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Stability for generalized Caputo proportional fractional delay integro-differential equations

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Abstract

A scalar nonlinear integro-differential equation with time-variable and bounded delays and generalized Caputo proportional fractional derivative is considered. The main goal of this paper is to study the stability properties of the zero solution. Results are given concerning stability, exponential stability, asymptotic stability, and boundedness of solutions. The investigations are based on an application of a quadratic Lyapunov function, its generalized Caputo proportional derivative, and a modification of the Razumikhin approach. Some auxiliary properties of the generalized Caputo proportional derivative are proved. Five illustrative examples are included.

MSC: 34A08; 34A37; 34E05

Keywords: Generalized Caputo proportional fractional derivative; Variable delays; Integro-differential fractional equations; Initial value problem

1 Introduction

In this paper, we study stability for a very general class of fractional delay integro-differential equations. We use a recently introduced concept [11] of the so-called generalized Caputo proportional fractional derivative. This derivative is a generalization of the Caputo fractional derivative. For the study of various types of fractional differential equations, we refer the reader to the classical book [19]. In [4, 6, 7, 9, 13, 14, 20], Lyapunov functions and functionals are used to study the stability of various fractional-order nonlinear dynamic systems with classical Caputo and Riemann–Liouville fractional derivatives and for ordinary differential equations with state delays in [21]. Following the introduction [11] of the generalized Caputo proportional fractional derivative, some studies concerning this derivative were contained in [1–3, 5, 10, 15–18].

In this paper, we apply Lyapunov functions to study some stability properties of integro-differential delay equations with generalized Caputo proportional fractional derivatives. We consider the case of bounded delays that are variable in time. We develop some necessary tools for generalized Caputo proportional fractional derivatives, starting with an important inequality concerning an estimate of that derivative of quadratic functions. A modified Razumikhin condition is developed and used to show the main result, conditions that guarantee the stability of the zero solution with zero initial function. From there, we give results concerning the boundedness of all solutions, as well as results about exponential stability and asymptotic stability.

The organization of the paper is as follows. In Sect. 2, the problem is stated and essential assumptions and definitions are given. In Sect. 3, we prove several auxiliary results about generalized Caputo proportional fractional derivatives. Section 4 contains the main results as well as a comparison with the recent paper [8]. The paper concludes with Sect. 5, in which five detailed examples are offered.

2 Problem statement

Consider the fractional delay integro-differential equation with generalized Caputo proportional fractional derivative

$$\begin{aligned}
\left({}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}x\right)(t) &= -F\big(t,x(t)\big) + \sum_{k=1}^{n}\psi_k\big(t,x(t),x\big(t-g_k(t)\big)\big) \\
&\quad + \sum_{k=1}^{n}\int_{t-g_k(t)}^{t} C_k(t,s)f_k\big(s,x(s)\big)\,\mathrm{d}s, \quad t>t_0, \\
x(t_0+\theta) &= \phi(\theta), \quad \theta\in[-\tau,0],
\end{aligned} \tag{2.1}$$

where we assume throughout this paper

(C_0) $t_0 \ge 0$, $\alpha\in(0,1)$, $\rho\in(0,1]$,

${}_{t_0}^{C}\mathscr{D}^{\alpha,\rho}x$ is the generalized Caputo proportional fractional derivative of $x\in C^1([t_0,\infty),\mathbb{R})$,

- $g_k:[t_0,\infty)\to[0,\tau_k]$, $\tau_k\ge 0$, $k=1,2,\dots,n$,
- $\phi\in C([-\tau,0],\mathbb{R})$, $\tau=\max_{k=1,2,\dots,n}\tau_k$,
- $F:[t_0,\infty)\times\mathbb{R}\to\mathbb{R}$, and
- $f_k:[t_0-\tau,\infty)\times\mathbb{R}\to\mathbb{R}$, $C_k:[t_0,\infty)\times[t_0-\tau,\infty)\to\mathbb{R}$, $\psi_k:[t_0,\infty)\times\mathbb{R}^2\to\mathbb{R}$ for $k=1,2,\dots,n$.

Remark 2.1 Everywhere in this paper, rather than giving conditions that imply the existence and uniqueness of solutions of (2.1), we will assume that for any initial function $\phi\in C([-\tau,0],\mathbb{R})$, the initial value problem (2.1) has a unique solution defined for $t\ge t_0$.

We recall that the generalized proportional fractional integral and the generalized Caputo proportional fractional derivative of a function $u:[a,\infty)\to\mathbb{R}$ are defined, respectively, by (as long as all integrals are well defined, see [11])

$$\left({}_{a}\mathscr{I}^{\beta,\rho}u\right)(t) = \frac{1}{\rho^{\beta}\Gamma(\beta)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\beta-1}u(s)\,\mathrm{d}s, \quad t>a$$

and

$$\begin{aligned}
\left({}_{a}^{C}\mathscr{D}^{\alpha,\rho}u\right)(t) &= \left({}_{a}\mathscr{I}^{1-\alpha,\rho}\left(\mathscr{D}^{\rho}u\right)\right)(t) \\
&= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}\frac{(\mathscr{D}^{\rho}u)(s)}{(t-s)^{\alpha}}\,\mathrm{d}s, \quad t>a,
\end{aligned}$$

where

$$\mathscr{D}^{\rho}u = (1-\rho)u + \rho u'.$$

Remark 2.2 The generalized proportional fractional integral ${}_a\mathscr{I}^{\beta,\rho}u$ is defined also for $\beta \in (-1,0)$ since $\Gamma(\beta) < 0$ exists.

Remark 2.3 If $\rho = 1$, then the generalized Caputo proportional fractional derivative is reduced to the classical Caputo fractional derivative.

Remark 2.4 If $u(t) = e^{\frac{\rho-1}{\rho}(t-c)}$ for $t \geq a$, where $c \in \mathbb{R}$, then the relation

$$\left({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}u\right)(t) = 0 \quad \text{for } t \geq a \tag{2.2}$$

is known from [11, Remark 3.2].

3 Auxiliary results

The next result about the generalized Caputo proportional fractional derivative is new, and it is used twice in the remainder of the paper. It also will be of valuable use any time the generalized Caputo proportional fractional derivative is studied.

Lemma 3.1 *If there exists $t > a$ such that $u \in \mathrm{C}^1([a,t],\mathbb{R})$ satisfies $u(t) = 0$, then*

$$\left({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}u\right)(t) = \left({}_a\mathscr{I}^{-\alpha,\rho}u\right)(t) - \frac{\rho^\alpha}{\Gamma(1-\alpha)} e^{\frac{\rho-1}{\rho}(t-a)} \frac{u(a)}{(t-a)^\alpha}. \tag{3.1}$$

Proof Let $u \in \mathrm{C}^1([a,t],\mathbb{R})$ and $u(t) = 0$. We denote

$$f(s) := e^{\frac{\rho-1}{\rho}(t-s)} \quad \text{and} \quad g(s) := \frac{u(s)}{(t-s)^\alpha} \quad \text{for } s \in [a,t).$$

Now, we integrate by parts to obtain

$$\begin{aligned}
\left({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}u\right)(t) &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_a^t f(s) \frac{(1-\rho)u(s) + \rho u'(s)}{(t-s)^\alpha}\,\mathrm{d}s \\
&= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t \left\{ \frac{1-\rho}{\rho} f(s)g(s) + f(s)\frac{u'(s)}{(t-s)^\alpha} \right\} \mathrm{d}s \\
&= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t \left\{ f'(s)g(s) + f(s)\left[g'(s) - \frac{\alpha u(s)}{(t-s)^{\alpha+1}} \right] \right\} \mathrm{d}s \\
&= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t (fg)'(s)\,\mathrm{d}s - \frac{\alpha\rho^\alpha}{\Gamma(1-\alpha)} \int_a^t f(s)\frac{u(s)}{(t-s)^{\alpha+1}}\,\mathrm{d}s \\
&= \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left\{ \lim_{s\to t^-} f(s)g(s) - f(a)g(a) \right\} \\
&\quad + \frac{\rho^\alpha}{\Gamma(-\alpha)} \int_a^t f(s)\frac{u(s)}{(t-s)^{\alpha+1}}\,\mathrm{d}s \\
&= \left({}_a\mathscr{I}^{-\alpha,\rho}u\right)(t) - \frac{\rho^\alpha}{\Gamma(1-\alpha)} f(a)g(a),
\end{aligned}$$

where we used L'Hôpital's rule to find

$$\lim_{s\to t^-} \frac{f(s)u(s)}{(t-s)^\alpha} = \lim_{s\to t^-} \frac{f'(s)u(s) + u'(s)f(s)}{\alpha(t-s)^{\alpha-1}} = 0,$$

completing the proof. □

Now, we prove three results that will be used in the derivation of our main results in the next section.

Lemma 3.2 *If* $u \in \mathrm{C}^1([a,\infty),\mathbb{R})$, *then*

$$\left({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}u^2\right)(t) \le 2u(t)\left({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}u\right)(t) \quad \text{for all } t > a. \tag{3.2}$$

Proof We fix $t > a$ and define the function $v : [a,t] \to [0,\infty)$ by

$$v(s) = \left(u(t) - u(s)\right)^2.$$

Note that $v \in \mathrm{C}^1([a,t],\mathbb{R})$ and $v(t) = 0$, so v satisfies the assumptions of Lemma 3.1, and therefore, from (3.1) and the inequalities $v(s) \ge 0$ and $\Gamma(-\alpha) < 0$ for $\alpha \in (0,1)$, it follows $({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}v)(t) \le 0$. Thus, since

$$\begin{aligned}
&\left(\mathscr{D}^{\rho}u^2\right)(s) - 2u(t)\left(\mathscr{D}^{\rho}u\right)(s) \\
&\quad = (1-\rho)u^2(s) + \rho\left(u^2\right)'(s) - 2u(t)\left\{(1-\rho)u(s) + \rho u'(s)\right\} \\
&\quad = (1-\rho)u^2(s) + 2\rho u(s)u'(s) - 2(1-\rho)u(t)u(s) - 2\rho u'(s)u(t) \\
&\quad = (1-\rho)\left(u(s) - u(t)\right)^2 + 2\rho u'(s)\left(u(s) - u(t)\right) - (1-\rho)u^2(t) \\
&\quad = (1-\rho)v(s) + \rho v'(s) - (1-\rho)u^2(t) \\
&\quad = \left(\mathscr{D}^{\rho}v\right)(s) - (1-\rho)u^2(t) \\
&\quad \le \left(\mathscr{D}^{\rho}v\right)(s),
\end{aligned}$$

we obtain

$$\left({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}u^2\right)(t) - 2u(t)\left({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}u\right)(t) \le \left({}_a^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}v\right)(t) \le 0,$$

where the last inequality follows from Lemma 3.1 because $v(s) \ge 0$ and since $\Gamma(-\alpha) < 0$ for $\alpha \in (0,1)$. □

Lemma 3.3 *Let* $\phi \in \mathrm{C}([-\tau,0],\mathbb{R})$ *and assume that*

$$x = x(\cdot, t_0, \phi) \in \mathrm{C}^1\left([-\tau,\infty),\mathbb{R}\right)$$

is a solution of (2.1). *If for any* $t \ge t_0$ *such that*

$$x^2(t+\theta)e^{\frac{1-\rho}{\rho}\theta} \le x^2(t) \quad \text{for all } \theta \in [-\tau,0), \tag{3.3}$$

the inequality

$$\left({}_{t_0}^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}x^2\right)(t) \le 0 \tag{3.4}$$

holds, then

$$\left|x(s)\right| \le \sqrt{\max_{\theta\in[-\tau,0]} \phi^2(\theta)e^{\frac{\rho-1}{\rho}(s-t_0)}} \quad \text{for all } s \ge t_0. \tag{3.5}$$

Proof Let $x = x(\cdot, t_0, \phi)$ be a solution of (2.1). We define

$$\eta(s) := x^2(s) - Be^{\frac{\rho-1}{\rho}(s-t_0)} - \varepsilon e^{\frac{\rho-1}{\rho}s} \quad \text{for } s \geq t_0,$$

where

$$B := \max_{\theta\in[-\tau,0]} \phi^2(\theta) \geq 0.$$

Observe that

$$\eta(t_0) = x^2(t_0) - B - \varepsilon e^{\frac{\rho-1}{\rho}t_0} < x^2(t_0) - B \leq B - B = 0.$$

We claim

$$\eta(s) < 0 \quad \text{for all } s \geq t_0, \tag{3.6}$$

from which (3.5) would follow. Assuming that (3.6) is not true, there exists $t > t_0$ such that

$$\eta(s) < 0 \quad \text{for all } s \in [t_0, t) \quad \text{and} \quad \eta(t) = 0. \tag{3.7}$$

Note that $\eta \in C^1([t_0, t], \mathbb{R})$ and $\eta(t) = 0$, hence η satisfies the assumptions of Lemma 3.1. According to Lemma 3.1 and the choice of the point t, we have $({}^{\mathrm{C}}_{t_0}\mathscr{D}^{\alpha,\rho}\eta)(t) > 0$. For any $\theta \in [-\tau, 0]$, we have

$$\begin{aligned} x^2(t+\theta)e^{\frac{1-\rho}{\rho}\theta} &= \left\{\eta(t+\theta) + Be^{\frac{\rho-1}{\rho}(t+\theta-t_0)} + \varepsilon e^{\frac{\rho-1}{\rho}(t+\theta)}\right\}e^{\frac{1-\rho}{\rho}\theta} \\ &= \eta(t+\theta)e^{\frac{1-\rho}{\rho}\theta} + Be^{\frac{\rho-1}{\rho}(t-t_0)} + \varepsilon e^{\frac{\rho-1}{\rho}t} \\ &= \eta(t+\theta)e^{\frac{1-\rho}{\rho}\theta} + x^2(t) - \eta(t) \\ &= x^2(t) + \eta(t+\theta)e^{\frac{1-\rho}{\rho}\theta} \\ &< x^2(t), \end{aligned}$$

so for this t, (3.3) holds, and thus

$$0 \overset{(3.4)}{\geq} \left({}^{\mathrm{C}}_{t_0}\mathscr{D}^{\alpha,\rho}x^2\right)(t) \overset{(2.2)}{=} \left({}^{\mathrm{C}}_{t_0}\mathscr{D}^{\alpha,\rho}\eta\right)(t) > 0,$$

where the last inequality follows from Lemma 3.1, contradicting (3.7), proving (3.6), and hence implying (3.5). □

Remark 3.4 The condition "if for a point t such that (3.3) holds, then (3.4) is satisfied" is a (modified) Razumikhin condition. In the case of the Caputo fractional derivative, i.e., $\rho = 1$, this condition reduces to the classical Razumikhin condition (see, for example [12]).

Remark 3.5 The main characteristic of Lyapunov functions applied to delay differential equations is the condition for its derivative, the so-called Razhumikhin condition (for example, the negativity of the derivative). It allows us to assume a restriction on their derivatives only at some point, not for all values of t. This is different from the applications to ordinary differential equations and to Caputo fractional differential equations without delays.

Corollary 3.6 *If the conditions of Lemma 3.3 are satisfied, then*

$$\left|x(s)\right| \le \sqrt{\max_{\theta\in[-\tau,0]} \phi^2(\theta)} \quad \textit{for all } s \ge t_0. \tag{3.8}$$

Proof This follows from (3.5), (C_0), and $e^{\frac{\rho-1}{\rho}(s-t_0)} \le 1$ for $s \ge t_0$. □

4 Stability and boundedness properties

We introduce the following assumptions.

(C_1) $F \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and for $k = 1,2,\dots,n$,
- $\psi_k \in C([t_0,\infty)\times\mathbb{R}^2,\mathbb{R})$,
- $f_k \in C([t_0-\tau_k,\infty)\times\mathbb{R},[0,\infty))$,
- $C_k \in C([t_0,\infty)\times[t_0-\tau,\infty),\mathbb{R})$.

(C_2) $F(t,0) = 0$, and for $k = 1,2,\dots,n$,
- $\psi_k(t,0,0) = 0$ for $t \ge t_0$,
- $f_k(t,0) = 0$ for $t \ge t_0 - \tau_k$.

(C_3) There exist $b \in C((t_0,\infty),[0,\infty))$ and, for $k = 1,2,\dots,n$, $A_k, B_k \in C((t_0,\infty),[0,\infty))$, $\ell_k \in C((t_0-\tau_k,\infty),[0,\infty))$ such that
- $xF(t,x) \ge b(t)x^2$ for $t \ge t_0$, $x \in \mathbb{R}$,
- $|\psi_k(t,x,y)| \le A_k(t)|x| + B_k(t)|y|$ for $t \ge t_0$, $x,y \in \mathbb{R}$,
- $|f_k(t,x)| \le \ell_k(t)|x|$ for $t \ge t_0 - \tau_k$, $x \in \mathbb{R}$.

(C_4) The inequality

$$\begin{aligned} b(t) \ge \sum_{k=1}^{n}\Bigg[& A_k(t) + \frac{1+e^{\frac{1-\rho}{\rho}g_k(t)}}{2}B_k(t) \\ & + \int_{t-g_k(t)}^{t} \frac{1+e^{\frac{1-\rho}{\rho}(t-s)}}{2}\left|C_k(t,s)\right|\ell_k(s)\,\mathrm{d}s\Bigg] \end{aligned}$$

holds for all $t \ge t_0$.

Our first main result is concerned with stability. Note that, in contrast to ordinary derivatives, fractional derivatives depend significantly on the initial time point t_0, which is equal to the lower limit of the derivative. Hence, any change of the initial time leads to a change of the fractional derivative. For this reason, we will not consider the case of uniform stability and study only the cases of stability, exponential stability, and asymptotic stability.

Theorem 4.1 (Stability) *If* (C_1)–(C_4) *hold, then the zero solution of* (2.1) *with zero initial function is stable.*

Proof Let $x = x(\cdot,t_0,\phi)$ be any solution of (2.1). We will show that the assumptions of Lemma 3.3 are satisfied. To this end, suppose $t \ge t_0$ is such that (3.3) is satisfied. If $s \in [t-g_k(t),t] \subset [t-\tau,t]$, then $\theta := s-t \in [-\tau,0]$, and from (3.3), we obtain

$$x^2(s)e^{\frac{1-\rho}{\rho}(s-t)} \le x^2(t) \quad \text{for all } s \in \left[t-g_k(t),t\right]. \tag{4.1}$$

In the calculation below, we also use the "trivial" inequality

$$|2ab| \le a^2 + b^2 \quad \text{for all } a, b \in \mathbb{R}. \tag{4.2}$$

Now,

$$\begin{aligned}
\left({}^{\mathrm{C}}_{t_0}\mathscr{D}^{\alpha,\rho}x^2\right)(t) &\overset{(3.2)}{\le} 2x(t)\left({}^{\mathrm{C}}_{t_0}\mathscr{D}^{\alpha,\rho}x\right)(t) \\
&= -2x(t)F\bigl(t,x(t)\bigr) + 2\sum_{k=1}^{n} x(t)\psi_k\bigl(t,x(t),x\bigl(t-g_k(t)\bigr)\bigr) \\
&\quad + 2\sum_{k=1}^{n}\int_{t-g_k(t)}^{t} C_k(t,s)x(t)f_k\bigl(s,x(s)\bigr)\,\mathrm{d}s \\
&\le -2x(t)F\bigl(t,x(t)\bigr) + 2\sum_{k=1}^{n}\bigl|x(t)\bigr|\bigl|\psi_k\bigl(t,x(t),x\bigl(t-g_k(t)\bigr)\bigr)\bigr| \\
&\quad + 2\sum_{k=1}^{n}\int_{t-g_k(t)}^{t} \bigl|C_k(t,s)\bigr|\bigl|x(t)\bigr|\bigl|f_k\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s \\
&\overset{(\mathrm{C}_3)}{\le} -2b(t)x^2(t) + 2\sum_{k=1}^{n}\bigl|x(t)\bigr|\bigl\{A_k(t)\bigl|x(t)\bigr| + B_k(t)\bigl|x\bigl(t-g_k(t)\bigr)\bigr|\bigr\} \\
&\quad + 2\sum_{k=1}^{n}\int_{t-g_k(t)}^{t} \bigl|C_k(t,s)\bigr|\ell_k(s)|x(t)x(s)|\,\mathrm{d}s \\
&= -2b(t)x^2(t) + \sum_{k=1}^{n} 2A_k(t)x^2(t) + \sum_{k=1}^{n} B_k(t)\bigl|2x(t)x\bigl(t-g_k(t)\bigr)\bigr| \\
&\quad + \sum_{k=1}^{n}\int_{t-g_k(t)}^{t} \bigl|C_k(t,s)\bigr|\ell_k(s)\bigl|2x(t)x(s)\bigr|\,\mathrm{d}s \\
&\overset{(4.2)}{\le} -2b(t)x^2(t) + \sum_{k=1}^{n} 2A_k(t)x^2(t) + \sum_{k=1}^{n} B_k(t)\bigl\{x^2(t) + x^2\bigl(t-g_k(t)\bigr)\bigr\} \\
&\quad + \sum_{k=1}^{n}\int_{t-g_k(t)}^{t} \bigl|C_k(t,s)\bigr|\ell_k(s)\bigl\{x^2(t) + x^2(s)\bigr\}\,\mathrm{d}s \\
&= -2b(t)x^2(t) + \sum_{k=1}^{n}\bigl\{2A_k(t) + B_k(t)\bigr\}x^2(t) + \sum_{k=1}^{n} B_k(t)x^2\bigl(t-g_k(t)\bigr) \\
&\quad + \sum_{k=1}^{n}\int_{t-g_k(t)}^{t} \bigl|C_k(t,s)\bigr|\ell_k(s)\bigl\{x^2(t) + x^2(s)\bigr\}\,\mathrm{d}s \\
&\overset{(4.1)}{\le} -2b(t)x^2(t) + \sum_{k=1}^{n}\bigl\{2A_k(t) + B_k(t)\bigr\}x^2(t) \\
&\quad + \sum_{k=1}^{n} B_k(t)x^2(t)e^{\frac{\rho-1}{\rho}(t-g_k(t)-t)} \\
&\quad + \sum_{k=1}^{n}\int_{t-g_k(t)}^{t} \bigl|C_k(t,s)\bigr|\ell_k(s)\bigl\{x^2(t) + x^2(t)e^{\frac{\rho-1}{\rho}(s-t)}\bigr\}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= 2x^2(t)\Bigg[-b(t) + \sum_{k=1}^{n}\left\{A_k(t) + \frac{1 + e^{\frac{1-\rho}{\rho}g_k(t)}}{2}B_k(t)\right\} \\
&\quad + \sum_{k=1}^{n}\int_{t-g_k(t)}^{t}\big|C_k(t,s)\big|\ell_k(s)\frac{1 + e^{\frac{1-\rho}{\rho}(t-s)}}{2}\,\mathrm{d}s\Bigg] \\
&\overset{(\mathrm{C}_4)}{\leq} 0,
\end{aligned}
$$

i.e., (3.4) holds. Hence, the assumptions of Lemma 3.3 are satisfied. According to Corollary 3.6, (3.8) holds. This completes the proof. □

Corollary 4.2 (Boundedness) *If* (C_1), (C_3), *and* (C_4) *hold, then the solutions of* (2.1) *are bounded.*

Proof As in the proof of Theorem 4.1, we obtain (3.8) for any solution $x = x(\cdot, t_0, \phi)$ of (2.1), i.e., boundedness is established. □

Remark 4.3 The above-obtained sufficient conditions for stability and boundedness are true for the case of delay Caputo fractional integro-differential equations ($\rho = 1$). Note that a partial case of (2.1), namely F in a special form, ψ_1 not depending on the third variable, $\psi_k = 0$ for all $2 \leq k \leq n$, constant delays, and the Caputo fractional derivative is studied in [8], and the stability of the zero solution and boundedness of solutions are proved. A modified Razhumikhin condition and results known in the literature (see [8, Lemma 1, Lemma 4]) are used. However, this condition is not applied adequately in the proof of the result for stability (see [8, Theorem 1]). In the proof of the boundedness [8, Theorem 2], the negativeness of the Caputo fractional derivative of the Lyapunov function is applied for all points t instead of only for those for which the Razhumikhin condition is true (compare with Remark 3.5). Hence, our results do not only generalize theorems from [8], they also remove the inconsistencies in the case of regular Caputo fractional differential equations.

In connection with the above-mentioned remarks and for future reference, we state our results for delay integro-differential equations with Caputo fractional derivative (with $\rho = 1$)

$$
\begin{aligned}
&\big({}^{\mathrm{C}}_{t_0}\mathscr{D}^{\alpha,1}x\big)(t) = -F\big(t,x(t)\big) + \sum_{k=1}^{n}\psi_k\big(t,x(t),x\big(t-g_k(t)\big)\big) \\
&\qquad + \sum_{k=1}^{n}\int_{t-g_k(t)}^{t}C_k(t,s)f_k\big(s,x(s)\big)\,\mathrm{d}s, \quad t > t_0, \\
&x(t_0+\theta) = \phi(\theta), \quad \theta \in [-\tau, 0].
\end{aligned}
\tag{4.3}
$$

As a partial case of Theorem 4.1 and Corollary 4.2, we obtain the following result.

Corollary 4.4 *Assume* (C_1)–(C_3) *hold and*

$$
b(t) \geq \sum_{k=1}^{n}\left[A_k(t) + B_k(t) + \int_{t-g_k(t)}^{t}\big|C_k(t,s)\big|\ell_k(s)\,\mathrm{d}s\right]. \tag{4.4}
$$

Then,

- *the solutions of* (4.3) *are bounded and the bound depends on the initial function*;
- *the zero solution of* (4.3) *with zero initial function is stable*.

From Theorem 4.1 and Corollary 4.2, we obtain the following results.

Corollary 4.5 (Exponential Stability) *Let* $\rho \in (0,1)$. *If* (C_1)–(C_4) *hold, then the zero solution of* (2.1) *is exponentially stable*.

Proof Let $x = x(\cdot, t_0, \phi)$ be any solution of (2.1), and suppose the point $t \geq t_0$ is such that (3.3) holds. If $s \in [t - g_k(t), t] \subset [t - \tau, t]$, then

$$\theta := s - t \in [-\tau, 0],$$

and from (3.3), we obtain (4.1), i.e.,

$$x^2(s)e^{\frac{1-\rho}{\rho}(s-t)} \leq x^2(t) \quad \text{for all } s \in \left[t - g_k(t), t\right]$$

holds. Following the ideas of the proof of Theorem 4.1, we obtain

$$\left({}_{t_0}^{C}\mathscr{D}^{\alpha,\rho} x^2\right)(t) \leq 0,$$

i.e., (3.4) holds. Therefore, the assumptions of Lemma 3.3 are satisfied, and thus

$$\left|x(s)\right| \leq \sqrt{\max_{\theta \in [-\tau, 0]} \phi^2(\theta) e^{\frac{\rho-1}{2\rho}(s-t_0)}} \quad \text{for all } s \geq t_0 \tag{4.5}$$

holds, i.e., since $\rho \in (0,1)$, exponential stability is established. □

Corollary 4.6 (Asymptotic Stability) *Let* $\rho \in (0,1)$. *If* (C_1)–(C_4) *hold, then the zero solution of* (2.1) *is asymptotically stable*.

Proof The proof follows from $\rho \in (0,1)$ and (4.5). □

Remark 4.7 The above-proved results about exponential stability and asymptotic stability are true only for $\rho \in (0,1)$, i.e., we do not provide sufficient conditions for the case of the Caputo fractional derivative (with $\rho = 1$). For Caputo fractional derivatives, results concerning the asymptotic stability [8, Theorem 3] and Mittag-Leffler stability [8, Theorem 4] are studied, but the negativeness of the Caputo fractional derivative of the Lyapunov function is applied for all points instead of only for those for which the Razumikhin condition holds (compare with Remark 3.5). Hence, the question about asymptotic stability and Mittag-Leffler stability is still open for delay Caputo fractional integro-differential equations.

5 Examples

We first look at [8, Example 1] with appropriate modifications.

Example 5.1 Consider

$$
\begin{aligned}
\left({}_{0.25}^{\mathrm{C}}\mathscr{D}^{\alpha,1}x\right)(t) &= -\left(24+\frac{1}{1+t}\right)\left(4x(t)+x(t)e^{-x^2(t)-t^2}\right) \\
&\quad -3x(t)-\sin\bigl(x(t)\bigr)+\frac{\sin(x(t))}{1+e^t} \\
&\quad +\int_{t-0.25}^{t} e^{s-t}\frac{\sin(x(s))}{1+e^{s^2}}\,\mathrm{d}s, \quad t>0.25, \\
x(0.25+\theta) &= \phi(\theta), \quad \theta\in[-0.25,0].
\end{aligned}
\tag{5.1}
$$

Problem (5.1) is a special case of (2.1) with

$$
\begin{aligned}
&n=1, \qquad \rho=1, \qquad t_0=0.25, \qquad g_1(t)\equiv\tau_1=0.25, \\
&C_1(t,s)=e^{s-t}, \qquad f_1(t,x)=\frac{\sin(x)}{1+e^{t^2}}, \qquad \psi_1(t,x,y)=\frac{\sin(x)}{1+e^t}, \\
&F(t,x)=\left(24+\frac{1}{1+t}\right)\left(4x+xe^{-x^2-t^2}\right)+3x+\sin(x).
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
&F(t,0)=0, \qquad xF(t,x)\ge\left(98+\frac{4}{1+t}\right)x^2, \qquad b(t)=98+\frac{4}{1+t}, \\
&\psi_1(t,0,0)=0, \qquad \bigl|\psi_1(t,x,y)\bigr|\le\frac{|x|}{1+e^t}, \qquad A_1(t)=\frac{1}{1+e^t}, \qquad B_1(t)\equiv 0, \\
&f_1(t,0)=0, \qquad \bigl|f_1(t,x)\bigr|\le\frac{|x|}{1+e^{t^2}}\le\frac{|x|}{2}, \qquad \ell_1(t)=\frac{1}{2}.
\end{aligned}
$$

Therefore, (C_1)–(C_3) are satisfied. Moreover, we have (recall that $B_1(t)\equiv 0$)

$$
\begin{aligned}
A_1(t)+\int_{t-g_1(t)}^{t}\bigl|C_1(t,s)\bigr|\ell_1(s)\,\mathrm{d}s &= \frac{1}{1+e^t}+\frac{1}{2}\int_{t-0.25}^{t}e^{s-t}\,\mathrm{d}s \\
&= \frac{1}{1+e^t}+\frac{1}{2(1-e^{-0.25})}<3.3<98<98+\frac{4}{1+t}=b(t),
\end{aligned}
$$

so that (4.4) is fulfilled. According to Corollary 4.4, the zero solution is stable.

Example 5.2 Let $\alpha\in(0,1)$ and let $\rho\in(0.0247292,1)$. Consider (5.1) with the generalized Caputo proportional fractional derivative instead of the Caputo fractional derivative, i.e., consider

$$
\begin{aligned}
\left({}_{0.25}^{\mathrm{C}}\mathscr{D}^{\alpha,\rho}x\right)(t) &= -\left(24+\frac{1}{1+t}\right)\left(4x(t)+x(t)e^{-x^2(t)-t^2}\right) \\
&\quad -3x(t)-\sin\bigl(x(t)\bigr)+\frac{\sin(x(t))}{1+e^t} \\
&\quad +\int_{t-0.25}^{t} e^{s-t}\frac{\sin(x(s))}{1+e^{s^2}}\,\mathrm{d}s, \quad t>0.25, \\
x(0.25+\theta) &= \phi(\theta), \quad \theta\in[-0.25,0].
\end{aligned}
\tag{5.2}
$$

As in Example 5.1, (C_1)–(C_3) are satisfied. Let us define

$$\kappa := \frac{1}{4\rho} - \frac{1}{2} \leq 9.6095062.$$

We have (recall that $B_1(t) \equiv 0$)

$$\begin{aligned}
&A_1(t) + \int_{t-g_1(t)}^{t} \frac{1 + e^{\frac{1-\rho}{\rho}(t-s)}}{2} \bigl|C_1(t,s)\bigr| \ell_1(s)\,\mathrm{d}s \\
&\quad = \frac{1}{1+e^t} + \int_{t-0.25}^{t} \frac{1 + e^{\frac{1-\rho}{\rho}(t-s)}}{2} e^{s-t} \frac{\mathrm{d}s}{2} \\
&\quad < 1 + \frac{1}{4} \int_{t-0.25}^{t} \bigl(e^{s-t} + e^{-4\kappa(s-t)}\bigr)\,\mathrm{d}s \\
&\quad = 1 + \frac{1}{4}\bigl(1 - e^{-0.25}\bigr) + \frac{1}{4} \int_{t-0.25}^{t} e^{-4\kappa(s-t)}\,\mathrm{d}s \\
&\quad = 1 + \frac{1}{4}\bigl(1 - e^{-0.25}\bigr) + \frac{1}{16} \begin{cases} 1 & \text{if } \kappa = 0, \\ \frac{e^{\kappa}-1}{\kappa} & \text{otherwise} \end{cases} \\
&\quad < 97.995812 < 98 < 98 + \frac{4}{1+t} = b(t),
\end{aligned}$$

and the first inequality in the last line is due to the fact that the function f defined by $f(x) = (e^x - 1)/x$ for $x \in \mathbb{R} \setminus \{0\}$ and $f(0) = 1$ is strictly increasing. Thus, the condition (C_4) is fulfilled. According to Corollary 4.5, the zero solution of (5.2) is exponentially stable.

Example 5.3 Consider

$$\begin{aligned}
&\bigl({}_0^{\mathrm{C}}\mathscr{D}^{\alpha,1}x\bigr)(t) = -\frac{2+t}{1+t}x(t) + \frac{x(t)}{1+t^2} + \int_{t-1}^{t} (t-s)x(s)\,\mathrm{d}s, \quad t > 0, \\
&x(\theta) = \phi(\theta), \quad \theta \in [-1,0].
\end{aligned} \tag{5.3}$$

Problem (5.3) is a special case of (2.1) with

$$\begin{aligned}
&n = 1, \qquad \rho = 1, \qquad t_0 = 0, \qquad g_1(t) \equiv \tau_1 = 1, \\
&C_1(t,s) = t - s, \qquad f_1(t,x) = x, \qquad \psi_1(t,x,y) = \frac{x}{1+t^2}, \\
&F(t,x) = \frac{2+t}{1+t}x.
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
&F(t,0) = 0, \qquad xF(t,x) = \frac{2+t}{1+t}x^2, \qquad b(t) = \frac{2+t}{1+t}, \\
&\psi_1(t,0,0) = 0, \qquad \bigl|\psi_1(t,x,y)\bigr| \leq \frac{|x|}{1+t^2}, \qquad A_1(t) = \frac{1}{1+t^2}, \qquad B_1(t) \equiv 0, \\
&f_1(t,0) = 0, \qquad \bigl|f_1(t,x)\bigr| = |x|, \qquad \ell_1(t) = 1.
\end{aligned}$$

Therefore, (C_1)–(C_3) are satisfied. Moreover, we have (recall that $B_1(t) \equiv 0$)

$$\begin{aligned} A_1(t) + \int_{t-g_1(t)}^{t} \left|C_1(t,s)\right| \ell_1(s)\,\mathrm{d}s &= \frac{1}{1+t^2} + \int_{t-1}^{t} |t-s|\,\mathrm{d}s \\ &= \frac{1}{1+t^2} + \frac{1}{2} < \frac{2+t}{1+t} = b(t), \end{aligned}$$

so that (4.4) is fulfilled. According to Corollary 4.4, the zero solution is stable.

Example 5.4 Let $\alpha \in (0,1)$ and let $\rho \in (0.3420471, 1)$. Consider (5.3) with the generalized Caputo proportional fractional derivative instead of the Caputo fractional derivative, i.e., consider

$$\begin{aligned} &\left({}_0^{\mathrm{C}}\mathscr{D}^{\alpha,\rho} x\right)(t) = -\frac{2+t}{1+t}x(t) + \frac{x(t)}{1+t^2} + \int_{t-1}^{t} (t-s)x(s)\,\mathrm{d}s, \quad t > 0, \\ &x(\theta) = \phi(\theta), \quad \theta \in [-1, 0]. \end{aligned} \tag{5.4}$$

As in Example 5.3, (C_1)–(C_3) are satisfied. Let us define

$$\begin{aligned} &\kappa := \frac{1}{\rho} - 1 \le 1.9235739, \qquad g(t) := \frac{1}{1+t^2} - \frac{2+t}{1+t}, \\ &\mu := \max_{t>0} g(t) = -1 - 2\left(\frac{1}{2} + \frac{\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)^2. \end{aligned}$$

We have for $t > 0$ (recall that $B_1(t) \equiv 0$)

$$\begin{aligned} &A_1(t) + \int_{t-g_1(t)}^{t} \frac{1 + e^{\frac{1-\rho}{\rho}(t-s)}}{2} \left|C_1(t,s)\right| \ell_1(s)\,\mathrm{d}s \\ &= \frac{1}{1+t^2} + \int_{t-1}^{t} \frac{1+e^{\kappa(t-s)}}{2}(t-s)\,\mathrm{d}s \\ &= \frac{1}{1+t^2} + \frac{1}{2}\int_0^1 \left(\nu + \nu e^{\kappa\nu}\right)\mathrm{d}\nu \\ &= \frac{1}{1+t^2} + \frac{1}{2}\left\{\frac{1}{2} + \frac{(\kappa-1)e^{\kappa}+1}{\kappa^2}\right\} \\ &= g(t) + b(t) + \frac{1}{2}\left\{\frac{1}{2} + \frac{(\kappa-1)e^{\kappa}+1}{\kappa^2}\right\} \\ &\le \mu + \frac{1}{2}\left\{\frac{1}{2} + \frac{(\kappa-1)e^{\kappa}+1}{\kappa^2}\right\} + b(t) \\ &< b(t), \end{aligned}$$

and the last inequality is due to the fact that the function f defined by $f(x) = ((x-1)e^x + 1)/x^2$ for $x > 0$ is strictly increasing. Thus, the condition (C_4) is fulfilled. According to Corollary 4.5, the zero solution of (5.4) is exponentially stable.

Example 5.5 Consider

$$\begin{aligned}
\left({}_0^{\mathrm{C}}\mathscr{D}^{0.3,0.8}x\right)(t) &= -\frac{2+t}{1+t}x(t) + 0.03x(t-1)\cos^2 t \\
&\quad + \frac{x(t-1-\sin t)}{10+t^2} \\
&\quad + \int_{t-1}^{t} (t-s)x(s)\,\mathrm{d}s, \quad t>0, \\
x(\theta) &= \phi(\theta), \quad \theta \in [-2,0].
\end{aligned} \tag{5.5}$$

Problem (5.5) is a special case of (2.1) with

$$\begin{aligned}
&n=2, \quad \rho = 0.8, \quad \alpha = 0.3, \quad t_0 = 0, \\
&g_1(t) \equiv 1, \quad g_2(t) = 1+\sin t, \quad \tau_1 = 1, \quad \tau_2 = 2, \quad \tau = 2, \\
&C_1(t,s) = t-s, \quad C_2(t,s) \equiv 0, \quad f_1(t,x) = f_2(t,x) = x, \\
&\psi_1(t,x,y) = 0.03y\cos^2 t, \quad \psi_2(t,x,y) = \frac{y}{10+t^2}, \quad F(t,x) = \frac{2+t}{1+t}x.
\end{aligned}$$

If we put

$$\begin{aligned}
&b(t) = \frac{2+t}{1+t}, \quad \ell_1(t) = \ell_2(t) \equiv 1, \\
&A_1(t) = A_2(t) \equiv 0, \quad B_1(t) = 0.03, \quad B_2(t) = 0.1,
\end{aligned}$$

then (C_1)–(C_3) are satisfied. We put $\kappa := (1-\rho)/\rho = 0.25$ and estimate for $t>0$ (recall that $A_1(t) = A_2(t) = C_2(t) \equiv 0$)

$$\begin{aligned}
&\frac{1+e^{\kappa g_1(t)}}{2}B_1(t) + \frac{1+e^{\kappa g_2(t)}}{2}B_2(t) + \int_{t-g_1(t)}^{t} \frac{1+e^{\kappa(t-s)}}{2}\left|C_1(t,s)\right|\ell_1(s)\,\mathrm{d}s \\
&\quad = \frac{1+e^{\kappa}}{2}0.03 + \frac{1+e^{\kappa(1+\sin t)}}{2}0.1 + \int_{t-1}^{t} \frac{1+e^{\kappa(t-s)}}{2}(t-s)\,\mathrm{d}s \\
&\quad \le \frac{1+e^{\kappa}}{2}0.03 + \frac{1+e^{2\kappa}}{2}0.1 + \frac{1}{2}\int_0^1 \left(v + ve^{\kappa v}\right)\mathrm{d}v \\
&\quad = \frac{1+e^{\kappa}}{2}0.03 + \frac{1+e^{2\kappa}}{2}0.1 + \frac{1}{2}\left\{\frac{1}{2} + \frac{(\kappa-1)e^{\kappa}+1}{\kappa^2}\right\} \\
&\quad < 0.712544 < 1 < 1 + \frac{1}{1+t} = b(t),
\end{aligned}$$

and thus (C_4) is fulfilled. From Corollary 4.5, follows the exponential stability and therefore the asymptotic stability of the zero solution of (5.5), and according to (4.5), any solution of (5.5) satisfies

$$\left|x(s)\right| \le \sqrt{\max_{\theta\in[-2,0]} \phi^2(\theta)e^{-0.125s}} \quad \text{for all } s \ge 0.$$

6 Conclusions

A scalar nonlinear fractional integro-differential equation with time-variable bounded delays and the generalized Caputo proportional fractional derivative is studied. Some qualitative properties such as stability, exponential stability, asymptotic stability, and boundedness are investigated. The main apparatus is the application of Lyapunov functions and a Razumikhin method, modified for our special situation. Five examples are provided to illustrate the usefulness of the obtained sufficient conditions for generalized Caputo proportional fractional delay integro-differential equations, as well as for classical Caputo fractional delay integro-differential equations. Also, the main lemma about the inequality for quadratic functions offers a tool for successful study of stability of various types of models described by generalized Caputo proportional fractional differential equations with and without delays.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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