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Global well-posedness of the compressible quantum magnetohydrodynamic model with small initial energy

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Abstract
In this paper, we investigate the three-dimensional Cauchy problem of the compressible quantum magnetohydrodynamic model. It is proved that the system admits a unique global solution, provided that the initial energy is suitably small. Furthermore, the large time behavior of the global solution is obtained.

Keywords: Compressible quantum; Magnetohydrodynamic model; Small initial energy; Global existence

1 Introduction

This paper is concerned with the compressible viscous quantum magnetohydrodynamic (vQMHD) model which has the following form (see, e.g., [7]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda)\nabla \operatorname{div} \mathbf{u} + \nabla P(\rho) - \frac{\hbar^2}{2}\rho \nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \\ \quad = (\operatorname{curl} \mathbf{B}) \times \mathbf{B}, \\ \mathbf{B}_t - \nu \Delta \mathbf{B} = \operatorname{curl}(\mathbf{u} \times \mathbf{B}), \qquad \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in \mathbb{R}^3 \times [0,\infty)$. Here ρ, $\mathbf{u}$, and $\mathbf{B}$ represent the fluid density, velocity, and magnetic field, respectively. $P(\rho) = \rho^\gamma$ with $\gamma > 1$ denotes the pressure. $\hbar$ is the Planck constant, and ν is the magnetic diffusivity. μ and λ are two viscosity constants satisfying the physical restrictions

$$\mu > 0 \quad \text{and} \quad 2\mu + 3\lambda \geq 0.$$

The expression $\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ indicates a quantum potential, i.e., Bohm potential, satisfying

$$2\rho \nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \operatorname{div}\left(\rho \nabla^2 \log \rho\right) = \Delta \nabla \rho + \frac{|\nabla \rho|^2 \nabla \rho}{\rho^2} - \frac{\nabla \rho \Delta \rho}{\rho} - \frac{\nabla \rho \cdot \nabla^2 \rho}{\rho},$$

which was considered for the thermodynamic equilibrium by Wigner [25]. The quantum hydrodynamic models are viewed as a quantum correction to the classical hydrodynamic equations. Manfredi and Haas [17] introduced a quantum hydrodynamic model for plasmas, and then this model was extended to a quantum magnetohydrodynamic model by Hass [7, 8] from a Wigner–Maxwell system. In dense astrophysical plasmas, such as atmospheres of neutron stars or interiors of massive white dwarfs, the quantum magnetohydrodynamic model plays an important role [18], where the dimensionless magnetic diffusivity $\nu > 0$ was considered. Later, Yang and Ju [30] induced the electric field E by moving conductive flow in the magnetic field, i.e.,

$$E = \nu\nabla \times \mathbf{B} - \mathbf{u} \times \mathbf{B} + \frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$$

and introduced vQMHD system (1.1). Although the electric field E does not appear in system (1.1) as in the compressible magnetohydrodynamic models, it is indeed induced according to the above relation. The main purpose of this paper is to study the Cauchy problem of system (1.1) supplemented with initial data

$$(\rho, \mathbf{u}, \mathbf{B})(x,t)|_{t=0} = \big(\rho_0(x), \mathbf{u}_0(x), \mathbf{B}_0(x)\big) \to (1,0,0) \quad \text{as } |x| \to \infty. \tag{1.2}$$

In the last decade, there have been many results on the well-posedness of the vQMHD system. Yang and Ju [30] studied the global weak solution in a three-dimensional torus by the Faedo–Galerkin method and weak compactness techniques. Then, Li et al. [15] investigated the large time behavior of the global weak solution. In [20, 21], the global well-posedness of classical solutions of compressible isentropic and nonisentropic vQMHD system under the condition that H^k-norm ($k \geq 3$) of initial data is small in a three-dimensional whole space was proved by Pu and Xu. They established the optimal time decay rates for the global solution as well. Later, Xie et al. [29] showed faster time convergence rates for the solutions with the initial data belonging to L^1. In [31], Yang proved that the classical solutions of the vQMHD equations converge to those of the incompressible magnetohydrodynamic equations with a sharp convergence rate as the Mach number goes to zero. Recently, Xi and her collaborators [26, 27] established the optimal time decay rates for higher-order spatial derivatives of solutions by using the Fourier splitting method. In [32], Yang et al. investigated the existence of the time-periodic solution combined with the topological degree theory. In addition, for related models of the vQMHD system, there is a rich body of literature concerned with the compressible fluid model of Korteweg type. We refer to [1–4, 12–14, 16, 19, 22, 24, 28] and the references therein for instance.

To our knowledge, the known results on the global smooth solutions of vQMHD equations need conditions as the initial perturbation is small in H^k ($k \geq 3$). Motivated by the study of compressible Navier–Stokes equations [10, 11], we discuss the global existence and large time behavior of the solution for the three-dimensional compressible vQMHD system with small initial energy. The condition of small initial energy in our paper is equivalent to the smallness for L^2-norm of the initial data. Our results improve previous works in [26, 27], in which the existence of global solutions is obtained for H^k ($k \geq 3$)-norm of the initial data. On the other hand, due to the strongly nonlinear degeneracy of the quantum Bohm potential and the nonlinear coupling of the velocity field and the magnetic field, it is more difficult to estimate high-order terms. In order to overcome these difficulties, we establish suitable *a priori* assumption (2.4) and introduce the continuity argument.

Throughout this paper, we use $H^s(\mathbb{R}^3)$ ($s \in \mathbb{N}$) to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^s}$ and $L^p(\mathbb{R}^3)$ ($1 \le p \le \infty$) to denote the L^p spaces with norm $\|\cdot\|_{L^p}$. For given initial data $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$, we define the initial energy E_0:

$$E_0 = \int \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{1}{2}|\mathbf{B}_0|^2 + G(\rho_0) \right) dx,$$

where $G(\rho) = \rho \int_1^\rho \frac{P(s)-P(1)}{s^2}\, ds$. For constants $\bar{\rho} \ge \underline{\rho} > 0$, it is clear that

$$c_1(\underline{\rho}, \bar{\rho})(\rho - 1)^2 \le G(\rho) \le c_2(\underline{\rho}, \bar{\rho})(\rho - 1)^2,$$

if $\underline{\rho} \le \rho \le \bar{\rho}$. Here positive constants $c_1(\underline{\rho}, \bar{\rho})$ and $c_2(\underline{\rho}, \bar{\rho})$ are dependent on $\underline{\rho}$ and $\bar{\rho}$.

Now, we state our main result.

Theorem 1.1 *For given $M_1 > 0$ (not necessarily small) and $\bar{\rho} \ge \underline{\rho} > 0$, suppose that the initial data $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ satisfy*

$$\begin{aligned} &\underline{\rho} \le \rho_0 \le \bar{\rho}, \qquad (\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0) \in H^4 \times H^3 \times H^3, \\ &\|\nabla^2 \rho_0\|_{L^2}^2 + \|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\mathbf{B}_0\|_{L^4}^4 \le M_1. \end{aligned} \tag{1.3}$$

Then there exists a unique global solution $(\rho, \mathbf{u}, \mathbf{B})$ satisfying

$$\begin{aligned} &\frac{1}{2}\underline{\rho} < \rho < 2\bar{\rho}, \qquad \rho - 1 \in L^\infty\big(0, T; H^4\big), \qquad \nabla\rho \in L^2\big(0, T; H^4\big), \\ &\mathbf{u}, \mathbf{B} \in L^\infty\big(0, T; H^3\big), \qquad \nabla\mathbf{u}, \nabla\mathbf{B} \in L^2\big(0, T; H^3\big), \end{aligned} \tag{1.4}$$

and enjoying the following large time behavior

$$\lim_{t\to 0}\big(\|\rho - 1\|_{L^q}^2 + \|\mathbf{u}\|_{L^q}^2 + \|\mathbf{B}\|_{L^q}^2\big) = 0 \quad \textit{for any } q \in (2, 6], \tag{1.5}$$

provided $E_0 \le \delta$, where δ is a positive constant depending on $\underline{\rho}$, $\bar{\rho}$, μ, λ, γ, $\hbar$, ν, and M_1, but independent of t.

Remark 1.1 If the initial data has more regularity $(\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0) \in H^5 \times H^4 \times H^4$ in Theorem 1.1, employing arguments similar to Sect. 4 (particular Lemma 4.5) in [10] and Lemma 2.1 in [27], one can further prove that the solution $(\rho, \mathbf{u}, \mathbf{B})$ belongs to

$$\rho - 1 \in L^\infty\big(0, T; H^5\big), \qquad \mathbf{u} \in L^\infty\big(0, T; H^4\big), \qquad \mathbf{B} \in L^\infty\big(0, T; H^4\big).$$

Thus, together with equations in (2.2) and using the following standard embedding:

$$L^\infty\big(0, T; H^1\big) \cap H^1\big(0, T; H^{-1}\big) \hookrightarrow C\big(0, T; L^q\big) \quad \text{for any } q \in [2, 6),$$

this solution is classical. For more details, one can see [10].

Remark 1.2 It is worth to point out that in previous works [26, 27] the smallness of $\|\rho_0 - 1\|_{H^5} + \|u_0\|_{H^4} + \|B_0\|_{H^4}$ was required, which means the initial data can only have small oscillations. Notice that in this paper we generalize the result in [26] to the case of large oscillations. Indeed, we only require the smallness of the initial energy E_0, which is equivalent to the smallness of the L^2-norm of $(\rho_0 - 1, u_0, B_0)$. Based on the difference of initial data, in [26, 27], the authors can establish higher derivative estimates directly, since some terms with H^k $(k \le 4)$ norms could be bounded by an appropriate small δ and be viewed as coefficients of other terms. It leads to the fact that the argument in [26, 27] cannot be used in this paper, since we do not have any smallness of higher derivative terms in assumptions. Therefore, in order to overcome difficulties from the higher derivative terms and nonlinear coupling, our strategy is to establish new and more precise estimates from lower to higher derivative to improve the regularity constantly.

The rest of the paper is organized as follows. In Sect. 2, we prove the local well-posedness of the solution and derive *a priori* estimates. In Sect. 3, the proof of Theorem 1.1 is provided.

2 A priori estimates

First of all, let us give the local existence of the solution for problem (1.1).

Lemma 2.1 *Assume that the initial data* $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ *satisfy* $(\rho_0 - 1, \mathbf{u}_0, \mathbf{B}_0) \in H^4 \times H^3 \times H^3$ *with* $0 < \underline{\rho} \le \rho_0$. *Then there exists a time* $T^* > 0$ *such that in* $0 \le t \le T^*$ *Cauchy problem* (1.1)–(1.2) *possesses a unique solution* $(\rho, \mathbf{u}, \mathbf{B})$ *satisfying*

$$\begin{aligned}
&\rho - 1 \in L^\infty\bigl(0, T^*; H^4\bigr) \cap L^2\bigl(0, T^*; H^5\bigr), \\
&\mathbf{u}, \mathbf{B} \in L^\infty\bigl(0, T^*; H^3\bigr), \qquad \nabla \mathbf{u}, \nabla \mathbf{B} \in L^2\bigl(0, T^*; H^3\bigr).
\end{aligned}$$

Proof Following the fixed point argument (for instance, Theorem 2.1 in [5] and Theorem 2.1 in [23]), using the technique in [9] to deal with the higher-order derivatives of density and the energy method to estimate the magnetic field $\mathbf{B}$ (see Lemma 2.3 in [6]), the local existence and uniqueness are quite standard. For completeness, we outline the proof here. Let us begin with the following auxiliary system:

$$\begin{cases}
\rho_t + \operatorname{div}(\rho \tilde{u}) = 0, \\
(\rho \mathbf{u})_t + \operatorname{div}(\rho \tilde{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + \nabla P(\rho) - \frac{\hbar^2}{2} \rho \nabla (\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}) \\
\quad = (\operatorname{curl} \mathbf{B}) \times \mathbf{B}, \\
\mathbf{B}_t - \nu \Delta \mathbf{B} = \operatorname{curl}(\tilde{u} \times \mathbf{B}), \qquad \operatorname{div} \mathbf{B} = 0,
\end{cases} \tag{2.1}$$

with initial condition (1.2). Here, $\tilde{u} \in R_{T^*}$ is a known function with $\tilde{u}(x, 0) = \mathbf{u}_0(x)$ and

$$R_{T^*} = \left\{ z \,\middle|\, \sup_{0 \le t \le T^*} \|z\|_{H^3}^2 + \int_0^{T^*} \|\nabla z\|_{H^3}^2 \, dt \le R \right\},$$

where $R > 1$ and $T^* > 0$ will be chosen later.

It is obvious that $(2.1)_1$ is a linear transport equation with regular $\tilde{u}$. The existence and uniqueness are well known. As did Lemma 2.2 in [6], let

$$\frac{dx(X,t)}{dt} = \tilde{u}\big(x(X,t),t\big) \quad \text{and} \quad x(X,0) = X,$$

then we have

$$\frac{d\rho(x(X,t),t)}{dt} = -\rho \operatorname{div} \tilde{u},$$

that is,

$$\rho(x,t) = \rho_0 \exp\left(-\int_0^t \operatorname{div} \tilde{u}\, ds\right).$$

Therefore,

$$\rho(x,t) \le \rho_0 \exp\left(\int_0^t \|\operatorname{div} \tilde{u}\|_{L^\infty}\, ds\right) \le \rho_0 \exp\left(\int_0^t C\|\tilde{u}\|_{H^3}\, ds\right).$$

Thus, if $T^* > 0$ is suitably small, we can get the bound of ρ. On the other hand, $(2.1)_3$ is a linear parabolic equation with respect to the function $\mathbf{B}$. Following the standard energy argument for linear parabolic equation, we have the existence of a unique solution $\mathbf{B}$ and the desired estimates. With the estimates for $(\rho, \mathbf{B})$ at hand and using the classical theory of linear parabolic equation again, we can get the existence and uniqueness of $\mathbf{u}$ by $(2.1)_2$, which implies that we obtain the existence and uniqueness for the linearized system (2.1).

Next, we can define a map Λ on R_{T^*} to the solution of $(2.1)_2$ as

$$\Lambda(\tilde{u}) = \mathbf{u}.$$

Taking advantage of the energy argument, one can get some desired estimates for $(\rho, \mathbf{u}, \mathbf{B})$, which yields that $\mathbf{u}$ belongs to R_{T^*} for some large positive constant R and an appropriately small T^*. Finally, Schauder's fixed point theorem gives us the local well-posedness of Cauchy problem (1.1)–(1.2). □

Now, we are ready to show *a priori* estimates on the solutions. Firstly, we need to rewrite system (1.1)–(1.2) as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \rho\mathbf{u}_t - \mu\Delta\mathbf{u} - (\mu+\lambda)\nabla\operatorname{div}\mathbf{u} + \nabla P - \frac{\hbar^2}{4}\Delta\nabla\rho = -\rho\mathbf{u}\cdot\nabla\mathbf{u} + F_1, \\ \mathbf{B}_t - \nu\Delta\mathbf{B} = \operatorname{curl}(\mathbf{u}\times\mathbf{B}), \qquad \operatorname{div}\mathbf{B} = 0, \end{cases} \tag{2.2}$$

where

$$F_1 = \frac{\hbar^2}{4}\left(\frac{|\nabla\rho|^2\nabla\rho}{\rho^2} + \frac{\nabla\rho\Delta\rho}{\rho} - \frac{\nabla\rho\nabla^2\rho}{\rho}\right) + (\operatorname{curl}\mathbf{B})\times\mathbf{B},$$

with the prescribed initial condition

$$(\rho,\mathbf{u},\mathbf{B})(x,t)|_{t=0} = \big(\rho_0(x),\mathbf{u}_0(x),\mathbf{B}_0(x)\big) \to (1,0,0) \quad \text{as } |x|\to\infty. \tag{2.3}$$

Define

$$A(T) = \sup_{0\le t\le T} \int \left(\left|\nabla^2\rho\right|^2 + |\nabla \mathbf{u}|^2 + |\mathbf{B}|^4 \right) dx.$$

In what follows, the generic constants $C > 0$ and $C_i > 0$ ($i = 1,2,3$) and a suitably small constant $\delta_1 > 0$ are dependent on some known constants $\underline{\rho}$, $\bar{\rho}$, μ, λ, γ, $\hbar$, and ν, but independent of t, respectively. Particularly, we write $C(M)$ to emphasize that C may depend on M, where $M = (1 + C_3)M_1$.

Next, let us establish necessary lower-order estimates for the global solution $(\rho, \mathbf{u}, \mathbf{B})$ independent of time. We assume that $(\rho, \mathbf{u}, \mathbf{B})$ is a solution of system (2.2)–(2.3) on $\mathbb{R}^3 \times (0, T)$ for positive time $T > 0$. Here, we set $E_0 \le 1$ without loss of generality.

Proposition 2.1 *Under the assumptions of Theorem 1.1, assume that the solution $(\rho, \mathbf{u}, \mathbf{B})$ satisfies*

$$A(T) \le 2M, \qquad \frac{1}{4}\underline{\rho} \le \rho \le 2\bar{\rho}, \tag{2.4}$$

for $(x,t) \in \mathbb{R}^3 \times (0,T)$, then it holds that

$$A(T) \le \frac{3}{2}M, \qquad \frac{1}{2}\underline{\rho} \le \rho \le \frac{3}{2}\bar{\rho}, \tag{2.5}$$

provided $E_0 \le \delta$, where δ is a positive constant depending on $\underline{\rho}$, $\bar{\rho}$, μ, λ, γ, ν, and M but independent of t.

In order to prove Proposition 2.1, we firstly investigate the following lemmas. Then, we finish the proof of Proposition 2.1 after Lemma 2.5.

Lemma 2.2 *Let $(\rho, \mathbf{u}, \mathbf{B})(x,t)$ be the solution of problem (2.2)–(2.3). Then it holds that*

$$\begin{aligned} &\sup_{0\le t\le T} \int \left(\frac{1}{2}\rho|\mathbf{u}|^2 + G(\rho) + \frac{\hbar^2}{8\rho}|\nabla\rho|^2 + \nu|\mathbf{B}|^2 \right) dx \\ &\quad + \int_0^T \int \left(\mu|\nabla\mathbf{u}|^2 + (\lambda+\mu)|\operatorname{div}\mathbf{u}|^2 + \nu|\nabla\mathbf{B}|^2 \right) dx\,dt \le E_0. \end{aligned} \tag{2.6}$$

Proof Multiplying $(2.2)_1$–$(2.2)_3$ by $G'(\rho)$, $\rho\mathbf{u}$ and $\mathbf{B}$ respectively, summing the resulting equations up and integrating them by parts, we obtain (2.6). □

Lemma 2.3 *Under the assumptions of Proposition 2.1, it holds that*

$$\frac{1}{2}\underline{\rho} \le \rho \le \frac{3}{2}\bar{\rho} \tag{2.7}$$

for $(x,t) \in \mathbb{R}^3 \times (0,T)$, provided $E_0 \le \min\{(\frac{1-\frac{1}{2}\underline{\rho}}{C(M)})^4, (\frac{\frac{3}{2}\bar{\rho}-1}{C(M)})^4\}$.

Proof By (2.4), (2.6), and the Sobolev inequality, we obtain

$$\|\rho - 1\|_{L^\infty} \le C\|\rho - 1\|_{L^2}^{\frac{1}{4}} \left\|\nabla^2\rho\right\|_{L^2}^{\frac{3}{4}} \le C(M)E_0^{\frac{1}{4}},$$

which implies

$$1 - C(M)E_0^{\frac{1}{4}} < \rho < 1 + C(M)E_0^{\frac{1}{4}}.$$

When $E_0 \leq \min\{(\frac{1-\frac{1}{2}\tilde{\rho}}{C(M)})^4, (\frac{\frac{3}{2}\tilde{\rho}-1}{C(M)})^4\}$, we get (2.7). □

Lemma 2.4 *Under the assumptions of Proposition 2.1, it holds that*

$$\int_0^T \|\nabla\rho\|_{L^2}^2\,dt + \int_0^T \left\|\nabla^2\rho\right\|_{L^2}^2 \leq C(M)E_0, \tag{2.8}$$

provided $E_0 \leq (\frac{\hbar^2}{32C(M)})^4$.

Proof Multiplying $(2.2)_2$ by $\frac{1}{\rho}\nabla\rho$ and using $(2.2)_1$, then integrating the resulting equation over $\mathbb{R}^3$, one has

$$\begin{aligned}
&\int \gamma\rho^{\gamma-2}|\nabla\rho|^2\,dx + \int \frac{\hbar^2}{4}\rho^{-1}\left|\nabla^2\rho\right|^2 dx \\
&\quad = -\frac{d}{dt}\int \mathbf{u}\cdot\nabla\rho\,dx + \int \operatorname{div}\mathbf{u}\cdot\operatorname{div}(\rho\mathbf{u})\,dx \\
&\qquad + \int \frac{1}{\rho}\nabla\rho\cdot\bigl(\mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\bigr)\,dx - \int \nabla\rho\cdot\mathbf{u}\cdot\nabla\mathbf{u}\,dx \\
&\qquad + \int \frac{1}{\rho}\nabla\rho\cdot F_1\,dx + \frac{\hbar^2}{4}\int\left(\frac{1}{\rho^2}\right)\nabla\rho\cdot\nabla\rho\cdot\nabla^2\rho\,dx \\
&\quad = -\frac{d}{dt}\int \mathbf{u}\cdot\nabla\rho\,dx + \sum_{i=1}^{5} J_i.
\end{aligned} \tag{2.9}$$

In view of (2.4), (2.6), and (2.7), and using Hölder, Young, and Sobolev inequalities, we obtain

$$\begin{aligned}
J_1 + J_2 + J_3 &\leq C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^6}\|\mathbf{u}\|_{L^3} + C\|\nabla\mathbf{u}\|_{L^2}^2 + C\left\|\nabla^2\rho\right\|_{L^2}\|\nabla\mathbf{u}\|_{L^2} \\
&\leq \frac{\hbar^2}{16}\left\|\nabla^2\rho\right\|_{L^2}^2 + C(M)\|\nabla\mathbf{u}\|_{L^2}^2, \\
J_4 + J_5 &\leq C\bigl(\|\nabla\rho\|_{L^4}\|\nabla\rho\|_{L^4}\left\|\nabla^2\rho\right\|_{L^2} + \|\nabla\rho\|_{L^4}^4 + \|\nabla\rho\|_{L^6}\|\mathbf{B}\|_{L^3}\|\nabla\mathbf{B}\|_{L^2}\bigr) \\
&\leq C(M)E_0^{\frac{1}{4}}\bigl(\left\|\nabla^2\rho\right\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2\bigr).
\end{aligned}$$

Substituting the estimates of J_i $(i = 1,\dots,5)$ into (2.9), for $E_0 \leq (\frac{\hbar^2}{32C(M)})^4$, it leads to

$$\|\nabla\rho\|_{L^2}^2 + \left\|\nabla^2\rho\right\|_{L^2}^2 \leq -C\frac{d}{dt}\int \mathbf{u}\cdot\nabla\rho\,dx + C(M)\bigl(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2\bigr). \tag{2.10}$$

Integrating (2.10) from 0 to T, combining with (2.6) and (2.7), we obtain the desired result (2.8). The proof of this lemma is complete. □

Lemma 2.5 *Under the assumptions of Proposition 2.1, it holds that*

$$\int_0^T \|\nabla^3\rho\|_{L^2}^2\,dt \le C(M)\Big(1+\int_0^T \|\nabla^2\mathbf{u}\|_{L^2}^2\,dt\Big)+C\int_0^T \|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2}^2\,dt, \tag{2.11}$$

provided $E_0 \le \Big(\frac{\hbar^2\underline{\rho}^{\frac{9}{2}}}{16(\underline{\rho}^2+\hbar^{\frac{3}{2}}(\underline{\rho}\bar{\rho})^{\frac{1}{4}})C(M)}\Big)^4$.

Proof Multiplying $(2.2)_2$ by $-\frac{1}{\rho}\Delta\nabla\rho$, integrating by parts over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{\hbar^2}{4}\int \frac{1}{\rho}|\Delta\nabla\rho|^2\,dx\\
&\quad=\frac{d}{dt}\int \Delta\nabla\rho\cdot\mathbf{u}\,dx+\int \Delta\nabla\operatorname{div}(\rho\mathbf{u})\cdot\mathbf{u}\,dx+\int \Delta\nabla\rho\cdot\big(\gamma\rho^{\gamma-2}\nabla\rho\big)\,dx\\
&\qquad+\int \Delta\nabla\rho\cdot\Big(\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{\rho}\big(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big)\Big)\,dx-\int \frac{1}{\rho}\Delta\nabla\rho\cdot F_1\,dx\\
&\quad=\frac{d}{dt}\int \Delta\nabla\rho\cdot\mathbf{u}\,dx+\sum_{i=1}^4 L_i.
\end{aligned}\tag{2.12}$$

Utilizing (2.4), (2.6), (2.7), Hölder, Sobolev, and Young inequalities, then integrating by parts, we have

$$\begin{aligned}
L_1&=\int \Delta\nabla\operatorname{div}(\rho\mathbf{u})\cdot\mathbf{u}\,dx=\int \nabla\operatorname{div}(\rho\mathbf{u})\cdot\Delta\mathbf{u}\,dx\\
&=\int \nabla^2\rho\cdot\mathbf{u}\Delta\mathbf{u}\,dx+2\int \nabla\rho\cdot\nabla\mathbf{u}\Delta\mathbf{u}\,dx+\int \rho\big|\nabla^2\mathbf{u}\big|^2\,dx\\
&\le C\|\rho\|_{L^\infty}\big\|\nabla^2\mathbf{u}\big\|_{L^2}^2+C\|\nabla\rho\|_{L^3}\|\nabla\mathbf{u}\|_{L^6}\big\|\nabla^2\mathbf{u}\big\|_{L^2}+C\|\mathbf{u}\|_{L^\infty}\big\|\nabla^2\rho\big\|_{L^2}\big\|\nabla^2\mathbf{u}\big\|_{L^2}\\
&\le C\big\|\nabla^2\mathbf{u}\big\|_{L^2}^2+C\|\nabla\rho\|_{L^2}^{\frac{1}{2}}\big\|\nabla^2\rho\big\|_{L^2}^{\frac{1}{2}}\big\|\nabla^2\mathbf{u}\big\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\big\|\nabla^2\mathbf{u}\big\|_{L^2}^{\frac{3}{2}}\big\|\nabla^2\rho\big\|_{L^2}^{\frac{1}{2}}\big\|\nabla^2\rho\big\|_{L^2}^{\frac{1}{2}}\\
&\le C(M)\big(\big\|\nabla^2\rho\big\|_{L^2}^2+\big\|\nabla^2\mathbf{u}\big\|_{L^2}^2\big),\\
L_2&=\int \Delta\nabla\rho\cdot\big(\gamma\rho^{\gamma-2}\nabla\rho\big)\,dx\\
&\le\gamma\big(\bar{\rho}^{\gamma-2}+\underline{\rho}^{\gamma-2}\big)\|\Delta\nabla\rho\|_{L^2}\|\nabla\rho\|_{L^2}\\
&\le\frac{\varepsilon}{2}\|\Delta\nabla\rho\|_{L^2}^2+\gamma^2\big(\bar{\rho}^{\gamma-2}+\underline{\rho}^{\gamma-2}\big)^2\frac{1}{2\varepsilon}\|\nabla\rho\|_{L^2}^2,\\
L_3&=\int \Delta\nabla\rho\cdot\Big(\mathbf{u}\cdot\nabla\mathbf{u}-\frac{1}{\rho}\big(\mu\Delta\mathbf{u}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big)\Big)\,dx\\
&\le\|\Delta\nabla\rho\|_{L^2}\|\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}\|_{L^6}+\frac{(2\mu+\lambda)}{\bar{\rho}}\|\Delta\nabla\rho\|_{L^2}\big(\|\Delta\mathbf{u}\|_{L^2}+\|\nabla\operatorname{div}\mathbf{u}\|_{L^2}\big)\\
&\le C\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\big\|\nabla^2\mathbf{u}\big\|_{L^2}\|\Delta\nabla\rho\|_{L^2}+\frac{\varepsilon}{2}\|\Delta\nabla\rho\|_{L^2}^2+\frac{(2\mu+\lambda)^2}{\bar{\rho}^2 2\varepsilon}\big\|\nabla^2\mathbf{u}\big\|_{L^2}^2\\
&\le\frac{C(M)}{\underline{\rho}^{\frac{1}{4}}}E_0^{\frac{1}{4}}\big(\|\Delta\nabla\rho\|_{L^2}^2+\big\|\nabla^2\mathbf{u}\big\|_{L^2}^2\big)+\frac{\varepsilon}{2}\|\Delta\nabla\rho\|_{L^2}^2+\frac{(2\mu+\lambda)^2}{\bar{\rho}^2 2\varepsilon}\big\|\nabla^2\mathbf{u}\big\|_{L^2}^2,
\end{aligned}$$

and

$$
\begin{aligned}
L_4 &= -\int \frac{1}{\rho}\Delta\nabla\rho\cdot F_1\,dx \\
&= -\int \frac{1}{\rho}\Delta\nabla\rho\cdot\left(\frac{\hbar^2}{4}\left(\frac{|\nabla\rho|^2\nabla\rho}{\rho^2}+\frac{\nabla\rho\Delta\rho}{\rho}-\frac{\nabla\rho\nabla^2\rho}{\rho}\right)+(\mathrm{curl}\mathbf{B})\times\mathbf{B}\right)dx \\
&\le C\frac{\hbar^2}{\underline{\rho}^3}\|\Delta\nabla\rho\|_{L^2}\|\nabla\rho\|_{L^6}^3 + C\frac{\hbar^2}{\underline{\rho}^2}\|\Delta\nabla\rho\|_{L^2}^2\|\nabla\rho\|_{L^3}+\frac{1}{\underline{\rho}}\|\Delta\nabla\rho\|_{L^2}\|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2} \\
&\le C\frac{\hbar^2}{\underline{\rho}^3}\|\Delta\nabla\rho\|_{L^2}\left\|\nabla^2\rho\right\|_{L^2}^3 + C\frac{\hbar^2}{\underline{\rho}^2}\|\Delta\nabla\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^{\frac{1}{2}}\left\|\nabla^2\rho\right\|_{L^2}^{\frac{1}{2}} \\
&\quad+\frac{1}{\underline{\rho}}\|\Delta\nabla\rho\|_{L^2}\|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2} \\
&\le C(M)\frac{\hbar^2}{\underline{\rho}^3}\|\Delta\nabla\rho\|_{L^2}\left\|\nabla^2\rho\right\|_{L^2} + C(M)\frac{\hbar^{\frac{3}{2}}\bar{\rho}^{\frac{1}{4}}}{\underline{\rho}^2}E_0^{\frac{1}{4}}\|\Delta\nabla\rho\|_{L^2}^2+\frac{1}{\underline{\rho}}\|\Delta\nabla\rho\|_{L^2}\|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2} \\
&\le \frac{\varepsilon}{2}\|\Delta\nabla\rho\|_{L^2}^2 + C(M)\frac{\hbar^4}{\underline{\rho}^6}\left\|\nabla^2\rho\right\|_{L^2}^2 + C(M)\frac{\hbar^{\frac{3}{2}}\bar{\rho}^{\frac{1}{4}}}{\underline{\rho}^2}E_0^{\frac{1}{4}}\|\Delta\nabla\rho\|_{L^2}^2+\frac{\varepsilon}{2}\|\Delta\nabla\rho\|_{L^2}^2 \\
&\quad+\frac{1}{\underline{\rho}^2 2\varepsilon}\|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2}^2.
\end{aligned}
$$

Therefore, it can be obtained that

$$
\begin{aligned}
&L_2+L_3+L_4 \\
&\le\left(2\varepsilon+\left(\frac{1}{\underline{\rho}^{\frac{1}{4}}}+\frac{\hbar^{\frac{3}{2}}\bar{\rho}^{\frac{1}{4}}}{\underline{\rho}^2}\right)C(M)E_0^{\frac{1}{4}}\right)\|\Delta\nabla\rho\|_{L^2}^2+\gamma^2\left(\bar{\rho}^{\gamma-2}+\underline{\rho}^{\gamma-2}\right)^2\frac{1}{2\varepsilon}\|\nabla\rho\|_{L^2}^2 \\
&\quad+C(M)\frac{\hbar^4}{\underline{\rho}^6}\left\|\nabla^2\rho\right\|_{L^2}^2+\left(\frac{C(M)}{\underline{\rho}^{\frac{1}{4}}}E_0^{\frac{1}{4}}+\frac{(2\mu+\lambda)^2}{\bar{\rho}^2 2\varepsilon}\right)\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2+\frac{1}{\underline{\rho}^2 2\varepsilon}\|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2}^2 \\
&\le\frac{\hbar^2}{8}\|\Delta\nabla\rho\|_{L^2}^2+C(M)\left(\|\nabla\rho\|_{H^1}^2+\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2\right)+C\|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2}^2
\end{aligned}
$$

provided

$$
\varepsilon=\frac{\hbar^2}{32}\quad\text{and}\quad E_0\le\left(\frac{\hbar^2\underline{\rho}^{\frac{9}{2}}}{16(\underline{\rho}^2+\hbar^{\frac{3}{2}}(\underline{\rho}\bar{\rho})^{\frac{1}{4}})C(M)}\right)^4.
$$

Substituting the estimates of L_i ($i=1,\dots,4$) into (2.12), we obtain

$$
\begin{aligned}
&\frac{\hbar^2}{8}\int|\Delta\nabla\rho|^2\,dx \\
&\le\frac{d}{dt}\int\Delta\nabla\rho\cdot\mathbf{u}\,dx+C(M)\left(\|\nabla\rho\|_{H^1}^2+\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2\right)+C\|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2}^2.
\end{aligned}
\tag{2.13}
$$

Integrating the above inequality over $(0,T)$, by (2.4), (2.6), and (2.8), we get (2.11). □

Lemma 2.6 *Under the assumptions of Proposition 2.1, it holds that*

$$\frac{d}{dt}\|\mathbf{B}\|_{L^4}^4 + \nu\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}^2 \le 0, \tag{2.14}$$

provided $E_0 \le (\frac{2\nu}{C(M)})^2$.

Proof Multiplying $(2.2)_3$ by $|\mathbf{B}|^2\mathbf{B}$, integrating the resulting equation over $\mathbb{R}^3$, combining with (2.4), we have

$$\frac{1}{4}\frac{d}{dt}\|\mathbf{B}\|_{L^4}^4 + 3\nu\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}^2 \le C\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}\left\||\mathbf{B}|^2\right\|_{L^6}\|\mathbf{u}\|_{L^3} \le C(M)E_0^{\frac{1}{2}}\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}^2.$$

Therefore, letting $E_0 \le (\frac{2\nu}{C(M)})^2$, we obtain (2.14). □

Proof of Proposition 2.1 Multiplying $(2.2)_2$ by $-\frac{1}{\rho}\Delta(\rho\mathbf{u})$, using $(2.2)_1$ and integrating it over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \frac{\hbar^2}{4}\left\|\nabla^2\rho\right\|_{L^2}^2\right) + \int\frac{1}{\rho}\Delta\mathbf{u}\cdot\big(\mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big)\,dx \\
&\quad= \int\mathbf{u}_t\cdot\big(\Delta(\rho\mathbf{u}) - \Delta\mathbf{u}\big)\,dx \\
&\qquad- \int\frac{1}{\rho}\big(\mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\operatorname{div}\mathbf{u}\big)\cdot(\Delta\rho\mathbf{u} + 2\nabla\rho\cdot\nabla\mathbf{u})\,dx \\
&\qquad+ \frac{\hbar^2}{4}\int\left(1 - \frac{1}{\rho}\right)\nabla\Delta\rho\cdot\Delta(\rho\mathbf{u})\,dx + \int(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta(\rho\mathbf{u})\,dx \\
&\qquad+ \int\frac{1}{\rho}\nabla P\Delta(\rho\mathbf{u})\,dx - \int\frac{1}{\rho}F_1\cdot\Delta(\rho\mathbf{u})\,dx \\
&\quad= \sum_{i=1}^{6}K_i.
\end{aligned} \tag{2.15}$$

Taking advantage of (2.4) and (2.6), we give the estimates about $\|\mathbf{u}_t\|_{L^2}$ and $\|\Delta(\rho\mathbf{u})\|_{L^2}$ as follows:

$$\|\mathbf{u}_t\|_{L^2} \le C\left(\left\|\nabla^2\mathbf{u}\right\|_{L^2} + \left\|\nabla^3\rho\right\|_{L^2} + \|\nabla\rho\|_{L^2} + \|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}\right), \tag{2.16}$$

$$\begin{aligned}
\left\|\Delta(\rho\mathbf{u})\right\|_{L^2} &\le C\left(\left\|\nabla^2\mathbf{u}\right\|_{L^2} + \|\nabla\rho\|_{L^3}\|\nabla\mathbf{u}\|_{L^6} + \|\mathbf{u}\|_{L^\infty}\left\|\nabla^2\rho\right\|_{L^2}\right) \\
&\le C\left\|\nabla^2\mathbf{u}\right\|_{L^2} + C(M)\left\|\nabla^2\rho\right\|_{L^2}.
\end{aligned} \tag{2.17}$$

Recalling (2.4), (2.7), (2.15), and (2.16) and utilizing Hölder, Young, and Sobolev inequalities, we obtain

$$\begin{aligned}
K_1 &\le C\|\mathbf{u}_t\|_{L^2}\left(\|\rho-1\|_{L^\infty}\left\|\nabla^2\mathbf{u}\right\|_{L^2} + \|\nabla\rho\|_{L^3}\|\nabla\mathbf{u}\|_{L^6} + \|\mathbf{u}\|_{L^3}\left\|\nabla^2\rho\right\|_{L^6}\right) \\
&\le C(M)E_0^{\frac{1}{2}}\|\mathbf{u}_t\|_{L^2}\left(\left\|\nabla^2\mathbf{u}\right\|_{L^2} + \left\|\nabla^3\rho\right\|_{L^2}\right) \\
&\le C(M)E_0^{\frac{1}{4}}\left(\left\|\nabla^2\mathbf{u}\right\|_{L^2}^2 + \left\|\nabla^3\rho\right\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}^2\right),
\end{aligned}$$

$$
\begin{aligned}
K_2 &\le C\|\nabla^2\mathbf{u}\|_{L^2}\big(\|\nabla\mathbf{u}\|_{L^6}\|\nabla\rho\|_{L^3}+\|\mathbf{u}\|_{L^\infty}\|\nabla^2\rho\|_{L^2}\big)\\
&\le\left(\frac{\mu}{8}+C(M)E_0^{\frac14}\right)\|\nabla^2\mathbf{u}\|_{L^2}^2+C(M)\|\nabla^2\rho\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
&K_3+K_4+K_5+K_6\\
&\quad\le C\big(\|\rho-1\|_{L^\infty}\|\nabla^3\rho\|_{L^2}+\|\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}\|_{L^6}\big)\big(\|\nabla^2\mathbf{u}\|_{L^2}+C(M)\|\nabla^2\rho\|_{L^2}\big)\\
&\qquad+C\big(\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}+\|\nabla\rho\|_{L^6}^3+\|\nabla\rho\|_{L^3}\|\nabla^2\rho\|_{L^6}\big)\big(\|\nabla^2\mathbf{u}\|_{L^2}+C(M)\|\nabla^2\rho\|_{L^2}\big)\\
&\quad\le C\big(\|\nabla\rho\|_{L^2}^{\frac12}\|\nabla^2\rho\|_{L^2}^{\frac12}\|\nabla^3\rho\|_{L^2}+\|\mathbf{u}\|_{L^2}^{\frac12}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{u}\|_{L^2}\big)\\
&\qquad\times\big(\|\nabla^2\mathbf{u}\|_{L^2}+C(M)\|\nabla^2\rho\|_{L^2}\big)\\
&\qquad+C\big(\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}+\|\nabla^2\rho\|_{L^2}^3\big)\big(\|\nabla^2\mathbf{u}\|_{L^2}+C(M)\|\nabla^2\rho\|_{L^2}\big)\\
&\quad\le\left(\frac{\mu}{8}+C(M)E_0^{\frac12}\right)\|\nabla^2\mathbf{u}\|_{L^2}^2+C\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}^2+C(M)E_0^{-\frac12}\|\nabla\rho\|_{H^1}^2.
\end{aligned}
$$

Substituting the estimates of K_i $(i=1,\dots,6)$ into (2.15), by (2.6) and (2.8), one has

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla^2\rho\|_{L^2}^2\big)+2C_1\|\nabla^2\mathbf{u}\|_{L^2}^2\\
&\quad\le C_2\|\mathbf{B}\cdot\nabla\mathbf{B}\|_{L^2}^2+C(M)E_0^{\frac12}\|\nabla^3\rho\|_{L^2}^2+C(M)E_0^{-\frac12}\|\nabla\rho\|_{H^1}^2.
\end{aligned}
\tag{2.18}
$$

Multiplying (2.14) by $\frac{2C_2}{\nu}$, then substituting the resulting equation and (2.18) into (2.11), combining with (2.8), we obtain

$$
\begin{aligned}
&\|\mathbf{B}\|_{L^4}^4+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla^2\rho\|_{L^2}^2+C_1\int_0^T\|\nabla^2\mathbf{u}\|_{L^2}^2\,dt\\
&\quad\le C(M)E_0^{\frac12}+C_3\|\mathbf{B}_0\|_{L^4}^4+\|\nabla\mathbf{u}_0\|_{L^2}^2+\|\nabla^2\rho_0\|_{L^2}^2\\
&\quad\le\frac32(C_3+1)M_1.
\end{aligned}
\tag{2.19}
$$

Thus we complete the proof of Proposition 2.1. □

Lemma 2.7 *Under the assumptions of Theorem 1.1, it holds that*

$$
\|\nabla\mathbf{B}\|_{L^2}^2+\int_0^T\big(\|\mathbf{B}_t\|_{L^2}^2+\|\nabla^2\mathbf{B}\|_{L^2}^2\big)\,dt\le C(M)E_0+C\|\nabla\mathbf{B}_0\|_{L^2}^2,
\tag{2.20}
$$

$$
\|\nabla^2\mathbf{B}\|_{L^2}^2+\int_0^T\big(\|\nabla\mathbf{B}_t\|_{L^2}^2+\|\nabla^3\mathbf{B}\|_{L^2}^2\big)\,dt\le C(M)\big(1+\|\nabla\mathbf{B}_0\|_{H^1}^2\big).
\tag{2.21}
$$

Proof Squaring both sides of (2.2)$_3$, then integrating the resulting equation over $\mathbb{R}^3$, by (2.4) and (2.6), we have

$$\begin{aligned} &\nu\frac{d}{dt}\|\nabla\mathbf{B}\|_{L^2}^2+\|\mathbf{B}_t\|_{L^2}^2+\nu^2\|\nabla^2\mathbf{B}\|_{L^2}^2\\ &\quad\le C\big(\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{B}\|_{L^\infty}^2+\|\nabla\mathbf{B}\|_{L^6}^2\|\mathbf{u}\|_{L^6}\|\mathbf{u}\|_{L^2}\big)\\ &\quad\le C\big(\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{B}\|_{L^2}\|\nabla^2\mathbf{B}\|_{L^2}+\|\nabla^2\mathbf{B}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^2}\big)\\ &\quad\le\left(C(M)E_0^{\frac12}+\frac{\nu^2}{4}\right)\|\nabla^2\mathbf{B}\|_{L^2}^2+C(M)\|\nabla\mathbf{B}\|_{L^2}^2. \end{aligned}\tag{2.22}$$

Integrating inequality (2.22) from 0 to T, we obtain (2.20). Similarly, combining with (2.4), (2.6), and (2.20), we obtain (2.21). Thus, we complete the proof of this lemma. □

At last, we prove the high-order estimates depending on time t.

Lemma 2.8 *Under the assumptions of Theorem 1.1, it holds that*

$$\|\nabla^3\rho\|_{H^1}^2+\|\nabla^2\mathbf{u}\|_{H^1}^2+\int_0^T\big(\|\nabla^4\rho\|_{H^1}^2+\|\nabla^3\mathbf{u}\|_{H^1}^2\big)\,dt\le C_T,\tag{2.23}$$

$$\|\nabla^3\mathbf{B}\|_{L^2}^2+\int_0^T\|\nabla^4\mathbf{B}\|_{L^2}^2\,dt\le C_T.\tag{2.24}$$

Proof Similar to the proofs of Lemma 3.10, Lemma 3.11, and Lemma 4.3 in [10], we can obtain (2.23). In fact, following Lemma 3.10 in [10], multiplying (2.2)$_2$ by $-\frac{\Delta^2(\rho u)}{\rho}$, integrating by part over $\mathbb{R}^3\times(0,T)$, we have

$$\begin{aligned} &\sup_{0\le t\le T}\big(\|\nabla^3\rho\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2\big)+\int_0^T\|\nabla^3u\|_{L^2}^2\,dt\\ &\quad\le C\delta\int_0^T\|\nabla^4\rho\|_{L^2}^2\,dt+C(M)-\int_0^T\int(\operatorname{curl}\mathbf{B})\times\mathbf{B}\cdot\frac{\Delta^2(\rho u)}{\rho}\,dx\,dt\\ &\quad=C\delta\int_0^T\|\nabla^4\rho\|_{L^2}^2\,dt+C(M)+I_1, \end{aligned}\tag{2.25}$$

where δ is small enough and determined later. Similar to Lemma 3.11 and Lemma 4.3 in [10], multiplying $-\frac{\nabla\Delta^2\rho}{\rho}$, integrating by part over $\mathbb{R}^3\times(0,T)$, we get

$$\begin{aligned} &\int_0^T\|\nabla^4\rho\|_{L^2}^2\,dt\\ &\quad\le C\sup_{0\le t\le T}\big(\|\nabla^3\rho\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2\big)+C\int_0^T\|\nabla^3u\|_{L^2}^2\,dt+C\\ &\qquad-\int_0^T\int(\operatorname{curl}\mathbf{B})\times\mathbf{B}\cdot\frac{\nabla\Delta^2\rho}{\rho}\,dx\,dt\\ &\quad=C\sup_{0\le t\le T}\big(\|\nabla^3\rho\|_{L^2}^2+\|\nabla^2u\|_{L^2}^2\big)+C\int_0^T\|\nabla^3u\|_{L^2}^2\,dt+C+I_2. \end{aligned}\tag{2.26}$$

Substituting (2.26) into (2.25) and choosing $\delta > 0$ sufficiently small gives

$$\sup_{0\le t\le T}\left(\left\|\nabla^3\rho\right\|_{L^2}^2+\left\|\nabla^2 u\right\|_{L^2}^2\right)+\int_0^T\left\|\nabla^3 u\right\|_{L^2}^2\,dt\le C(M)+I_1+C+CI_2. \tag{2.27}$$

According to integrating by parts and the Cauchy inequality, we obtain

$$\begin{aligned}
I_1&=-\int_0^T\int(\operatorname{curl}\mathbf{B})\times\mathbf{B}\cdot\frac{\Delta^2(\rho u)}{\rho}\,dx\,dt\\
&=-\int_0^T\int\Delta\left(\frac{(\operatorname{curl}\mathbf{B})\times\mathbf{B}}{\rho}\right)\cdot\Delta(\rho u)\,dx\,dt\\
&\le C\int_0^T\int\left|\nabla^2\left(\frac{(\operatorname{curl}\mathbf{B})\times\mathbf{B}}{\rho}\right)\right|^2dx\,dt+\int_0^T\int\left|\nabla^2(\rho u)\right|^2dx\,dt.
\end{aligned}$$

Utilizing (2.19), (2.20), (2.21), Hölder, Young, and Sobolev inequalities, we get

$$\begin{aligned}
&\int\left|\nabla^2\left(\frac{(\operatorname{curl}\mathbf{B})\times\mathbf{B}}{\rho}\right)\right|^2dx\\
&\quad\le C\|\nabla\rho\|_{L^6}^4\|\nabla\mathbf{B}\|_{L^6}^2\|\mathbf{B}\|_{L^\infty}^2+C\left\|\nabla^2\rho\right\|_{L^6}^2\|\nabla\mathbf{B}\|_{L^6}^2\|\mathbf{B}\|_{L^6}^2+C\|\nabla\rho\|_{L^6}^2\|\nabla\mathbf{B}\|_{L^6}^4\\
&\qquad+C\left\|\nabla^3\mathbf{B}\right\|_{L^2}^2\|\mathbf{B}\|_{L^\infty}^2+C\left\|\nabla^2\mathbf{B}\right\|_{L^6}^2\|\nabla\mathbf{B}\|_{L^6}\|\nabla\mathbf{B}\|_{L^2}\\
&\quad\le C\left\|\nabla^2\rho\right\|_{L^2}^4\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\|\nabla\mathbf{B}\|_{H^1}^2+C\left\|\nabla^3\rho\right\|_{L^2}^2\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2\|\nabla\mathbf{B}\|_{L^2}^2+C\left\|\nabla^2\rho\right\|_{L^2}^2\left\|\nabla^2\mathbf{B}\right\|_{L^2}^4\\
&\qquad+C\left\|\nabla^3\mathbf{B}\right\|_{L^2}^2\|\nabla\mathbf{B}\|_{H^1}^2+C\left\|\nabla^3\mathbf{B}\right\|_{L^2}^2\left\|\nabla^2\mathbf{B}\right\|_{L^2}\|\nabla\mathbf{B}\|_{L^2}\\
&\quad\le C(M)\left(\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2+\left\|\nabla^3\rho\right\|_{L^2}^2+\left\|\nabla^3\mathbf{B}\right\|_{L^2}^2\right).
\end{aligned}$$

Thus, by (2.11), (2.14), (2.19), and (2.20), it is obtained that

$$\begin{aligned}
&\int_0^T\int\left|\nabla^2\left(\frac{(\operatorname{curl}\mathbf{B})\times\mathbf{B}}{\rho}\right)\right|^2dx\,dt\\
&\quad\le C(M)\left(\int_0^T\left\|\nabla^2\mathbf{B}\right\|_{L^2}^2dt+\int_0^T\left\|\nabla^3\rho\right\|_{L^2}^2dt+\int_0^T\left\|\nabla^3\mathbf{B}\right\|_{L^2}^2dt\right)\\
&\quad\le C(M)\left(C(M)+C\|\nabla\mathbf{B}_0\|_{L^2}^2+\int_0^T\left\|\nabla^2 u\right\|_{L^2}^2dt\right.\\
&\qquad\left.+\int_0^T\|\nabla\mathbf{B}\cdot\mathbf{B}\|_{L^2}^2dt+\|\nabla\mathbf{B}_0\|_{H^1}^2\right)\\
&\quad\le C(M)\left(C(M)+\|\mathbf{B}_0\|_{L^4}^4+\|\nabla\mathbf{B}_0\|_{H^1}^2\right)\\
&\quad\le C(M).
\end{aligned}\tag{2.28}$$

In addition, using (2.4) and (2.19), we have

$$\begin{aligned}
&\int\left|\nabla^2(\rho u)\right|^2dx\\
&\quad\le C\int\left|\nabla^2\rho\right|^2u^2\,dx+C\int|\nabla\rho|^2|\nabla u|^2\,dx+C\int\rho^2\left|\nabla^2 u\right|dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|\nabla^2\rho\|_{L^2}\|\nabla^2\rho\|_{L^6}\|u\|_{L^6}^2 + C\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\|\nabla\rho\|_{L^6}^2 + C\|\nabla^2 u\|_{L^2}^2 \\
&\leq C\|\nabla^2\rho\|_{L^2}\|\nabla^3\rho\|_{L^2}\|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla^2\rho\|_{L^2}\|\nabla^2\rho\|_{L^2} + C\|\nabla^2 u\|_{L^2}^2 \\
&\leq C(M)\big(\|\nabla^2\rho\|_{L^2}^2 + \|\nabla^3\rho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\big).
\end{aligned}
$$

Hence, using (2.8), (2.20), (2.14), and (2.19), it leads to

$$
\begin{aligned}
\int_0^T\int|\nabla^2(\rho u)|^2\,dx\,dt &\leq C(M)\int_0^T\big(\|\nabla^2\rho\|_{L^2}^2 + \|\nabla^3\rho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\big)\,dt \\
&\leq C(M)\big(C(M) + E_0 + \|\mathbf{B}_0\|_{L^4}^4\big) \\
&\leq C(M).
\end{aligned}
\tag{2.29}
$$

Therefore,

$$
I_1 \leq C(M). \tag{2.30}
$$

On the other hand, for I_2, integrating by parts, the Cauchy inequality, (2.14), (2.20), (2.26), and (2.28), one has

$$
\begin{aligned}
I_2 &= -\int_0^T\int(\operatorname{curl}\mathbf{B})\times\mathbf{B}\cdot\frac{\nabla\Delta^2\rho}{\rho}\,dx\,dt \\
&= -\int_0^T\int\Delta\left(\frac{(\operatorname{curl}\mathbf{B})\times\mathbf{B}}{\rho}\right)\cdot\nabla\Delta\rho\,dx\,dt \\
&\leq C\int_0^T\int\left|\nabla^2\left(\frac{(\operatorname{curl}\mathbf{B})\times\mathbf{B}}{\rho}\right)\right|^2 dx\,dt + C\int_0^T\int|\nabla^3\rho|^2\,dx\,dt \\
&\leq C(M).
\end{aligned}
\tag{2.31}
$$

Substituting (2.30) and (2.31) into (2.27), we obtain

$$
\sup_{0\leq t\leq T}\big(\|\nabla^3\rho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2\big) + \int_0^T\|\nabla^3 u\|_{L^2}^2\,dt \leq C(M).
$$

Together with (2.26), it yields

$$
\int_0^T\|\nabla^4\rho\|_{L^2}^2\,dt \leq C(M).
$$

Therefore, we get the desired estimate (2.23).

Using (2.4), (2.6), (2.20), (2.21), and (2.23), similar to the proof of Lemma 2.7, one can obtain (2.24). □

3 Proof of Theorem 1.1

Lemma 3.1 *Under the assumptions of Proposition 2.1, it holds that*

$$
\int_0^T\left|\frac{d}{dt}\big(\|\nabla\rho\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{B}\|_{L^2}^2\big)\right|dt \leq C(M). \tag{3.1}
$$

Proof Combining with (2.20) and (2.21), we get

$$\int_0^T \left| \frac{d}{dt} \|\nabla \mathbf{B}\|_{L^2}^2 \right| dt \le C(M).$$

In view of $(2.2)_1$, (2.5), and using Hölder, Young, and Sobolev inequalities, one has

$$\begin{aligned}
\|\nabla \rho_t\|_{L^2}^2 &\le C \int \left|\nabla^2 \rho\right|^2 |\mathbf{u}|^2 + |\nabla \rho|^2 |\nabla \mathbf{u}|^2 + |\rho|^2 \left|\nabla^2 \mathbf{u}\right|^2 dx \\
&\le C\big(\|\mathbf{u}\|_{L^\infty}^2 \left\|\nabla^2 \rho\right\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^3} \|\nabla \mathbf{u}\|_{L^6} \|\nabla \rho\|_{L^6}^2 + \|\rho\|_{L^\infty}^2 \left\|\nabla^2 \mathbf{u}\right\|_{L^2}^2\big) \\
&\le C(M)\big(\left\|\nabla^2 \rho\right\|_{L^2}^2 + \left\|\nabla^2 \mathbf{u}\right\|_{L^2}^2\big).
\end{aligned}$$

Noticing (2.8) and the above inequality leads to

$$\int_0^T \left| \frac{d}{dt} \|\nabla \rho\|_{L^2}^2 \right| dt \le C(M).$$

Multiplying $(2.2)_2$ by $\nabla^2 \mathbf{u}$, integrating by parts, and using Hölder, Sobolev, and Young inequalities, we get

$$\left| \frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 \right| \le C \left\|\nabla^2 \mathbf{u}\right\|_{L^2}^2 + C \|\nabla \rho\|_{H^2}^2 + \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^2}^2.$$

Integrating the above inequality over 0 to T, it follows from (2.6), (2.16), (2.13), and (2.14) that

$$\int_0^T \left| \frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 \right| dt \le C(M).$$

Thus, we complete the proof of this lemma. □

Finally, we prove Theorem 1.1.

Proof of Theorem 1.1 Applying all *a priori* estimates above, we arrive at

$$\|\rho - 1\|_{H^4}^2 + \|\mathbf{u}\|_{H^3}^2 + \|\mathbf{B}\|_{H^3}^2 + \int_0^T \|\nabla \rho\|_{H^4}^2 + \|\nabla \mathbf{u}\|_{H^3}^2 + \|\nabla \mathbf{B}\|_{H^3}^2 \, dt \le C_T. \tag{3.2}$$

Now, let $[0, T^*)$ be the maximal existence interval of the solution to system (1.1)–(1.2). Based on the local existence result (Lemma 2.1) and (3.2) and using the standard continuity argument, we have that $T^* = +\infty$. Hence, the global existence of the solution is obtained.

Next, we investigate the large time behavior for solution. Following (2.6) and (2.8), we have

$$\int_0^\infty \big(\|\nabla \rho\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\big) dt \le C(M). \tag{3.3}$$

Together with (3.1) and (3.3), it gives

$$\lim_{t \to \infty} \big(\|\nabla \rho\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2\big) = 0. \tag{3.4}$$

Combining this with

$$\|\rho - 1\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2 + \|\mathbf{B}\|_{H^1}^2 \le C(M)$$

and using the Sobolev inequality, we prove (1.5). Thus, we complete the proof of Theorem 1.1. □

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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