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Numerical resolution of the wave equation using the spectral method

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Abstract

We present a new procedure for the numerical study of the wave equation. We use the spectral discretization method associated with the Euler scheme for spatial and temporal discretization. A detailed numerical analysis leads to an a priori error estimate. We confirm the high precision of the method presented by a numerical study.

Keywords: Wave equation; Spectral method; Backward Euler scheme; A priori error estimate; Implementation

1 Introduction

During the last decade, many works have focused on the numerical analysis of second-order parabolic and hyperbolic partial differential equations such as the heat and wave equations [1–8].

The time and space approximation for the wave equation has been studied in [9] using the finite element method. We present a new study of this problem using the spectral method associated with backward Euler discretization scheme. The spectral method is known for its high precision [10, 11]. For the space discretization, the discrete spaces are constructed from spaces of polynomials of high degree. Then the discrete problem is obtained using the Galerkin method combined with numerical integration.

The outline of the paper is as follows. In Sect. 2, we study the continuous problem and present some energy estimate properties. In Sect. 3, we are interested to the time-semidiscrete problem. We discretize the second time derivative by using a second difference quotient of the solution on a nonuniform temporal grid. Then we obtain an optimal a priori error estimate. In Sect. 4, we study the fully discrete problem and establish an optimal a priori error estimate. Finally, in Sect. 5, we present a numerical study.

2 The continuous problem

Consider an open bounded connected domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with Lipschitz continuous boundary Γ , and let T be a positive real number. Let $H^s(\Omega)$, $s > 0$, be the Sobolev spaces associated with the norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. The space $H_0^1(\Omega)$ is the clo-

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sure in the space $H^1(\Omega)$ of infinitely differentiable functions with compact support in Ω , and $H^{-1}(\Omega)$ is its dual space. We denote by (\cdot, \cdot) and $\|\cdot\|$, respectively, the scalar product and associated norm in $L^2(\Omega)$. By $H^{\frac{1}{2}}(\partial\Omega)$ we denote the trace space of functions in $H^1(\Omega)$. Let $\gamma \subset \partial\Omega$, $H_{00}^{\frac{1}{2}}(\gamma)$ be the space of functions in $H^{\frac{1}{2}}(\gamma)$ such that their extension by zero to $\partial\Omega/\gamma$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$.

In the following, we define

- $u(\mathbf{x}, t)$ on $\Omega \times]0, T[$ as

$$\begin{aligned} u :]0, T[&\longrightarrow X \\ t &\longmapsto u(t) = u(\cdot, t), \end{aligned}$$

where X is a separable Banach space.

- $\mathcal{C}^j(0, T; X)$ represents the set of functions of time of class \mathcal{C}^j with values in X . It is a Banach space with norm

$$\|u\|_{\mathcal{C}^j(0, T; X)} = \sup_{0 \leq t \leq T} \sum_{l=0}^j \|\partial_t^l u\|_X,$$

where $\partial_t^l u$ is the time partial derivative of u of order l .

- $L^p(0, T; X) = \{v \text{ measurable on }]0, T[\text{ such that } \int_0^T \|v(t)\|_X^p dt < \infty\}$ is the Banach space with norm

$$\|v\|_{L^p(0, T; X)} = \begin{cases} \left(\int_0^T \|v(t)\|_X^p dt \right)^{\frac{1}{p}} & \text{for } 1 \leq p < +\infty, \\ \sup_{0 \leq t \leq T} \|v(t)\|_X & \text{for } p = +\infty. \end{cases}$$

- $H^s(0, T; X) = \{v \in L^2(0, T; X); \partial^k v \in L^2(0, T; X); k \leq s\}$ is the Hilbert space with scalar product

$$(u, v) = \left((u, v)_{L^2(0, T; X)} + \sum_{k=0}^s (\partial^k u, \partial^k v)_{L^2(0, T; X)} \right)^{\frac{1}{2}}.$$

- $W^{m,1}(0, T, X)$ is the space of functions in $L^1(0, T, X)$ such that all their derivatives up to the order m belong to $L^1(0, T, X)$.

Consider the wave equation problem

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } \Omega \times]0, T[, \\ u = 0 & \text{on } \Gamma \times]0, T[, \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \\ \partial_t u(\cdot, 0) = v_0 & \text{in } \Omega, \end{cases} \quad (1)$$

where the wave u is the unknown defined on $\Omega \times]0, T[$, and (u_0, v_0) are the data functions defined on Ω .

This problem can be written in a more general form,

$$\begin{cases} \partial_t \Phi - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \Phi = F & \text{in } \Omega \times]0, T[, \\ u = 0 & \text{on } \Gamma \times]0, T[, \\ \Phi(\cdot, 0) = \Phi_0 & \text{in } \Omega, \end{cases} \quad (2)$$

where $\Phi = \begin{pmatrix} u \\ v = \partial_t u \end{pmatrix}$, $F = \begin{pmatrix} f \\ g \end{pmatrix}$, and $\Phi_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

Lemma 1 *If $(f, g) \in L^1(0, T; H_0^1(\Omega)) \times L^1(0, T; L^2(\Omega))$ and $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$, then*

$$(\|v\|^2 + \|\nabla u\|^2)^{\frac{1}{2}} \leq (\|v_0\|^2 + \|\nabla u_0\|^2)^{\frac{1}{2}} + \int_0^t (\|f\| + \|g\|)(s) ds, \quad 0 \leq t \leq T. \quad (3)$$

Proof 1 Taking the inner product of the first equation of problem (2) and $\begin{pmatrix} -\Delta u \\ v \end{pmatrix}$ and integrating by parts the second term, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|\partial_t u\|^2) + \frac{1}{2} \frac{d}{dt} (\|\nabla u\|^2) \leq \|f\| + \|g\|.$$

By integrating this inequality between 0 and t we get estimate (3).

Remark 1 Consider the following Laplace problem:

$$\begin{cases} -\Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \quad (4)$$

Let $u = (\Delta)^{-1}h$ be the solution of problem (4). The operator $(\Delta)^{-1}$ is a self-adjoint positive definite isometry of the space $H^{-1}(\Omega)$ into $H_0^1(\Omega)$. Thus, for $h \in H^{-1}(\Omega)$, we obtain $\|((\Delta)^{-1})^{\frac{1}{2}}h\| = \|h\|_{H^{-1}(\Omega)}$ (see [12], Chap. 1, Thm. 12.3, for the proof). Then, for $(f, g) \in L^1(0, T, L^2(\Omega)) \times L^1(0, T, H^{-1}(\Omega))$, taking the inner product of the first equation of system (2) and $\begin{pmatrix} u \\ (\Delta)^{-1}v \end{pmatrix}$, we obtain the following estimate:

$$(\|v\|_{H^{-1}(\Omega)}^2 + \|u\|^2)^{\frac{1}{2}} \leq (\|v_0\|_{H^{-1}(\Omega)}^2 + \|u_0\|^2)^{\frac{1}{2}} + \int_0^t (\|f\| + \|g\|_{H^{-1}(\Omega)})(s) ds.$$

Finally, we have the following result proved in [12, Chap. 1].

Proposition 1 *For any data $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$, system (1) has a unique solution $u \in \mathcal{C}^1(0, T; L^2(\Omega)) \cap \mathcal{C}^0(0, T; H_0^1(\Omega))$. Moreover, this solution satisfies*

$$\|\partial_t u\|^2 + \|\nabla u\|^2 = \|\nabla u_0\|^2 + \|u_0\|^2.$$

3 Discretization on time

Consider a partition of the time interval $[0, T]$ into subintervals $[t_k, t_{k+1}]$, $1 \leq k \leq I$, such that $0 = t_0 < t_1 < \dots < t_K = T$. We denote $\delta t_k = t_{k+1} - t_k$, $\delta t = (\delta t_1, \dots, \delta t_K)$, and $|\delta t| = \max_{1 \leq k \leq K} |\delta t_k|$.

To formulate the time semidiscrete problem, we apply the Euler implicit method to system (1). Then it consists in finding the sequence of functions $(u^k)_{0 \leq k \leq K}$ in the space $H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)^{K-1}$ such that

$$\begin{cases} \frac{u^{k+1}-u^k}{\delta t_k} - \frac{u^k-u^{k-1}}{\delta t_{k-1}} - \delta t_k \Delta u^{k+1} = 0 & \text{in } \Omega, \ 1 \leq k \leq K, \\ u^{k+1} = 0 & \text{on } \Gamma, \ 1 \leq k \leq K, \\ u^0 = u_0 & \text{in } \Omega, \\ u^1 = u_0 + \delta t_0 v_0 & \text{in } \Omega. \end{cases} \quad (5)$$

We suppose that the data $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$. Then, if u^0 and v^0 are known, then we easily show that u^{k+1} , $k \geq 1$, is a solution of the following weak problem:

Find $u^{k+1} \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$,

$$\begin{aligned} & \int_{\Omega} u^{k+1}(\mathbf{x}) v(\mathbf{x}) \, d\mathbf{x} + \delta t_k^2 \int_{\Omega} \nabla u^{k+1}(\mathbf{x}) \nabla v(\mathbf{x}) \, d\mathbf{x} \\ &= \int_{\Omega} \left(u^k + \frac{\delta t_k}{\delta t_{k-1}} (u^k - u^{k-1}) \right) (\mathbf{x}) v(\mathbf{x}) \, d\mathbf{x}. \end{aligned} \quad (6)$$

Proposition 2 *If $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$ is known, then problem (6) has a unique solution u^{k+1} , $k \geq 1$, in $H_0^1(\Omega)$. Moreover, the solution $(u^k)_{0 \leq k \leq K}$ of problem (5) verifies for $0 \leq k \leq K$ the following stability condition:*

$$\left\| \frac{u^{k+1} - u^k}{\delta t_k} \right\|^2 + \|\nabla u^{k+1}\|^2 \leq \|v_0\|^2 + 2\|\nabla u_0\|^2 + 2\delta t_0^2 \|\nabla v_0\|^2. \quad (7)$$

Proof 2 Using the Lax–Milgram theorem, we show that problem (6) has a unique solution. Then, by iteration on k , we deduce that problem (5) has a unique solution.

Taking the inner product of $\frac{u^{k+1}-u^k}{\delta t_k}$ and the first equation in (5), we obtain

$$\left\| \frac{u^{k+1} - u^k}{\delta t_k} \right\|^2 + \|\nabla u^{k+1}\|^2 = \left(\frac{u^{k+1} - u^k}{\delta t_k}, \frac{u^k - u^{k-1}}{\delta t_{k-1}} \right) + (\nabla u^{k+1}, \nabla u^k). \quad (8)$$

Applying the Cauchy–Schwarz inequality leads to

$$\left\| \frac{u^{k+1} - u^k}{\delta t_k} \right\|^2 + \|\nabla u^{k+1}\|^2 \leq \left\| \frac{u^k - u^{k-1}}{\delta t_{k-1}} \right\|^2 + \|\nabla u^k\|^2. \quad (9)$$

Then by iteration on k we have

$$\left\| \frac{u^{k+1} - u^k}{\delta t_k} \right\|^2 + \|\nabla u^{k+1}\|^2 \leq \left\| \frac{u^1 - u^0}{\delta t_0} \right\|^2 + \|\nabla u^1\|^2.$$

Finally, we conclude by using the third and fourth equations of system (5).

Remark 2 1) We notice that for $k \geq 1$, the solution u^{k+1} of problem (6) belongs to $H^{s+1}(\Omega)$, $s \geq \frac{1}{2}$. When the domain Ω is convex or of dimension 1, $s \geq 1$ is explicitly known. For

$\frac{1}{2} \leq s \leq 1$, from condition (7) we derive the inequality

$$\|u^{k+1}\|^2 \leq C\delta t_k^{-2s}(\|v_0\|^2 + 2\|\nabla u_0\|^2 + 2\delta t_0^2\|\nabla v_0\|^2), \quad (10)$$

where the constant C is independent of the step δt .

This inequality is not optimal since $\|u^{k+1}\|^2$ is not bounded independently of the step δt .

2) Using the implicit Euler scheme for the time discretization, problem (2) is written as follows: Find $\Phi^k = \begin{pmatrix} u^k \\ v^k \end{pmatrix}$ such that,

$$\begin{cases} \frac{\Phi^{k+1} - \Phi^k}{\delta t_k} - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \Phi^{k+1} = F^{k+1} & \text{in } \Omega, \ 0 \leq k \leq K, \\ u^{k+1} = 0 & \text{on } \Gamma, \ 0 \leq k \leq K, \\ \Phi^0 = \Phi_0 & \text{in } \Omega, \end{cases} \quad (11)$$

where $F^{k+1} = \begin{pmatrix} f^{k+1} \\ g^{k+1} \end{pmatrix}$.

For $n \neq 0$, systems (11) and (5) coincide if $F^{k+1} = \mathbf{0}$, $k \geq 1$. When $n = 0$, we propose the following two cases where the two systems completely coincide:

i) We replace the fourth equation of system (5) by the following implicit equation:

$$\begin{cases} u^1 - \delta t_0^2 \Delta u^1 = u_0 + \delta t_0 v_0 & \text{in } \Omega, \\ u^1 = 0 & \text{on } \Gamma. \end{cases} \quad (12)$$

ii) We replace the third equation of system (11):

$$\begin{cases} \frac{\Phi^{k+1} - \Phi^k}{\delta t_k} - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \Phi^{k+1} = F^{k+1} & \text{in } \Omega, \ 1 \leq k \leq K, \\ u^{k+1} = 0 & \text{on } \Gamma, \ 1 \leq k \leq K, \\ v^1 = \begin{pmatrix} u_0 + \delta t_0 v_0 \\ v_0 \end{pmatrix} & \text{in } \Omega. \end{cases} \quad (13)$$

Multiplying the first equation in (13) by $\begin{pmatrix} -\Delta u^{k+1} \\ u^{k+1} \end{pmatrix}$, we obtain the following stability condition:

$$\|v^{k+1}\|^2 + \|\nabla u^{k+1}\|^2 \leq 2(\|v^1\|^2 + \|\nabla u^1\|^2) + 2\left(\sum_{j=1}^k \delta t_j (\|g^{j+1}\| + \|\nabla f^{j+1}\|)\right)^2. \quad (14)$$

However, if we take the inner product of the first equation of system (13) and $\begin{pmatrix} u^{k+1} \\ (\Delta)^{-1} u^{k+1} \end{pmatrix}$, we obtain the following stability condition in terms of weaker norms:

$$\begin{aligned} & \|v^{k+1}\|_{H^{-1}(\Omega)}^2 + \|u^{k+1}\|^2 \\ & \leq 2(\|v^1\|_{H^{-1}(\Omega)}^2 + \|u^1\|^2) + 2\left(\sum_{j=1}^k \delta t_j (\|g^{j+1}\|_{H^{-1}(\Omega)} + \|f^{j+1}\|)\right)^2. \end{aligned} \quad (15)$$

To obtain the error estimate between the solutions u of (1) and $(u)^k_{0 \leq k \leq K}$ of (5), we define the error $\Upsilon^k = \begin{pmatrix} e(u)^k \\ e(v)^k \end{pmatrix}$ such that $e(u)^k = u(t_k) - u^k$ and $e(v)^k = v(t_k) - v^k$. We can easily show

that $(\Upsilon)_{0 \leq k \leq K}^k$ is a solution of (13), where the two components of F^{k+1} are the following consistency errors:

$$\varepsilon(u)^k = \frac{u(t_{k+1}) - u(t_k)}{\delta t_k} - \partial_t u(t_{k+1}), \quad \varepsilon(v)^k = \frac{v(t_{k+1}) - v(t_k)}{\delta t_k} - \partial_t v(t_{k+1}). \quad (16)$$

Theorem 1 *If a solution u of problem (1) belongs to $W^{3,1}(0, T; L^2(\Omega)) \cap W^{2,1}(0, T; H_0^1(\Omega))$, then*

$$\begin{aligned} & \|\varepsilon(u)^k\|^2 + \|\nabla(u(t_k) - u^k)\|^2 \\ & \leq C\delta t^2 \left(\int_0^{t_k} (\|\partial_t^3 u\| + \|\partial_t^2 \nabla u\|)(s) ds \right)^2, \quad 0 \leq k \leq K, \end{aligned} \quad (17)$$

where C is a positive constant independent of δt .

Proof 3 Since $(\Upsilon)_{0 \leq k \leq K}^k$ is a solution of (13), where the second member is F^{k+1} . Then applying the stability condition (14) leads to

$$\begin{aligned} & \|e(u)^k\|^2 + \|\nabla e(u)^k\|^2 \\ & \leq 2(\|e(v)^0\|^2 + \|\nabla e(u)^0\|^2) + 2 \left(\sum_{j=1}^k \delta t_j (\|\varepsilon(v)^j\| + \|\nabla \varepsilon(u)^j\|) \right)^2 \end{aligned} \quad (18)$$

thanks to the Taylor theorem with integral remainder to bound the terms $\|\varepsilon(v)^j\|$, $\|\nabla \varepsilon(u)^j\|$, $\|e(v)^0\|$, and $\|\nabla e(u)^0\|$. Then we conclude the desired estimate (17).

We can find the following error estimate in weaker norms by using the same technique as the proof of Theorem 1 and replacing condition (14) by (15).

Corollary 1 *Suppose that the solution u of system (1) belongs to $W^{3,1}(0, T; L^2(\Omega)) \cap W^{2,1}(0, T; H_0^1(\Omega))$. Then the following a priori error estimate between the solution u and the solution $(u^k)_{0 \leq k \leq K}$ of system (5) holds for $0 \leq k \leq K$:*

$$\|\varepsilon(u)^k\|_{H^{-1}(\Omega)}^2 + \|\nabla(u(t_k) - u^k)\|^2 \leq C\delta t^2 \left(\int_0^{t_k} (\|\partial_t^3 u\|_{H^{-1}(\Omega)} + \|\partial_t^2 \nabla u\|)(s) ds \right)^2, \quad (19)$$

where C is a positive constant independent of δt .

We remark that the obtained estimates (18) and (19) are optimal of order 1 in time, since the discretization is based on the implicit Euler scheme, which is of order 1.

4 The spectral discretization

We further suppose that Ω is a rectangle for $d = 2$ or a parallelepiped rectangle for $d = 3$.

Let $\mathbb{P}_N(\Omega)$ the space of polynomials of degree $\leq N$ ($N \geq 2$) for each variable, and let $\mathbb{P}_N^0(\Omega) = \mathbb{P}_N(\Omega) \cap H_0^1(\Omega)$. We define ζ_i , $0 \leq i \leq N$, the set of nodes, roots of the polynomial $(1 - x^2)L_N'$, where L_N is the Legendre polynomial, and ϱ_i , $0 \leq i \leq N$, are the weight set of the following Gauss–Lobatto quadrature formula on the interval $]-1, 1[$:

$$\forall \eta_N \in \mathbb{P}_{2N-1}([-1, 1]), \quad \int_{-1}^1 \eta_N(x) dx = \sum_{i=0}^N \eta_N(\zeta_i) \varrho_i. \quad (20)$$

We recall the following property (see [10, 13]):

$$\forall \eta_N \in \mathbb{P}_N(-1, 1], \quad \|\eta_N\|_{L^2(-1, 1]}^2 \leq \sum_{i=0}^N \eta_N^2(\zeta_i) \varrho_i \leq 3 \|\eta_N\|_{L^2[-1, 1]}^2. \quad (21)$$

The reference domain $]-1, 1]^d$ ($d = 2, 3$) is transformed to the domain Ω using the affine mapping T , and the scalar product is defined on continuous functions u and v by

$$(u, v)_N = \begin{cases} \frac{\text{meas}(\Omega)}{4} \sum_{i=0}^N \sum_{j=0}^N (u \circ T)(\zeta_i, \zeta_j) (v \circ T)(\zeta_i, \zeta_j) \varrho_i \varrho_j & \text{if } d = 2, \\ \frac{\text{meas}(\Omega)}{8} \sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N (u \circ T)(\zeta_i, \zeta_j, \zeta_k) (v \circ T)(\zeta_i, \zeta_j, \zeta_k) \varrho_i \varrho_j \varrho_k & \text{if } d = 3. \end{cases} \quad (22)$$

Remark 3 For simplicity of analysis, we suppose that the spectral discretization is fixed over time.

We suppose that u_0 and v_0 are continuous on $\bar{\Omega}$. The discrete problem is deduced from (5) by applying the Galerkin method combined with numerical integration:

$$\text{For } u_N^0 = \mathcal{I}_N(u_0) \quad \text{and} \quad u_N^1 = \mathcal{I}_N(u_0) + \delta t_0 \mathcal{I}_N(v_0) \quad \text{in } \Omega, \quad (23)$$

where \mathcal{I}_N is the interpolating operator from $L^2(\Omega)$ into $\mathbb{P}_N(\Omega)$, find $u_N^k \in \mathbb{P}_N^0(\Omega) \times \mathbb{P}_N(\Omega) \times (\mathbb{P}_N^0(\Omega))^{K-1}$, $1 \leq k \leq K$, such that for all $v_N \in \mathbb{P}_N^0(\Omega)$,

$$\left(\frac{u_N^{k+1} - u_N^k}{\delta t_k} - \frac{u_N^k - u_N^{k-1}}{\delta t_{k-1}}, v_N \right)_N + \delta t_k (\nabla u_N^{k+1}, \nabla v_N)_N = 0. \quad (24)$$

As in (6), u_N^{k+1} , $1 \leq k \leq K$, is the solution of the discrete weak problem

$$(u_N^{k+1}, v_N)_N + \delta t_k^2 (\nabla u_N^{k+1}, \nabla v_N)_N = \left(u_N^k + \frac{\delta t_k}{\delta t_{k-1}} (u_N^k - u_N^{k-1}), v_N \right)_N. \quad (25)$$

Proposition 3 Let the data $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$. If u_N^0 and v_N^0 are known, then problem (25) has a unique solution u_N^{k+1} , $k \geq 1$, in $H_0^1(\Omega)$. Moreover, the solution $(u_N^k)_{0 \leq k \leq K}$ of problem (23)–(24) satisfies for $0 \leq k \leq K$ the following stability condition:

$$\begin{aligned} & \left\| \frac{u_N^{k+1} - u_N^k}{\delta t_k} \right\|^2 + \|\nabla u_N^{k+1}\|^2 \\ & \leq (3^d)^K (\|\mathcal{I}_N(v_0)\|^2 + 2\|\nabla \mathcal{I}_N(u_0)\|^2 + 2\delta t_0^2 \|\nabla \mathcal{I}_N(v_0)\|^2). \end{aligned} \quad (26)$$

Proof 4 We show that problem (25) has a unique solution using the Lax–Milgram theorem and property (21).

To prove the stability condition (26), we define $\|\cdot\|_d$ the discrete norm deduced from the discrete scalar product $(\cdot, \cdot)_N$. Now letting $v_N = \frac{u_N^{k+1} - u_N^k}{\delta t_k}$ in (24) leads to

$$\left\| \frac{u_N^{k+1} - u_N^k}{\delta t_k} \right\|_d^2 + \|\nabla u_N^{k+1}\|_d^2 = \left(\frac{u_N^{k+1} - u_N^k}{\delta t_k}, \frac{u_N^k - u_N^{k-1}}{\delta t_{k-1}} \right)_N + (\nabla u_N^{k+1}, \nabla u_N^k)_N.$$

Using the Cauchy–Schwarz inequality and (21), we have

$$\left\| \frac{u_N^{k+1} - u_N^k}{\delta t_k} \right\|^2 + \|\nabla u_N^{k+1}\|^2 \leq 3^d \left(\left\| \frac{u_N^k - u_N^{k-1}}{\delta t_{k-1}} \right\|^2 + \|\nabla u^k\|^2 \right).$$

Then iterating over k , we obtain

$$\left\| \frac{u_N^{k+1} - u_N^k}{\delta t_k} \right\|^2 + \|\nabla u_N^{k+1}\|^2 \leq (3^d)^K \left(\left\| \frac{u^1 - u^0}{\delta t_0} \right\|^2 + \|\nabla u^1\|^2 \right).$$

Finally, estimate (26) is deduced from (23).

Proposition 4 Let u_0, v_0 be continuous on $\overline{\Omega}$, and let u_N^0, v_N^0 be known. The error estimate between solutions $u^{k+1}, k \geq 1$, and $u_N^{k+1}, k \geq 1$, of problems (6) and (25), respectively, is

$$\begin{aligned} \|u^{k+1} - u_N^{k+1}\| \leq C & \left(\inf_{\chi_N^{k+1} \in \mathbb{P}_N^0(\Omega)} \|u^{k+1} - \chi_N^{k+1}\| + \left[\|u_0 - u_N^0\| + \|v_0 - v_N^0\| \right. \right. \\ & \left. \left. + \sum_{j=1}^k (T^{1,j} + T^{2,j} + T^{3,j}) \right] \right), \end{aligned} \quad (27)$$

where

$$\begin{aligned} T^{1,j} &= \frac{1}{\delta t_j^2} \sup_{v_N \in \mathbb{P}_N^0(\Omega)} \frac{\int_{\Omega} (u^{j+1} - u^j) v_N \, d\mathbf{x} - (\chi_N^{j+1} - \chi_N^j, v_N)_N}{\|v_N\|}, \\ T^{2,j} &= \sup_{v_N \in \mathbb{P}_N^0(\Omega)} \frac{\int_{\Omega} \nabla u^{j+1} \nabla v_N \, d\mathbf{x} - (\nabla \chi_N^{j+1}, \nabla v_N)_N}{\|v_N\|}, \\ T^{3,j} &= \sup_{v_N \in \mathbb{P}_N^0(\Omega)} \frac{\int_{\Omega} (u^j - u^{j-1}) v_N \, d\mathbf{x} - (\mathfrak{I}_N(u^j - u^{j-1}), v_N)_N}{\|v_N\|}, \end{aligned}$$

and C is a positive constant independent of N .

Proof 5 Consider $\chi_N^{k+1} \in \mathbb{P}_N^0(\Omega)$. By the triangle inequality we have

$$\|u^{k+1} - u_N^{k+1}\| \leq \|u^{k+1} - \chi_N^{k+1}\| + \|\chi_N^{k+1} - u_N^{k+1}\|.$$

To estimate $\|u_N^{k+1} - \chi_N^{k+1}\|$, we begin by writing problems (5) and (25) for $v_N \in \mathbb{P}_N^0(\Omega)$. Then we consider $\tau_k = \frac{\delta t_k}{\delta t_{k-1}}$ and doing the difference term by term, we obtain

$$(u_N^{k+1} - \chi_N^{k+1}, v_N)_N + \delta t_k^2 (\nabla(u_N^{k+1} - \chi_N^{k+1}), \nabla v_N)_N = (u_N^k - \chi_N^k, v_N)_N + \tau_k \mathcal{H}^k(v_N),$$

where

$$\begin{aligned} \mathcal{H}^k(v_N) &= \frac{1}{\delta t_k^2} \left(\int_{\Omega} (u^{k+1} - u^k) v_N \, d\mathbf{x} - (\chi_N^{k+1} - \chi_N^k, v_N)_N \right) \\ &\quad + \int_{\Omega} \nabla u^{k+1} \nabla v_N \, d\mathbf{x} - (\nabla \chi_N^{k+1}, \nabla v_N)_N \\ &\quad + \int_{\Omega} (u^k - u^{k-1}) v_N \, d\mathbf{x} - (\mathfrak{I}_N(u^k - u^{k-1}), v_N)_N. \end{aligned}$$

Since \mathcal{K}^k is linear and continuous on $\mathbb{P}_N^0(\Omega)$, by the Riesz theorem there exists ϑ_N^k in $\mathbb{P}_N^0(\Omega)$ such that

$$\mathcal{K}^k(v_N) = (\vartheta_N^k, v_N)_N.$$

Applying the result proved in [14, Prop. 4.1] and [15], we get

$$\|u_N^{k+1} - \chi_N^{k+1}\| \leq C \left(\|u_0 - u_N^0\| + \|v_0 - v_N^0\| + \sum_{j=1}^k \|\vartheta_N^j\|^2 \right)^{1/2},$$

where C is a positive constant independent of N .

So we conclude (27), since

$$\|\vartheta_N^j\| \leq C \sup_{v_N \in \mathbb{P}_N^0(\Omega)} \frac{(\vartheta_N^j, v_N)_N}{\|v_N\|},$$

where C is a positive constant independent of N .

To find the order of convergence as a function of N , it is necessary to estimate each of the terms of the second member of inequality (27).

• *Estimation of T^{1j}*

We consider $\varpi^{j+1} = u^{j+1} - u^j$ and $\chi_N^{j+1} - \chi_N^j = \Pi_{N-1}^{1,0}(\varpi^{j+1})$. By the exactness of the Gauss–Lobatto quadrature formula of (20), $\int_{\Omega} \Pi_{N-1}^{1,0}(\varpi^{j+1}) v_N \, d\mathbf{x}$ and $(\Pi_{N-1}^{1,0}(\varpi^{j+1}), v_N)_N$ are equal, and thus

$$T^{1j} \leq \|\varpi^j - \Pi_{N-1}^{1,0}(\varpi^j)\|, \quad (28)$$

where $\Pi_N^{1,0}$ is the orthogonal projection operator from $H_0^1(\Omega)$ into $\mathbb{P}_N^0(\Omega)$ related to the inner product defined by the semi norm $|\cdot|_{1,\Omega}$. (See ([13], Lemma VI.2.5) and [10] for all the properties of this operator.)

• *Estimation of T^{2j}*

Since the Gauss–Lobatto quadrature formula is exact for a polynomial of degree $\leq 2N - 1$, we have

$$\begin{aligned} & \int_{\Omega} \nabla u^{j+1} \nabla v_N \, d\mathbf{x} - (\nabla \chi_N^{j+1}, \nabla v_N)_N \\ &= \int_{\Omega} \nabla (u^{j+1} - \Pi_{N-1}^{1,0} u^{j+1}) \nabla v_N \, d\mathbf{x} - (\nabla (\chi_N^{j+1} - \Pi_{N-1}^{1,0} \chi_N^{j+1}), \nabla v_N)_N. \end{aligned} \quad (29)$$

Thanks to the triangle and Cauchy–Schwarz inequalities, we have

$$\begin{aligned} & \sup_{v_N \in \mathbb{P}_N^0(\Omega)} \frac{\int_{\Omega} \nabla u^{j+1} \nabla v_N \, d\mathbf{x} - (\nabla \chi_N^{j+1}, \nabla v_N)_N}{\|v_N\|} \\ & \leq (|u^{j+1} - \Pi_{N-1}^{1,0} u^{j+1}|_{1,\Omega} + |\chi_N^{j+1} - \Pi_{N-1}^{1,0} \chi_N^{j+1}|_{1,\Omega}). \end{aligned} \quad (30)$$

Thus we conclude using the properties of $\Pi_{N-1}^{1,0}$.

• *Estimation of $T^{3,j}$*

Let $\theta^j = u^j - u^{j-1}$. We use for this estimation Π_{N-1} the orthogonal projection from $L^2(\Omega)$ into $\mathbb{P}_{N-1}(\Omega)$. By the same argument above for the Gauss–Lobatto formula, we have

$$\begin{aligned} & \int_{\Omega} \theta^j(\mathbf{x}) v_N(\mathbf{x}) d\mathbf{x} - (\mathcal{I}_N \theta^j, v_N)_N \\ &= \int_{\Omega} (\theta^j - \Pi_{N-1} \theta^j)(\mathbf{x}) v_N(\mathbf{x}) d\mathbf{x} - (\mathcal{I}_N \theta^j - \Pi_{N-1} \theta^j, v_N)_N. \end{aligned}$$

Using inequality (21) in each direction leads to

$$\int_{\Omega} \theta^j(\mathbf{x}) v_N(\mathbf{x}) d\mathbf{x} - (\mathcal{I}_N \theta^j, v_N)_N \leq [\|\theta^j - \Pi_{N-1} \theta^j\|^2 + 9\|\theta^j - \mathcal{I}_N \theta^j\|^2] \|v_N\|.$$

Thanks to the approximation properties of operator Π_{N-1} (see [10, Thm. 7.1]) and \mathcal{I}_N (see [10, Thm. 14.2]), for $\theta^j \in H^s(\Omega)$; $s > 1$, we obtain

$$\sup_{v_N \in \mathbb{P}_N(\Omega)} \frac{\int_{\Omega} \theta^j(\mathbf{x}) v_N(\mathbf{x}) d\mathbf{x} - (\theta^j, v_N)_N}{\|v_N\|} \leq CN^{-2s} \|\theta^j\|_{s,\Omega}^2. \quad (31)$$

Finally, to estimate

$$\inf_{\chi_N^{k+1} \in \mathbb{P}_N^0(\Omega)} \|u^{k+1} - \chi_N^{k+1}\|, \quad \|u_0 - u_N^0\| \quad \text{and} \quad \|v_0 - v_N^0\|, \quad (32)$$

we choose, respectively, $\chi_N^{k+1} = \Pi_N^{1,0} u^{k+1}$, $u_N^0 = \Pi_N^{1,0} u_0$, and $v_N^0 = \Pi_N v_0$. Then we conclude using properties of operators $\Pi_N^{1,0}$ and Π_N .

So, from estimates (28), (30), (31), and (32) we obtain the following main theorem.

Theorem 2 For (u_0, v_0) continuous on $\bar{\Omega}$, solution $(u^k)_{0 \leq k \leq K}$ of problem (5) belongs to $H^s(\Omega)$; $s > 1$. The error between solutions u^{k+1} and (u_N^{k+1}) of problems (6) and (25), respectively, satisfies

$$\begin{aligned} \|u^{k+1} - u_N^{k+1}\| &\leq C \left[N^{-s} \left(\|u^{k+1}\|_{s,\Omega} + \sum_{j=1}^k (\delta t_j^{-2} \|u^{j+1} - u^j\|_{s,\Omega} + \|u^j - u^{j-1}\|_{s,\Omega}) \right) \right. \\ &\quad \left. + N^{1-s} \sum_{j=1}^k \|u^{j+1}\|_{s,\Omega} \right], \end{aligned} \quad (33)$$

where C is a positive constant independent of N .

5 Numerical results

Consider the interpolating Lagrange polynomial φ_j defined by

$$\varphi_j \in \mathbb{P}([-1, 1]), \quad \varphi_j(\zeta_i) = \delta_{ij}, \quad 0 \leq i, j \leq N,$$

where δ_{ij} is the Kronecker symbol. The solution u_N^{k+1} of problem (25) is written as

$$u_N^{k+1}(x, y) = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} u_N^{k+1}(\zeta_i, \zeta_j) \varphi_i(x) \varphi_j(y).$$

Let U^{k+1} be the admissible solution vector of components $u_N^{k+1}(\zeta_i, \zeta_j)$. The matrix system deduced form of the discrete problem (25) is written

$$(D^{k+1} + \delta t_k^2 A^{k+1})U^{k+1} = F^k, \quad (34)$$

where D^{k+1} is a diagonal matrix of coefficients $\varrho_r \varrho_s$, $1 \leq r, s \leq N-1$, A^{k+1} is the matrix with coefficients $(\nabla(\varphi_i \varphi_j); \nabla(\varphi_r \varphi_s))$, $1 \leq i, j, r, s \leq N-1$, and F^k is the vector with components

$$u_N^k(\zeta_r, \zeta_s) + \frac{\delta t_k}{\delta t_{k-1}} (u_N^k(\zeta_r, \zeta_s) - u_N^{k-1}(\zeta_r, \zeta_s)) \varrho_r \varrho_s, \quad 1 \leq r, s \leq N-1.$$

We remark that the matrix $D^{k+1} + \delta t_k^2 A^{k+1}$ is symmetric and positive definite. Then system (34) is solved using the gradient conjugate method at each iteration.

5.1 Iterative algorithm

Step 1: Let

$$u_N^0 = \mathcal{I}_N(u_0) \quad \text{and} \quad u_N^1 = \mathcal{I}_N(u_0) + \delta t_0 \mathcal{I}_N(v_0).$$

Step 2: Suppose u_N^{k-1} and u_N^k are known. The linear system (34) is solved by the gradient conjugate method

$$(D^{k+1} + \delta t_k^2 A^{k+1})U^{k+1} = F^k.$$

We do the iterations until the following condition is satisfied:

$$\|U_N^{k+1} - U_N^k\|_{H^1(\Omega)} \leq \xi,$$

where $\xi = 10^{-10}$ for all the following numeric tests.

5.2 Time convergence

We consider the domain $\Omega =]-1, 1[^2$. Two exact solutions are tested.

The first one is

$$u(t, x, y) = e^t \sin(\pi x) \sin(\pi y). \quad (35)$$

We choose $T = 1$, $N = 20$, and $\delta t = 10^{-k}$, $k = 1, \dots, 4$. Figure 1 deals with the quantities $\log_{10} \|u - u_{N\delta t}\|_{H^1(\Omega)}$ (in blue) and $\log_{10} \|u - u_{N\delta t}\|_{L^2(\Omega)}$ (in red) as functions of $\log_{10}(\delta t)$.

For test 2, we study the singular solution

$$u(t, x, y) = t^{3/2} (1 - x^2)^{5/2} (1 - y^2)^{5/2}. \quad (36)$$

Figure 2 deals with the same curves as in Fig. 1 tested for the solution (36) when $N = 20$, $T = 0.1$, and $\delta t = 5 \cdot 10^{-2}, 10^{-2}, 5 \cdot 10^{-3}, 10^{-4}$. The obtained results show the convergence of the method with an order of convergence almost equal to 1.

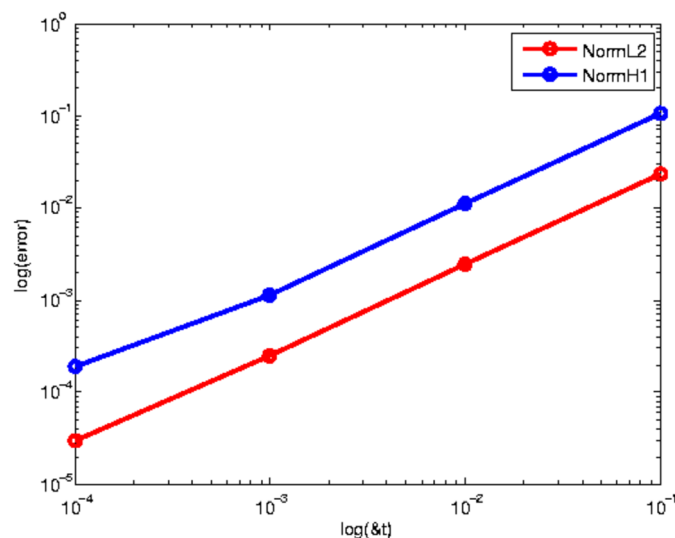


Figure 1 Time error for solution (35)

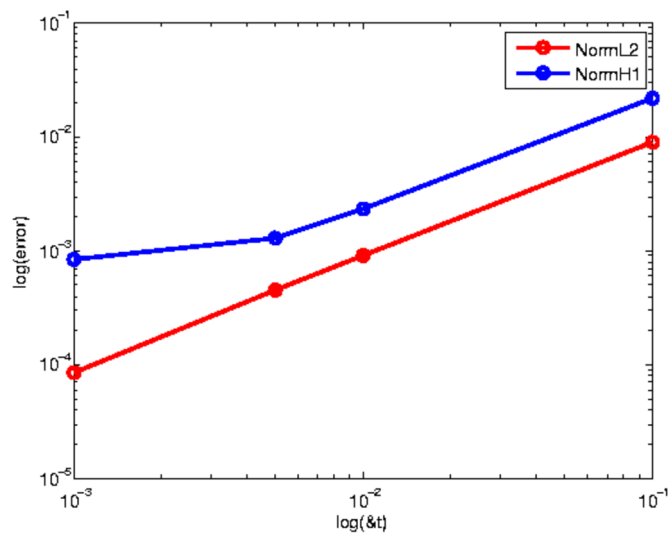


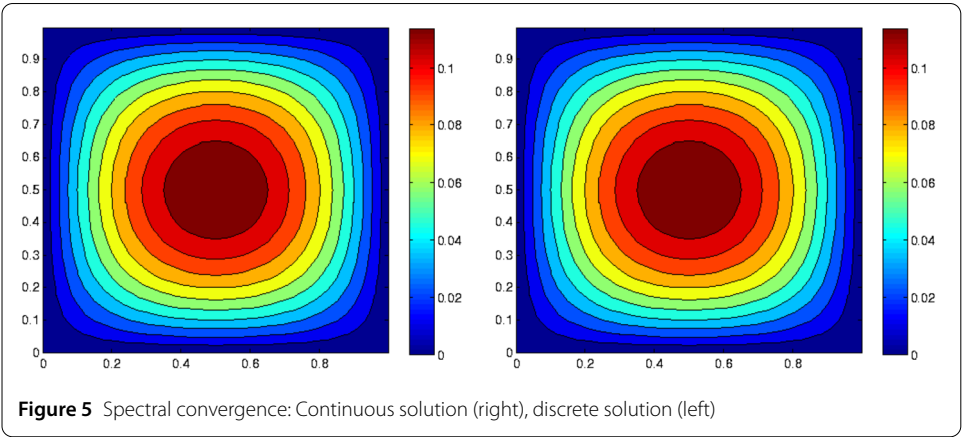
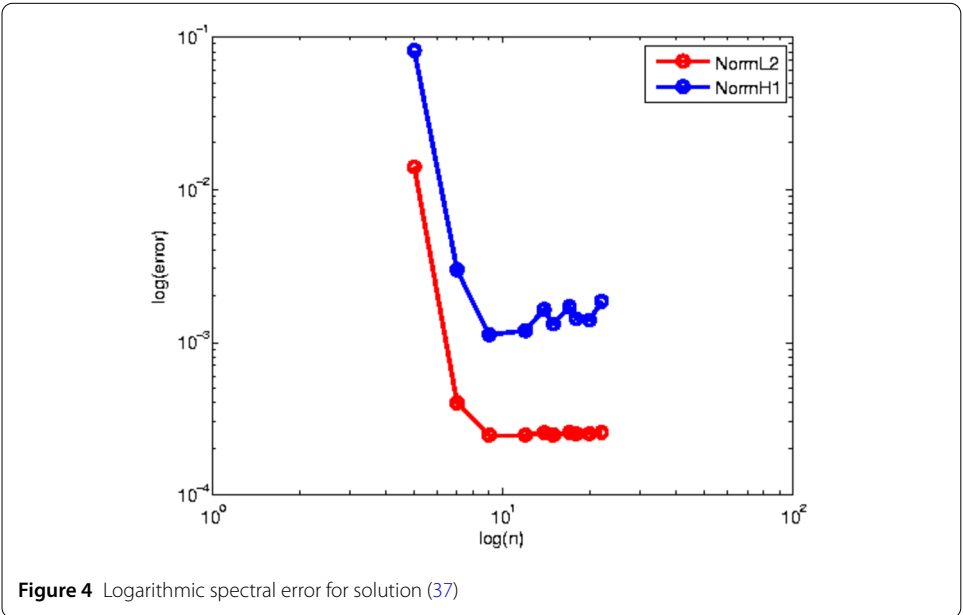
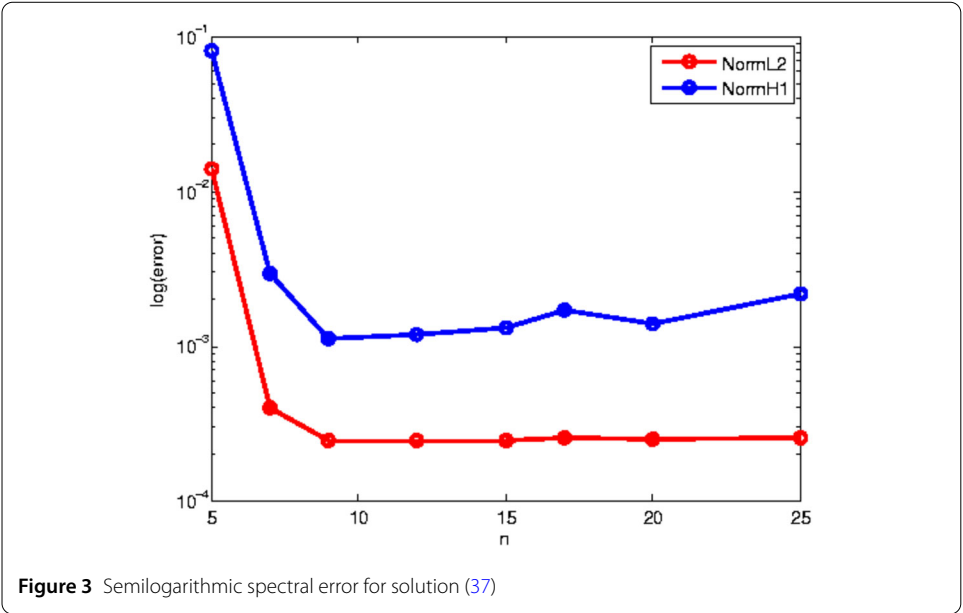
Figure 2 Time error for solution (36)

5.3 Spectral convergence

In this test, we fix $\delta t = 0.01$ and take $N = 5, 7, 9, 12, 14, 15, 17, 18, 20, 22$ and $T = 1$. We consider

$$u = (1 + t)(1 - x^2)(1 - y^2). \quad (37)$$

Figure 3 (respectively, Fig. 4) deals with $\log_{10} \|u - u_{N\delta t}\|_{H^1(\Omega)}$ (in blue) and $\log_{10} \|u - u_{N\delta t}\|_{L^2(\Omega)}$ (in red) as functions of N (respectively, $\log_{10}(N)$). We remark that the error norms $\log_{10} \|u - u_{N\delta t}\|_{H^1(\Omega)}$ decrease exponentially until $N = 10$ and stagnate for $N > 10$. The errors $\log_{10} \|u - u_{N\delta t}\|_{L^2(\Omega)}$ decrease until $N = 10$ and stagnate for $N > 10$. We remark



that convergence stagnates. This is due to the fact that the time order of convergence is less than the order of the spectral method.

Finally, the isovalues of the exact and discrete solutions (37) are presented in Fig. 5.

6 Conclusion and future work

This work concerns the numerical analysis of the implicit Euler scheme in time and the spectral discretization in space of the second-order wave equation. We prove an optimal error estimate in time and space. These estimates depend only on the regularity of the solution. Although the spectral methods are known as high-order methods in space, we remark that they have the disadvantage of losing part of this accuracy due to lower order of temporal discretization (often of order 1 or 2). The numerical analysis and implementation of the more difficult case where the spectral discretization depends on time using the second-order BDF method for the time discretization will be the subject of our forthcoming work.

Acknowledgements

Researchers Supporting Project number (RSP-2021/153), King Saud University, Riyadh, Saudi Arabia.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. Both authors read and approved the final manuscript.

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Received: 21 February 2022 Accepted: 17 March 2022 Published online: 31 March 2022

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