rho \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \nabla \rho (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \\ &- \int \rho^{-1} \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) \cdot \nabla \rho (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx. \end{aligned}$$

Based on the integration by parts, the first term on the right-hand side can be written as

$$\begin{aligned} & - \mu \int \rho^{-1} \Delta \mathbf{u} \cdot \nabla \rho (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \\ &= - \mu \int \Delta \mathbf{u} \cdot \nabla \ln \rho (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \\ &= \mu \int (\partial_i u^j \partial_j (\ln \rho)) \partial_i (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx + \mu \int (\partial_i u^j \partial_{ij} (\ln \rho)) (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \\ &= \mu \int (\partial_i u^j \partial_j (\ln \rho)) \partial_i (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx - \mu \int (\partial_i \operatorname{div} \mathbf{u}) (\partial_i (\ln \rho)) (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \\ &- \mu \int (\partial_i u^j \partial_i (\ln \rho)) \partial_j (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx, \end{aligned}$$

hence

$$\begin{aligned}
 J_1 = & - \int \mathbf{B} \cdot \nabla \mathbf{B} \cdot \nabla (\ln \rho) (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) \, dx \\
 & + \frac{1}{2} \int \rho^{-1} \nabla |\mathbf{B}|^2 \cdot \nabla \rho (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) \, dx \\
 & + \mu \int (\partial_i u^j \partial_j (\ln \rho)) \partial_i (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) \, dx \\
 & - \mu \int (\partial_i \operatorname{div} \mathbf{u}) (\partial_i (\ln \rho)) (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) \, dx \\
 & - \mu \int (\partial_i u^j \partial_j (\ln \rho)) \partial_j (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) \, dx \\
 & - (\mu + \lambda) \int \nabla \operatorname{div} \mathbf{u} \cdot \nabla (\ln \rho) (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) \, dx \\
 & + \int \rho^{-1} \nabla P \cdot \nabla \rho (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) \, dx + \int (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \nabla \rho (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) \, dx \\
 \leq & C \int |\nabla \rho| |\mathbf{B}| |\nabla \mathbf{B}| |\operatorname{div} \mathbf{u}|^2 \, dx + C \int |\nabla \rho| |\nabla \mathbf{u}| |\operatorname{div} \mathbf{u}| |\nabla \operatorname{div} \mathbf{u}| \, dx \\
 & + C \int |\nabla \rho|^2 |\operatorname{div} \mathbf{u}|^2 \, dx + C \int |\nabla \rho| |\nabla \mathbf{u}| |\mathbf{u}| |\operatorname{div} \mathbf{u}|^2 \, dx \triangleq \sum_{i=1}^4 J_{1,i}.
 \end{aligned} \tag{2.53}$$

The right-hand side terms (2.53) can be estimated as follows. By virtue of (2.7) and (2.34), we obtain

$$\begin{aligned}
 J_{1,1} & \leq C \|\nabla \rho\|_{L^3} \|\mathbf{B}\|_{L^6} \|\nabla \mathbf{B}\|_{L^6} \|\operatorname{div} \mathbf{u}\|_{L^6}^2 \\
 & \leq C \|\nabla^2 \mathbf{B}\|_{L^2} \|\operatorname{div} \mathbf{u}\|_{L^3}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^9}^{3/2} \\
 & \leq C \|\nabla^2 \mathbf{B}\|_{L^2} \|\operatorname{div} \mathbf{u}\|_{L^3}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2} \\
 & \leq \frac{2\mu + \lambda}{8} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 + C \|\nabla^2 \mathbf{B}\|_{L^2}^2 \|\operatorname{div} \mathbf{u}\|_{L^3},
 \end{aligned} \tag{2.54}$$

where the following simple fact was used:

$$\begin{cases} \|\operatorname{div} \mathbf{u}\|_{L^6} \leq C \|\operatorname{div} \mathbf{u}\|_{L^3}^{1/4} \|\operatorname{div} \mathbf{u}\|_{L^9}^{3/4}, \\ \|\operatorname{div} \mathbf{u}\|_{L^9} = \|\operatorname{div} \mathbf{u}\|_{L^6}^{3/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^{2/3} \leq C \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^{2/3}. \end{cases} \tag{2.55}$$

Next, for the second term $J_{1,2}$, we have from (2.35), (2.55), and the Cauchy–Schwarz inequality that

$$\begin{aligned}
 J_{1,2} & \leq C \|\nabla \rho\|_{L^p} \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}} \|\operatorname{div} \mathbf{u}\|_{L^9}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2} \\
 & \leq \frac{2\mu + \lambda}{16} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^2 \|\operatorname{div} \mathbf{u}\|_{L^9} \\
 & \leq \frac{2\mu + \lambda}{8} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3,
 \end{aligned} \tag{2.56}$$

and similarly,

$$\begin{aligned}
 J_{1,3} + J_{1,4} &\leq C \|\nabla \rho\|_{L^3}^2 \|\operatorname{div} \mathbf{u}\|_{L^6}^2 + C \|\nabla \rho\|_{L^3} \|\nabla \mathbf{u}\|_{L^6} \|\mathbf{u}\|_{L^6} \|\operatorname{div} \mathbf{u}\|_{L^6}^2 \\
 &\leq C \|\nabla^2 \mathbf{u}\|_{L^2}^2 + C \|\nabla^2 \mathbf{u}\|_{L^2} \|\operatorname{div} \mathbf{u}\|_{L^6}^2 \\
 &\leq C \|\nabla^2 \mathbf{u}\|_{L^2}^2 + C \|\nabla^2 \mathbf{u}\|_{L^2} \|\operatorname{div} \mathbf{u}\|_{L^3}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^6}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2} \\
 &\leq \frac{2\mu + \lambda}{8} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + C \|\nabla^2 \mathbf{u}\|_{L^2}^2 \|\operatorname{div} \mathbf{u}\|_{L^3} + C \|\nabla^2 \mathbf{u}\|_{L^2}^2.
 \end{aligned} \tag{2.57}$$

Inserting (2.54)–(2.57) into (2.53), we see that

$$\begin{aligned}
 J_1 &\leq \frac{3(2\mu + \lambda)}{8} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + C (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3) \\
 &\quad + C (\|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2) (1 + \|\operatorname{div} \mathbf{u}\|_{L^3}^2), \quad 3 < p < 6.
 \end{aligned} \tag{2.58}$$

For J_2 , we have from (2.4), (2.7), (2.35), and (2.54) that

$$\begin{aligned}
 J_2 &\leq C \int |\nabla \rho| |\mathbf{u}| |\nabla \mathbf{u}| |\operatorname{div} \mathbf{u}|^2 dx + C \int |\rho| |\nabla \mathbf{u}|^2 |\operatorname{div} \mathbf{u}|^2 dx \\
 &\leq C \|\nabla \rho\|_{L^3} \|\mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^6} \|\operatorname{div} \mathbf{u}\|_{L^6}^2 + C \|\nabla \mathbf{u}\|_{L^6}^2 \|\operatorname{div} \mathbf{u}\|_{L^3}^2 \\
 &\leq \frac{2\mu + \lambda}{8} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + C \|\nabla^2 \mathbf{u}\|_{L^2}^2 (1 + \|\operatorname{div} \mathbf{u}\|_{L^3}^2),
 \end{aligned} \tag{2.59}$$

and similarly,

$$\begin{aligned}
 J_3 &\leq C \int |\nabla \rho| |\operatorname{div} \mathbf{u}| |\nabla \operatorname{div} \mathbf{u}| dx \\
 &\leq \|\nabla \rho\|_{L^3} \|\operatorname{div} \mathbf{u}\|_{L^3}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^6}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2} \\
 &\leq \frac{2\mu + \lambda}{4} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + C (1 + \|\operatorname{div} \mathbf{u}\|_{L^3}^2).
 \end{aligned} \tag{2.60}$$

Noticing that $\operatorname{div} \mathbf{B} = 0$, we have from (2.5) and the integration by parts that

$$\begin{aligned}
 J_4 + J_5 &= -C \int B^i \partial_i B^j \partial_j (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx + C \int \partial_j B^i B^j \partial_i (|\operatorname{div} \mathbf{u}| \operatorname{div} \mathbf{u}) dx \\
 &\leq C \int |\mathbf{B}| |\nabla \mathbf{B}| |\operatorname{div} \mathbf{u}| |\nabla \operatorname{div} \mathbf{u}| dx \\
 &\leq \|\mathbf{B}\|_{L^3} \|\nabla \mathbf{B}\|_{L^6} \|\operatorname{div} \mathbf{u}\|_{L^3}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^6}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2} \\
 &\leq \frac{2\mu + \lambda}{4} \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + C \|\nabla^2 \mathbf{B}\|_{L^2}^2 (1 + \|\operatorname{div} \mathbf{u}\|_{L^3}^2).
 \end{aligned} \tag{2.61}$$

Substituting (2.58)–(2.61) into (2.52), we have

$$\begin{aligned}
 &\frac{d}{dt} \int \rho |\operatorname{div} \mathbf{u}|^3 dx + \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\operatorname{div} \mathbf{u}\|_{L^2}^2 \\
 &\leq (1 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2) (1 + \|\operatorname{div} \mathbf{u}\|_{L^3}^2) + C \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3.
 \end{aligned} \tag{2.62}$$

Step II. Estimation of $\|\operatorname{curl} \mathbf{u}\|_{L^3}$.

Multiplying (2.51) by $|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}$ and integrating by parts over \mathbb{R}^3 , we get

$$\begin{aligned}
 & \frac{1}{3} \frac{d}{dt} \int \rho |\operatorname{curl} \mathbf{u}|^3 dx + \mu \int (|\operatorname{curl} \mathbf{u}| |\nabla \operatorname{curl} \mathbf{u}|^2 + |\operatorname{curl} \mathbf{u}| |\nabla |\operatorname{curl} \mathbf{u}||^2) dx \\
 &= \int (\nabla B^i) \times (\partial_i \mathbf{B}) \cdot (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &+ \int \mathbf{B} \cdot \nabla (\operatorname{curl} \mathbf{B}) \cdot (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &- \int (\nabla \rho \times \mathbf{u}_t) (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &- \int \nabla (\rho u^i) \times (\partial_i \mathbf{u}) (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &\triangleq \sum_{i=1}^4 N_i.
 \end{aligned} \tag{2.63}$$

The right-hand side terms of (2.63) can be estimated as follows. Since it holds by the Cauchy–Schwarz inequality that

$$N_1 \leq C \int |\nabla \mathbf{B}|^2 |\operatorname{curl} \mathbf{u}|^2 dx \leq C \|\nabla \mathbf{B}\|_{L^6}^2 \|\operatorname{curl} \mathbf{u}\|_{L^3}^2 \leq C \|\nabla^2 \mathbf{B}\|_{L^2}^2 \|\operatorname{curl} \mathbf{u}\|_{L^3}^2. \tag{2.64}$$

Thanks to $\operatorname{div} \mathbf{B} = 0$, we obtain from the integration by parts that

$$\begin{aligned}
 N_2 &\leq C \int |\mathbf{B}| |\nabla \mathbf{B}| |\operatorname{curl} \mathbf{u}| |\nabla \operatorname{curl} \mathbf{u}| dx \\
 &\leq C \|\mathbf{B}\|_{L^3} \|\nabla \mathbf{B}\|_{L^6} \|\operatorname{curl} \mathbf{u}\|_{L^3}^{1/2} \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^{1/2} \|\operatorname{curl} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^{1/2} \\
 &\leq \frac{\mu}{4} \|\operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 (1 + \|\operatorname{curl} \mathbf{u}\|_{L^3}^2).
 \end{aligned} \tag{2.65}$$

In terms of the following simple fact that

$$\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} = (2\mu + \lambda) \nabla \operatorname{div} \mathbf{u} - \mu \nabla \times (\operatorname{curl} \mathbf{u}),$$

we have from (2.1)₂ that

$$\begin{aligned}
 N_3 &= - \int (\nabla \ln \rho) \times \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) \cdot (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &+ \mu \int (\nabla \ln \rho) \times (\nabla \times \operatorname{curl} \mathbf{u}) \cdot (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &- (2\mu + \lambda) \int (\nabla \ln \rho) \times (\nabla \operatorname{div} \mathbf{u}) \cdot (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &+ \int (\nabla \ln \rho) \times (\nabla P) \cdot (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &+ \int (\nabla \ln \rho) \times (\rho \mathbf{u} \cdot \nabla \mathbf{u}) \cdot (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}) dx \\
 &\triangleq \sum_{i=1}^5 N_{3,i}.
 \end{aligned} \tag{2.66}$$

It is easy to get from (2.35) and the Cauchy–Schwarz inequality that

$$\begin{aligned} N_{3,1} &\leq C \int |\nabla \rho| |\mathbf{B}| |\nabla \mathbf{B}| |\operatorname{curl} \mathbf{u}|^2 dx \\ &\leq C \|\nabla \rho\|_{L^3} \|\mathbf{B}\|_{L^6} \|\nabla \mathbf{B}\|_{L^6} \|\operatorname{curl} \mathbf{u}\|_{L^6}^2 \\ &\leq \frac{\mu}{16} \|\operatorname{curl} \mathbf{u}\|_{L^2}^2 \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 (1 + \|\operatorname{curl} \mathbf{u}\|_{L^3}^2). \end{aligned} \quad (2.67)$$

By virtue of (2.35) and (2.55),

$$\begin{aligned} N_{3,2} &\leq C \int |\nabla \rho| |\nabla \operatorname{curl} \mathbf{u}| |\nabla \mathbf{u}| |\operatorname{curl} \mathbf{u}| dx \\ &\leq C \|\nabla \rho\|_{L^p} \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}} \|\operatorname{curl} \mathbf{u}\|_{L^9}^{1/2} \|\operatorname{curl} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2} \\ &\leq \frac{\mu}{16} \|\operatorname{curl} \mathbf{u}\|_{L^2}^2 \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3. \end{aligned} \quad (2.68)$$

A key observation for dealing with $N_{3,3}$ lies in the fact that for smooth scalar/vector functions f , g , and h ,

$$\int (\nabla f \times \nabla g) \cdot h dx = - \int g (\nabla f) \cdot (\nabla \times h) dx,$$

so that, by taking $f = \ln \rho$, $g = \operatorname{div} \mathbf{u}$, and $h = |\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u}$, we find

$$\begin{aligned} N_{3,3} &= (2\mu + \lambda) \int (\nabla \times (|\operatorname{curl} \mathbf{u}| \operatorname{curl} \mathbf{u})) \cdot (\nabla \ln \rho) (\operatorname{div} \mathbf{u}) dx \\ &\leq C \int |\nabla \rho| |\nabla \operatorname{curl} \mathbf{u}| |\nabla \mathbf{u}| |\operatorname{curl} \mathbf{u}| dx \\ &\leq C \|\nabla \rho\|_{L^p} \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}} \|\operatorname{curl} \mathbf{u}\|_{L^9}^{1/2} \|\operatorname{curl} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2} \\ &\leq \frac{\mu}{16} \|\operatorname{curl} \mathbf{u}\|_{L^2}^2 \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3. \end{aligned} \quad (2.69)$$

Next, for $N_{3,4}$, we have from (2.4) and (2.57) that

$$N_{3,4} \leq C \int |\nabla \rho|^2 |\operatorname{curl} \mathbf{u}|^2 dx \leq C \|\nabla^2 \mathbf{u}\|_{L^2}^2, \quad (2.70)$$

and

$$\begin{aligned} N_{3,5} &\leq C \int |\nabla \rho| |\rho| |\mathbf{u}| |\nabla \mathbf{u}| |\operatorname{curl} \mathbf{u}|^2 dx \\ &\leq C \|\nabla^2 \mathbf{u}\|_{L^2} \|\operatorname{curl} \mathbf{u}\|_{L^6}^2 \\ &\leq \frac{\mu}{16} \|\operatorname{curl} \mathbf{u}\|_{L^2}^2 \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 (1 + \|\operatorname{curl} \mathbf{u}\|_{L^3}^2). \end{aligned} \quad (2.71)$$

Inserting (2.67)–(2.71) into (2.66), we have

$$\begin{aligned} N_3 &\leq \frac{\mu}{4} \|\operatorname{curl} \mathbf{u}\|_{L^2}^2 \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + C (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2) (1 + \|\operatorname{curl} \mathbf{u}\|_{L^3}^2) \\ &\quad + C \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3. \end{aligned} \quad (2.72)$$

Similarly, it is easily seen from (2.59) that

$$\begin{aligned} N_4 &\leq C \int (|\nabla \rho| |\mathbf{u}| |\nabla \mathbf{u}| + |\rho| |\nabla \mathbf{u}|^2) |\operatorname{curl} \mathbf{u}|^2 dx \\ &\leq C \|\nabla^2 \mathbf{u}\|_{L^2} \|\operatorname{curl} \mathbf{u}\|_{L^6}^2 dx \\ &\leq \frac{\mu}{4} \|\operatorname{curl} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2 (1 + \|\operatorname{curl} \mathbf{u}\|_{L^3}^2). \end{aligned} \quad (2.73)$$

Thus, putting (2.64), (2.65), (2.72), and (2.73) into (2.63), we obtain

$$\begin{aligned} &\frac{d}{dt} \int \rho |\operatorname{curl} \mathbf{u}|^3 dx + \|\operatorname{curl} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 \\ &\leq C (\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2) (1 + \|\operatorname{curl} \mathbf{u}\|_{L^3}^2) + C \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3. \end{aligned} \quad (2.74)$$

Step III. Closing the estimations.

In view of (2.62) and (2.74), we have

$$\begin{aligned} &\frac{d}{dt} (\|\rho^{1/3} \operatorname{div} \mathbf{u}\|_{L^3}^3 + \|\rho^{1/3} \operatorname{curl} \mathbf{u}\|_{L^3}^3 dx) \\ &\quad + \|\operatorname{div} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 + \|\operatorname{curl} \mathbf{u}\|_{L^2}^{1/2} \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 \\ &\leq (1 + \|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2) (1 + \|\operatorname{div} \mathbf{u}\|_{L^3}^2 + \|\operatorname{curl} \mathbf{u}\|_{L^3}^2) + C \|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3. \end{aligned} \quad (2.75)$$

Since it holds that

$$1 < \frac{18-2p}{p} \leq 2 \quad \text{and} \quad 1 \leq \frac{5p-18}{p} < 2 \quad \text{for} \quad \frac{9}{2} \leq p < 6,$$

thus

$$\|\nabla \mathbf{u}\|_{L^{\frac{9p}{4p-9}}}^3 \leq C \|\nabla \mathbf{u}\|_{L^3}^{\frac{5p-18}{p}} \|\nabla \mathbf{u}\|_{L^6}^{\frac{18-2p}{p}} \leq C (1 + \|\nabla \mathbf{u}\|_{L^3}^2) (1 + \|\nabla \mathbf{u}\|_{H^1}^2). \quad (2.76)$$

Since it holds that

$$\|\nabla \mathbf{u}\|_{L^p} \leq C (\|\operatorname{div} \mathbf{u}\|_{L^p} + \|\operatorname{curl} \mathbf{u}\|_{L^p}), \quad \forall p > 1,$$

together with (2.7), (2.45), (2.75), (2.76), we have

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\nabla \mathbf{u}\|_{L^3}^3 &\leq C \sup_{0 \leq t \leq T} (\|\operatorname{div} \mathbf{u}\|_{L^3}^3 + \|\operatorname{curl} \mathbf{u}\|_{L^3}^3) \\ &\leq C + C \int_0^T (\|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2) (1 + \|\nabla \mathbf{u}\|_{L^3}^2) dt \\ &\quad + C \int_0^T (1 + \|\nabla \mathbf{u}\|_{L^3}^2) (1 + \|\nabla \mathbf{u}\|_{H^1}^2) dt \\ &\leq C + C \int_0^T (\|\nabla^2 \mathbf{B}\|_{L^2}^2 + \|\nabla^2 \mathbf{u}\|_{L^2}^2) \|\nabla \mathbf{u}\|_{L^3}^2 dt, \end{aligned} \quad (2.77)$$

so that, combined with (2.7), (2.45), (2.77), and Gronwall's inequality, this leads to (2.49). \square

3 Proof of Theorem 1.1

By virtue of all the a priori estimates stated in Sect. 2, we are now ready to prove the main results in our paper. To be continued, we first need to prove the local existence of smooth solutions of (1.1)–(1.5). Fortunately, thanks to Li, Xu, and Zhang's research (cf. [17, Proposition 5.1]), the details can be omitted here for simplicity. Next, we need to prove the uniqueness of the solutions.

Proof of uniqueness Let $(\rho_1, \mathbf{u}_1, \mathbf{B}_1)$ and $(\rho_2, \mathbf{u}_2, \mathbf{B}_2)$, belonging to the class of functions (1.15) and enjoying the same initial data, be two solutions of problem (2.1), (1.4), and (1.5) on $\mathbb{R}^3 \times [0, T]$. Define

$$\varrho \triangleq \rho_1 - \rho_2, \quad \mathbf{v} \triangleq \mathbf{u}_1 - \mathbf{u}_2, \quad \mathcal{B} \triangleq \mathbf{B}_1 - \mathbf{B}_2.$$

Then it is easily derived from the above representations for $\varrho = \varrho(x, t)$ that

$$\varrho_t + \mathbf{u}_2 \cdot \nabla \varrho + \varrho \operatorname{div} \mathbf{u}_2 + \rho_1 \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \nabla \rho_1 = 0,$$

thus, multiplying by ϱ in L^2 and integrating by parts yields

$$\begin{aligned} \frac{d}{dt} \|\varrho\|_{L^2}^2 &\leq C \|\operatorname{div} \mathbf{u}_2\|_{L^\infty} \|\varrho\|_{L^2}^2 + C (\|\nabla \mathbf{v}\|_{L^2} + \|\mathbf{v}\|_{L^6} \|\nabla \rho_1\|_{L^3}) \|\varrho\|_{L^2} \\ &\leq C \|\operatorname{div} \mathbf{u}_2\|_{L^\infty} \|\varrho\|_{L^2}^2 + C \|\nabla \mathbf{v}\|_{L^2} \|\varrho\|_{L^2}. \end{aligned} \quad (3.1)$$

Thanks to (1.15), we know that $\|\operatorname{div} \mathbf{u}_2\|_{L^\infty} \in L^1(0, T)$, thus, with the help of (3.1) and Gronwall's inequality, we get

$$\|\varrho(t)\|_{L^2} \leq C \int_0^t \|\nabla \mathbf{v}\|_{L^2} ds \leq C t^{1/2} \left(\int_0^t \|\nabla \mathbf{v}\|_{L^2}^2 ds \right)^{1/2}, \quad \forall t \in [0, T]. \quad (3.2)$$

On the other hand, since $\dot{\mathbf{u}}_2 = \mathbf{u}_{2t} + \mathbf{u}_2 \cdot \nabla \mathbf{u}_2$, we have from (2.1)₂ that

$$\begin{aligned} &\rho_1 \mathbf{v}_t + \rho_1 \mathbf{u}_1 \cdot \nabla \mathbf{v} - \mu \Delta \mathbf{v} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{v} \\ &= -\varrho \dot{\mathbf{u}}_2 - \rho_1 \mathbf{v} \cdot \nabla \mathbf{u}_2 - \nabla (P(\rho_1) - P(\rho_2)) + \mathbf{B}_1 \cdot \nabla \mathcal{B} - \frac{1}{2} \nabla (|\mathbf{B}_1|^2 - |\mathbf{B}_2|^2). \end{aligned} \quad (3.3)$$

Multiplying (3.3) by \mathbf{v} in L^2 and integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2 + \mu \|\nabla \mathbf{v}\|_{L^2}^2 + (\mu + \lambda) \|\operatorname{div} \mathbf{v}\|_{L^2}^2 \\ &\leq C \|\varrho\|_{L^2} \|\dot{\mathbf{u}}_2\|_{L^3} \|\mathbf{v}\|_{L^6} + C \|\mathbf{v}\|_{L^2} \|\nabla \mathbf{u}_2\|_{L^3} \|\mathbf{v}\|_{L^6} + C \|\varrho\|_{L^2} \|\nabla \mathbf{v}\|_{L^2} \\ &\quad \times C (\|\mathbf{B}_1\|_{L^\infty} + \|\mathbf{B}_2\|_{L^\infty}) \|\mathcal{B}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2} \\ &\leq \frac{\mu}{2} \|\nabla \mathbf{v}\|_{L^2}^2 + C (1 + \|\dot{\mathbf{u}}_2\|_{L^3}^2) \|\varrho\|_{L^2}^2 \\ &\quad + C (\|\mathbf{B}_1\|_{L^\infty}^2 + \|\mathbf{B}_2\|_{L^\infty}^2 + \|\nabla \mathbf{u}_2\|_{L^3}^2) (\|\mathcal{B}\|_{L^2}^2 + \|\mathbf{v}\|_{L^2}^2). \end{aligned}$$

Thus, it follows from (1.15) and (2.49) that

$$\begin{aligned} \frac{d}{dt} \|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^2 &\leq C(1 + \|\dot{\mathbf{u}}_2\|_{L^3}^2) \|\varrho\|_{L^2}^2 \\ &\quad + C(1 + \|\nabla^2 \mathbf{B}_1\|_{L^2}^2 + \|\nabla^2 \mathbf{B}_2\|_{L^2}^2) (\|\mathcal{B}\|_{L^2}^2 + \|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2). \end{aligned} \quad (3.4)$$

Finally, note that

$$\mathcal{B}_t - \nu \Delta \mathcal{B} = -\mathbf{u}_1 \cdot \nabla \mathcal{B} - \mathbf{v} \cdot \nabla \mathbf{B}_2 + \mathcal{B} \cdot \nabla \mathbf{u}_1 + \mathbf{B}_2 \cdot \nabla \mathbf{v} - \mathcal{B} \operatorname{div} \mathbf{u}_1 - \mathbf{B}_2 \operatorname{div} \mathbf{v}. \quad (3.5)$$

Multiplying (3.5) by \mathcal{B} and integrating by parts over \mathbb{R}^3 , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathcal{B}\|_{L^2}^2 + \|\nabla \mathcal{B}\|_{L^2}^2 \\ = \int (-\mathbf{u}_1 \cdot \nabla \mathcal{B} - \mathbf{v} \cdot \nabla \mathbf{B}_2 + \mathcal{B} \cdot \nabla \mathbf{u}_1 + \mathbf{B}_2 \cdot \nabla \mathbf{v} - \mathcal{B} \operatorname{div} \mathbf{u}_1 - \mathbf{B}_2 \operatorname{div} \mathbf{v}) \cdot \mathcal{B} \, dx \\ \triangleq \sum_{i=1}^6 J_i. \end{aligned} \quad (3.6)$$

Now, we estimate J_i ($i = 1, 2, \dots, 6$) in (3.6) as follows. By virtue of (2.49), we have

$$J_1 = -\frac{1}{2} \int \mathbf{u}_1 \cdot \nabla (|\mathcal{B}|^2) \, dx = \frac{1}{2} \int \operatorname{div} \mathbf{u}_1 |\mathcal{B}|^2 \, dx \leq \frac{\nu}{8} \|\nabla \mathcal{B}\|_{L^2}^2 + \|\mathcal{B}\|_{L^2}^2 \quad (3.7)$$

and

$$\begin{aligned} J_2 &\leq C \|\mathbf{v}\|_{L^2} \|\nabla \mathbf{B}_2\|_{L^3} \|\mathcal{B}\|_{L^6} \\ &\leq \frac{\nu}{8} \|\nabla \mathcal{B}\|_{L^2}^2 + C(1 + \|\nabla^2 \mathbf{B}_2\|_{L^2}^2) \|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2, \end{aligned} \quad (3.8)$$

and

$$J_3 + J_4 + J_5 \leq \frac{\nu}{4} \|\nabla \mathcal{B}\|_{L^2}^2 + C(1 + \|\nabla^2 \mathbf{B}_2\|_{L^2}^2) \|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2 + \|\mathcal{B}\|_{L^2}^2. \quad (3.9)$$

Next, for J_6 , we have from (1.15) that

$$\begin{aligned} J_6 &\leq C \|\mathbf{B}_2\|_{L^\infty} \|\nabla \mathbf{v}\|_{L^2} \|\mathcal{B}\|_{L^2} \\ &\leq C_1 \|\nabla \mathbf{v}\|_{L^2}^2 + C \|\mathbf{B}_2\|_{L^6} \|\nabla \mathbf{B}_2\|_{L^6} \|\mathcal{B}\|_{L^2}^2 \\ &\leq C_1 \|\nabla \mathbf{v}\|_{L^2}^2 + C(1 + \|\nabla^2 \mathbf{B}_2\|_{L^2}^2) \|\mathcal{B}\|_{L^2}^2. \end{aligned} \quad (3.10)$$

Putting (3.6)–(3.10) into (3.5), we have

$$\begin{aligned} \frac{d}{dt} \|\mathcal{B}\|_{L^2}^2 + \|\nabla \mathcal{B}\|_{L^2}^2 \\ \leq C_1 \|\nabla \mathbf{v}\|_{L^2}^2 + C(1 + \|\nabla^2 \mathbf{B}_2\|_{L^2}^2) (\|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2 + \|\mathcal{B}\|_{L^2}^2). \end{aligned} \quad (3.11)$$

Multiplying (3.3) by $(C_1 + 1)$ and combining with (3.2) and (3.11), we obtain

$$\begin{aligned} & \frac{d}{dt} (\|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2 + \|\mathcal{B}\|_{L^2}^2) + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathcal{B}\|_{L^2}^2 \\ & \leq C(1 + \|\nabla^2 \mathbf{B}_1\|_{L^2}^2 + \|\nabla^2 \mathbf{B}_2\|_{L^2}^2) (\|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2 + \|\mathcal{B}\|_{L^2}^2) \\ & \leq Ct(1 + \|\dot{\mathbf{u}}_2\|_{L^3}^2) \left(\int_0^t \|\nabla \mathbf{v}\|_{L^2}^2 ds \right). \end{aligned} \quad (3.12)$$

Let

$$\Phi(t) \triangleq (\|\sqrt{\rho_1} \mathbf{v}\|_{L^2}^2 + \|\mathcal{B}\|_{L^2}^2) + \int_0^t (\|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathcal{B}\|_{L^2}^2) ds.$$

Then, due to the fact that ρ_1 has a positive lower bound, we infer from (3.12) that

$$\Phi'(t) \leq A(t)\Phi(t) \quad \text{with } \Phi(0) = 0, \quad (3.13)$$

where

$$\begin{aligned} A(t) & \triangleq Ct(1 + \|\dot{\mathbf{u}}_2\|_{L^3}^2) + C(1 + \|\nabla^2 \mathbf{B}_1\|_{L^2}^2 + \|\nabla^2 \mathbf{B}_2\|_{L^2}^2) \\ & \leq Ct(1 + \|\dot{\mathbf{u}}_2\|_{L^2}^2 + \|\nabla \dot{\mathbf{u}}_2\|_{L^2}^2) + C(1 + \|\nabla^2 \mathbf{B}_1\|_{L^2}^2 + \|\nabla^2 \mathbf{B}_2\|_{L^2}^2). \end{aligned}$$

Since it holds by (1.15) that $A(t) \in L^1(0, T)$, thus, we can infer from (3.13) and Gronwall's inequality that

$$(\|\mathbf{v}\|_{L^2}^2 + \|\mathcal{B}\|_{L^2}^2) + \int_0^T (\|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla \mathcal{B}\|_{L^2}^2) dt = 0, \quad \forall t \in [0, T],$$

so that

$$\mathbf{v}(x, t) = 0, \quad \mathcal{B} = 0, \quad \text{a.e. on } \mathbb{R}^3 \times [0, T],$$

which, combined with (3.2), yields

$$\varrho(x, t) = 0, \quad \text{a.e. on } \mathbb{R}^3 \times [0, T].$$

Thus, we complete the proof of Theorem 1.1. \square

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The author declares that she has no competing interests.

Authors' contributions

This entire work has been completed by the author Dr. MZ. The author read and approved the final manuscript.

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References

1. Beale, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Commun. Math. Phys.* **94**, 61–66 (1984)
2. Cabannes, H.: *Theoretical Magnetofluidynamics*. Academic Press, New York (1970)
3. Chen, G.Q., Wang, D.: Global solution of nonlinear magnetohydrodynamics with large initial data. *J. Differ. Equ.* **182**, 344–376 (2002)
4. Chen, G.Q., Wang, D.: Existence and continuous dependence of large solutions for the magnetohydrodynamic equations. *Z. Angew. Math. Phys.* **54**, 608–632 (2003)
5. Cho, Y., Choe, H.J., Kim, H.: Unique solvability of the initial boundary value problems for compressible viscous fluids. *J. Math. Pures Appl.* **83**(2), 243–275 (2004)
6. Ducomet, B., Feireisl, E.: The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars. *Commun. Math. Phys.* **266**, 595–629 (2006)
7. Fan, J., Jiang, S., Nakamura, G.: Vanishing shear viscosity limit in the magnetohydrodynamic equations. *Commun. Math. Phys.* **270**, 691–708 (2007)
8. Fan, J., Li, F.: Global strong solutions to the 3D compressible non-isentropic MHD equations with zero resistivity. *Z. Angew. Math. Phys.* **71**(2), 41 (2020)
9. Fan, J., Yu, W.: Global variational solutions to the compressible magnetohydrodynamic equations. *Nonlinear Anal.* **69**, 3637–3660 (2008)
10. Fan, J., Yu, W.: Strong solutions to the compressible MHD equations with vacuum. *Nonlinear Anal., Real World Appl.* **10**, 392–409 (2009)
11. Hoff, D.: Global solutions of the Navier–Stokes equations for the multidimensional compressible flow with discontinuous initial data. *J. Differ. Equ.* **120**, 215–254 (1995)
12. Hu, X., Wang, D.: Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. *Commun. Math. Phys.* **283**, 255–284 (2008)
13. Hu, X., Wang, D.: Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. *Arch. Ration. Mech. Anal.* **197**, 203–238 (2010)
14. Huang, X.D., Li, J., Xin, Z.P.: Serrin type criterion for the three-dimensional viscous compressible flows. *SIAM J. Math. Anal.* **43**, 1872–1886 (2011)
15. Huang, X.D., Li, J., Xin, Z.P.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations. *Commun. Pure Appl. Math.* **65**, 549–585 (2012)
16. Kulikovskiy, A.F., Lyubimov, G.A.: *Magnetohydrodynamics*. Addison-Wesley, Reading (1965)
17. Li, H.L., Xu, X.Y., Zhang, J.W.: Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum. *SIAM J. Math. Anal.* **45**, 1356–1387 (2013)
18. Lu, L., Huang, B.: On local strong solutions to the Cauchy problem of the two-dimensional full compressible magnetohydrodynamic equations with vacuum and zero heat conduction. *Nonlinear Anal., Real World Appl.* **31**, 409–430 (2016)
19. Lv, B.Q., Shi, X.D., Xu, X.Y.: Global existence and large-time asymptotic behavior of strong solutions to the compressible magnetohydrodynamic equations with vacuum. *Indiana Univ. Math. J.* **65**, 925–975 (2016)
20. Vol'pert, A.I., Khudiaev, S.I.: On the Cauchy problem for the composite systems of nonlinear equations. *Mat. Sb.* **87**, 504–528 (1972)
21. Xu, H., Zhang, J.W.: Regularity and uniqueness for the compressible full Navier–Stokes equations. *J. Differ. Equ.* **272**, 46–73 (2021)
22. Zhang, J.W., Jiang, S., Xie, F.: Global weak solutions of an initial boundary value problem for screw pinches in plasma physics. *Math. Models Methods Appl. Sci.* **19**, 833–875 (2009)
23. Zhang, J.W., Zhao, J.N.: Some decay estimates of solutions for the 3-D compressible isentropic magnetohydrodynamics. *Commun. Math. Sci.* **8**, 835–850 (2010)