Park *Boundary Value Problems* (2022) 2022:23
https://doi.org/10.1186/s13661-022-01602-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

General decay for a viscoelastic von Karman equation with delay and variable exponent nonlinearities

Sun-Hye Park[1*]

*Correspondence:
sh-park@pusan.ac.kr
[1]Office for Education Accreditation, Pusan National University, Busan 46241, South Korea

Abstract
In this paper, we consider a viscoelastic von Karman equation with damping, delay, and source effects of variable exponent type. Firstly, we show the global existence of solution applying the potential well method. Then, by making use of the perturbed energy method and properties of convex functions, we derive general decay results for the solution under more general conditions of a relaxation function. General decay results of solutions for viscoelastic von Karman equations with variable exponent nonlinearities have not been discussed before. Our results extend and complement many results for von Karman equations in the literature.

MSC: 35B40; 35L70; 74D99

Keywords: Von Karman equation; Variable exponent; Time delay; General decay; Convex function

1 Introduction

In this work, we study the following viscoelastic von Karman equation with damping, delay, and source terms of variable exponents:

$$u_{tt} + \Delta^2 u - \int_0^t k(t-s)\Delta^2 u(s)\,ds + \alpha|u_t|^{m(\cdot)-2}u_t + \beta\big|u_t(t-\tau)\big|^{m(\cdot)-2}u_t(t-\tau)$$
$$= \big[u, \chi(u)\big] + \gamma|u|^{p(\cdot)-2}u \quad \text{in } \Omega\times(0,\infty), \tag{1.1}$$

$$\Delta^2\chi(u) = -[u,u] \quad \text{in } \Omega\times(0,\infty), \tag{1.2}$$

$$u = \frac{\partial u}{\partial\nu} = 0, \qquad \chi(u) = \frac{\partial\chi(u)}{\partial\nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{1.3}$$

$$u(0) = u_0, \qquad u_t(0) = u_1 \quad \text{in } \Omega, \tag{1.4}$$

$$u_t(x,t-\tau) = j_0(x,t-\tau) \quad \text{in } \Omega\times(0,\tau), \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^2$ complementing a smooth boundary $\partial\Omega$, ν is the unit normal vector outward to $\partial\Omega$, $\alpha > 0$, $\beta \in \mathbb{R}$, $\gamma > 0$, $\tau > 0$ is time delay, u_0, u_1, and j_0 are given initial data, and the von Karman bracket $[\cdot,\cdot]$ is defined as $[v,\tilde{v}] = v_{x_1x_1}\tilde{v}_{x_2x_2} + v_{x_2x_2}\tilde{v}_{x_1x_1} -$

$2v_{x_1x_2}\tilde{v}_{x_1x_2}$, where $(x_1,x_2)=x\in\Omega$. The relaxation function k and the exponents $m(\cdot)$ and $p(\cdot)$ will be specified later. This type of von Karman equations arises in many applications to the modeling of engineering and physical phenomena such as shells, nonlinear elastic plates, bifurcation theory, and so on.

In the absence of damping, delay, and source effects($\alpha=\beta=\gamma=0$) in (1.1), many authors [5, 22, 25, 26, 28, 30] have developed decay results by weakening the conditions of the relaxation function k. Munoz Rivera and Menzala [22] showed an exponential decay result when k satisfies

$$k'(t)\le -\zeta k(t),\quad \zeta>0, \tag{1.6}$$

and a polynomial decay result when

$$k'(t)\le -\zeta k^{1+\frac{1}{q}}(t),\quad \zeta>0, q>2. \tag{1.7}$$

The authors of [25, 30] improved those results under the condition

$$k'(t)\le -\zeta(t)k(t), \tag{1.8}$$

where ζ fulfills $\zeta(t)>0$ and $\zeta'(t)\le 0$ on $[0,\infty)$. Cavalcanti et al. [5] established a general decay result for a wider class of relaxation functions satisfying

$$k'(t)\le -K\bigl(k(t)\bigr), \tag{1.9}$$

where K is an increasing convex function on an interval $[0,\varepsilon)$ and satisfies $K(0)=0$. Recently, Park [26] established a more general decay result under the very general condition

$$k'(t)\le -\zeta(t)K\bigl(k(t)\bigr), \tag{1.10}$$

where K is an increasing convex function satisfying certain conditions, invoked by the pioneering work of Mustafa [23]. In [23], the author introduced condition (1.10) and established an explicit and general decay result for a viscoelastic wave equation when k satisfies (1.10).

On one hand, the study of elliptic, parabolic, and hyperbolic problems with nonlinearities of variable exponent type has been attracting much interest [1, 2, 19, 21, 31]. The nonlinearities of such type describe various physical applications, for example, electrorheological fluids [29], nonlinear elastics [33], non-Newtonian fluids [3], and image precessing [1]. In recent years, some authors studied the following wave equation with such nonlinearities:

$$u_{tt}-\Delta u+\int_0^t k(t-s)\Delta u(s)\,ds+\alpha|u_t|^{m(x)-2}u_t=\gamma|u|^{p(x)-2}u. \tag{1.11}$$

In case $k=0$ in (1.11), Messaoudi et al. [21] proved the local existence of solution and showed a blow-up result of the solution with negative initial energy when the exponents $m(x)\ge 2$ and $p(x)\ge 2$ satisfy some hypotheses. Later, in [12], the authors proved the global existence of solution for the same equation by giving some conditions on initial data. Moreover, they showed that the solution decays exponentially when $m(x) = 2$ and polynomially when $m(x) \geq 2$ and $m_2 > 2$, where $m_2 = \operatorname{ess\,sup}_{x\in\Omega} m(x)$, by using an integral inequality introduced by Komornik [17]. In case $k \neq 0$ in (1.11), Park and Kang [27] obtained similar results of [21] for the solution with certain positive initial energy. Most recent, Messaoudi et al. [20] established very general decay results when $m(x) > 1$ and the relaxation function k satisfies (1.10). Their results generalize and extend the previous results for problem (1.11).

Inspired by these works, in this article, we consider the viscoelastic von Karman system (1.1)–(1.5) with damping, source, and time delay effects of variable-exponent type. Time delay appears in the phenomena depending on some past occurrences as well as on the present state, and may cause instability. We refer to [7, 32] for more applications of time delay and [11, 16, 24] for various decay results of delayed equations. In the absence of memory and time delay($k = \beta = 0$) in (1.1), Ha and Park [13] proved the global existence of solution and showed exponential or polynomial decay results depending on $m_2 \geq 2$. At this point, it is worth to say that there are no works on the global existence of solution and general decay of the solution for viscoelastic von Karman equations with variable exponent damping and source terms. Due to the presence of source effect, we have some difficulty in deriving desired general decay results. We overcome this by giving some conditions on initial data. Moreover, as far as we know, the global existence and decay of solutions for viscoelastic von Karman equations with delay of variable exponent type have not been considered before. Thus, we intend to discuss the issues for problem (1.1)–(1.5).

Here are the contents of this paper. We give preliminaries in Sect. 2. We show a global existence result in Sect. 3. We establish general decay results for both cases $1 < m_1 < 2$ and $m_1 > 2$, where $m_1 = \operatorname{ess\,inf}_{x\in\Omega} m(x)$.

2 Preliminaries

In this section, we present notations, review necessary materials, give assumptions, and state a local existence result.

We denote by $\|\cdot\|_Y$ the norm of a norm space Y. To simplify notations, we denote $\|\cdot\|_{L^s(\Omega)}$ as $\|\cdot\|_s$ for $1 \leq s \leq \infty$. We use the letter B_s to denote the embedding constant satisfying

$$\|v\|_s \leq B_s \|\Delta v\|_2 \quad \text{for } s \in [2,\infty], v \in H_0^2(\Omega). \tag{2.1}$$

Here, we recall Lebesgue and Sobolev spaces of variable exponents (see e.g. [8, 9, 18]). Let D be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, and $r : \Omega \to [1,\infty]$ be a measurable function. The Lebesque space

$$L^{r(\cdot)}(D) = \left\{ v : D \to \mathbb{R} | v \text{ is measurable in } D, \int_D \left|\mu v(x)\right|^{r(x)} dx < \infty \text{ for some } \mu > 0 \right\}$$

is a Banach space with respect to the Luxembourg-type norm

$$\|v\|_{r(\cdot)} = \inf \left\{ \mu > 0 | \int_D \left| \frac{v(x)}{\mu} \right|^{r(x)} dx \leq 1 \right\}.$$

It is said that $r(\cdot)$ satisfies the log-Hölder continuity condition if

$$\left|r(x)-r(\tilde{x})\right| \le -\frac{b_1}{\log|x-\tilde{x}|} \tag{2.2}$$

for all $x, \tilde{x} \in D$ with $|x-\tilde{x}| < b_2$, where $b_1 > 0$ and $0 < b_2 < 1$.

Throughout this paper, we let

$$r_1 := \operatorname*{ess\,inf}_{x\in D} r(x) \quad \text{and} \quad r_2 := \operatorname*{ess\,sup}_{x\in D} r(x).$$

We remind the following property of von Karman bracket.

Lemma 2.1 ([6]) *Let $v_1, v_2, v_3 \in H^2(\Omega)$. If at least one of them is an element of $H_0^2(\Omega)$, then*

$$\int_\Omega [v_1, v_2]v_3\,dx = \int_\Omega [v_1, v_3]v_2\,dx.$$

We give the following assumptions.

(A_1) The exponents $p(\cdot)$ and $m(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying (2.2) and

$$2 < p_1 \le p(x) \le p_2 < \infty, \tag{2.3}$$

$$1 < m_1 \le m(x) \le m_2 < \infty. \tag{2.4}$$

(A_2) The coefficients α and β have the relation

$$|\beta| < \frac{\alpha}{1+\frac{1}{m_1}-\frac{1}{m_2}}. \tag{2.5}$$

(A_3) $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuously differentiable function and satisfies

$$k(0) > 0, \quad 1-\int_0^\infty k(s)\,ds = k_l > 0,$$

and

$$k'(t) \le -\zeta(t)K\bigl(k(t)\bigr) \quad \text{for all } t \ge 0, \tag{2.6}$$

where $K : (0,\infty) \to (0,\infty)$ is a continuously differentiable function, which is either a linear function or a strictly increasing and strictly convex C^2-function on $(0,\varepsilon]$, $\varepsilon \le k(0)$, $K(0) = K'(0) = 0$, and ζ is positive, differentiable, and nonincreasing.

Remark 2.1 1. For examples of the function k satisfying (A_3), we refer to [23].

2. Since K is a strictly increasing and strictly convex C^2-function on $(0,\varepsilon]$ satisfying $K(0) = K'(0) = 0$, there exists an extension $\overline{K}$ of K, which is a strictly increasing and strictly convex C^2 function on $(0,\infty)$. We mention [23] for details.

As in [24], we introduce a function y as

$$y(x,\rho,t) = u_t(x, t-\rho\tau) \quad \text{for } (x,\rho,t) \in \Omega \times (0,1) \times (0,T).$$

Then, problem (1.1)–(1.5) reads as

$$\begin{aligned} & u_{tt} + \Delta^2 u - \int_0^t k(t-s)\Delta^2 u(s)\,ds + \alpha |u_t|^{m(\cdot)-2}u_t + \beta \bigl|y(x,1,t)\bigr|^{m(\cdot)-2} y(x,1,t) \\ & \quad = \bigl[u, \chi(u)\bigr] + \gamma |u|^{p(\cdot)-2}u \quad \text{in } \Omega \times (0,\infty), \end{aligned} \tag{2.7}$$

$$\Delta^2 \chi(u) = -[u,u] \quad \text{in } \Omega \times (0,\infty), \tag{2.8}$$

$$\tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0 \quad \text{for } (x,\rho,t) \in \Omega \times (0,1) \times (0,T), \tag{2.9}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \qquad \chi(u) = \frac{\partial \chi(u)}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2.10}$$

$$u(0) = u_0, \qquad u_t(0) = u_1 \quad \text{in } \Omega, \tag{2.11}$$

$$y(x,\rho,0) = j_0(x,-\rho\tau) := y_0 \quad \text{in } \Omega \times (0,1), \tag{2.12}$$

By virtue of the arguments of [4, 15, 27], we have the following local existence result.

Theorem 2.1 (Local existence) *Under* (A_1), (A_2), *and* (A_3), *problem* (2.7)–(2.12) *has a unique local solution* (u,y) *satisfying*

$$\begin{aligned} & u \in L^\infty\bigl(0,T;H_0^2(\Omega)\bigr), \qquad u_t \in L^\infty\bigl(0,T;L^2(\Omega)\bigr) \cap L^{m(\cdot)}\bigl(\Omega \times (0,T)\bigr), \\ & y \in L^\infty\bigl(0,T;L^{m(\cdot)}\bigl(\Omega \times (0,1)\bigr)\bigr) \end{aligned}$$

for every $(u_0,u_1,y_0) \in H_0^2(\Omega) \times L^2(\Omega) \times L^{m(\cdot)}(\Omega \times (0,1))$.

3 Global existence

In this section, we derive the global existence of solution to problem (2.7)–(2.12). In the proof of global existence and decay results, we will use the following lemma several times.

Lemma 3.1 *Let* r *be a continuous function on* $\overline{\Omega}$ *with* $2 \le r_1 \le r(x) \le r_2 < \infty$. *Then, for* $v \in H_0^2(\Omega)$, *it holds*

$$\int_\Omega |v|^{r(x)}\,dx \le \bigl(B_{r_1}^{r_1}\|\Delta v\|_2^{r_1-2} + B_{r_2}^{r_2}\|\Delta v\|_2^{r_2-2}\bigr)\|\Delta v\|_2^2. \tag{3.1}$$

Proof Using (2.1), we have

$$\begin{aligned} \int_\Omega |v|^{r(x)}\,dx &= \int_{\{x\in\Omega:|v(x)|<1\}} |v|^{r(x)}\,dx + \int_{\{x\in\Omega:|v(x)|\ge 1\}} |v|^{r(x)}\,dx \\ &\le \int_{\{x\in\Omega:|v(x)|<1\}} |v|^{r_1}\,dx + \int_{\{x\in\Omega:|v(x)|\ge 1\}} |v|^{r_2}\,dx \\ &\le \|v\|_{r_1}^{r_1} + \|v\|_{r_2}^{r_2} \le B_{r_1}^{r_1}\|\Delta v\|_2^{r_1} + B_{r_2}^{r_2}\|\Delta v\|_2^{r_2}. \end{aligned}$$ □

We define the energy E of the solution to (2.7)–(2.12) as

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t k(s)\,ds\right)\|\Delta u\|_2^2 + \frac{1}{4}\bigl\|\Delta\chi(u)\bigr\|_2^2 + \frac{1}{2}(k\Box\Delta u) \\ & + \frac{\xi\tau}{2}\int_\Omega\int_0^1 \bigl|y(x,\rho,t)\bigr|^{m(x)}\,d\rho\,dx - \gamma\int_\Omega \frac{|u|^{p(x)}}{p(x)}\,dx, \end{aligned} \tag{3.2}$$

where

$$(k\square\Delta u)(t) = \int_0^t k(t-s)\left\|\Delta u(t)-\Delta u(s)\right\|_2^2\,ds$$

and

$$\frac{2|\beta|(m_2-1)}{m_2} < \xi < \frac{2(\alpha m_1 - |\beta|)}{m_1}. \tag{3.3}$$

Let

$$J(t) = \frac{1}{2}\left(1-\int_0^t k(s)\,ds\right)\|\Delta u\|_2^2 + \frac{1}{2}(k\square\Delta u) - \frac{\gamma}{p_1}\int_\Omega |u|^{p(x)}\,dx \tag{3.4}$$

and

$$I(t) = \left(1-\int_0^t k(s)\,ds\right)\|\Delta u\|_2^2 + (k\square\Delta u) - \gamma\int_\Omega |u|^{p(x)}\,dx. \tag{3.5}$$

Then we see

$$J(t) = \left(\frac{1}{2}-\frac{1}{p_1}\right)\left\{\left(1-\int_0^t k(s)\,ds\right)\|\Delta u\|_2^2 + (k\square\Delta u)\right\} + \frac{1}{p_1}I(t) \tag{3.6}$$

and

$$E(t) \geq J(t) + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{4}\left\|\Delta\chi(u)\right\|_2^2 + \frac{\xi\tau}{2}\int_\Omega\int_0^1 \left|y(x,\rho,t)\right|^{m(x)}\,d\rho\,dx. \tag{3.7}$$

Lemma 3.2 *Let* (A_1), (A_2), *and* (A_3) *hold. Then there exists* $C_0 > 0$ *satisfying*

$$\begin{aligned} E'(t) &\leq -C_0\left(\int_\Omega |u_t|^{m(x)}\,dx + \int_\Omega \left|y(x,1,t)\right|^{m(x)}\,dx\right) + \frac{1}{2}\left(k'\square\Delta u\right) - \frac{k(t)}{2}\|\Delta u\|_2^2 \\ &\leq 0. \end{aligned} \tag{3.8}$$

Proof From (2.7), (2.8), (2.10), we have

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1-\int_0^t k(s)\,ds\right)\|\Delta u\|_2^2 + \frac{1}{4}\left\|\Delta\chi(u)\right\|_2^2 \right. \\ &\qquad \left. + \frac{1}{2}(k\square\Delta u) - \gamma\int_\Omega \frac{|u|^{p(x)}}{p(x)}\,dx\right\} \\ &\quad = -\alpha\int_\Omega |u_t|^{m(x)}\,dx - \beta\int_\Omega u_t\left|y(x,1,t)\right|^{m(x)-2}y(x,1,t)\,dx \\ &\qquad + \frac{1}{2}\left(k'\square\Delta u\right) - \frac{k(t)}{2}\|\Delta u\|_2^2. \end{aligned} \tag{3.9}$$

Using (2.9) and the equality $y(x,0,t) = u_t(x,t)$, we get

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\int_\Omega\int_0^1 \left|y(x,\rho,t)\right|^{m(x)}\,d\rho\,dx\right) \\ &\quad = \int_\Omega\int_0^1 m(x)\left|y(x,\rho,t)\right|^{m(x)-2}y(x,\rho,t)y_t(x,\rho,t)\,d\rho\,dx \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\tau}\int_\Omega\int_0^1 m(x)\left|y(x,\rho,t)\right|^{m(x)-2}y(x,\rho,t)y_\rho(x,\rho,t)\,d\rho\,dx \\
&= -\frac{1}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\left(\left|y(x,\rho,t)\right|^{m(x)}\right)d\rho\,dx \\
&= -\frac{1}{\tau}\int_\Omega\left|y(x,1,t)\right|^{m(x)}\,dx + \frac{1}{\tau}\int_\Omega |u_t|^{m(x)}\,dx.
\end{aligned}
\tag{3.10}
$$

Using Young's inequality with $\frac{m(x)-1}{m(x)} + \frac{1}{m(x)} = 1$ and the fact that the function $f(s) = \frac{s-1}{s}$ is increasing for $s > 0$, we find

$$
\begin{aligned}
&-\beta\int_\Omega u_t\left|y(x,1,t)\right|^{m(x)-2}y(x,1,t)\,dx \\
&\le |\beta|\int_\Omega \frac{|u_t|^{m(x)}}{m(x)}\,dx + |\beta|\int_\Omega \frac{m(x)-1}{m(x)}\left|y(x,1,t)\right|^{m(x)}\,dx \\
&\le \frac{|\beta|}{m_1}\int_\Omega |u_t|^{m(x)}\,dx + \frac{|\beta|(m_2-1)}{m_2}\int_\Omega\left|y(x,1,t)\right|^{m(x)}\,dx.
\end{aligned}
\tag{3.11}
$$

Combining (3.9), (3.10), and (3.11), we obtain

$$
\begin{aligned}
E'(t) \le &-\left(\alpha - \frac{\xi}{2} - \frac{|\beta|}{m_1}\right)\int_\Omega |u_t|^{m(x)}\,dx - \left(\frac{\xi}{2} - \frac{|\beta|(m_2-1)}{m_2}\right)\int_\Omega\left|y(x,1,t)\right|^{m(x)}\,dx \\
&+ \frac{1}{2}\left(k'\Box\Delta u\right) - \frac{k(t)}{2}\|\Delta u\|_2^2.
\end{aligned}
$$

From this, (3.3), and (A_2), we finish the proof. □

Lemma 3.3 *Let* (A_1), (A_2), (A_3) *hold. If*

$$
I(0) > 0 \quad \text{and} \quad B_{p_1}^{p_1}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_1-2}{2}} + B_{p_2}^{p_2}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_2-2}{2}} < \frac{k_l}{\gamma},
\tag{3.12}
$$

then

$$
I(t) > 0 \quad \text{for all } t \ge 0.
\tag{3.13}
$$

Proof Suppose that there exists $0 < t_0 \le T$ with $I(t_0) \le 0$. Let T_m be the first time satisfying

$$
I(T_m) = 0.
\tag{3.14}
$$

Then $I(t) \ge 0$ for $t \in [0, T_m]$. Thus, from (3.6), (3.7), and (3.8), we observe

$$
\begin{aligned}
k_l\|\Delta u\|_2^2 &\le \left(1 - \int_0^t k(s)\,ds\right)\|\Delta u\|_2^2 \\
&\le \frac{2p_1}{p_1-2}J(t) \le \frac{2p_1}{p_1-2}E(t) \le \frac{2p_1}{p_1-2}E(0)
\end{aligned}
\tag{3.15}
$$

for $t \in [0, T_m]$. Using (3.1), (3.15), and (3.12), we see

$$\begin{aligned}\gamma \int_\Omega |u|^{p(x)}\,dx &\le \gamma \left(B_{p_1}^{p_1}\|\Delta u\|_2^{p_1-2} + B_{p_2}^{p_2}\|\Delta u\|_2^{p_2-2}\right)\|\Delta u\|_2^2 \\ &\le \gamma \left\{ B_{p_1}^{p_1}\left(\frac{2p_1 E(0)}{k_l(p_1-2)}\right)^{\frac{p_1-2}{2}} + B_{p_2}^{p_2}\left(\frac{2p_1 E(0)}{k_l(p_1-2)}\right)^{\frac{p_2-2}{2}} \right\}\|\Delta u\|_2^2 \\ &< k_l\|\Delta u\|_2^2 \le \left(1 - \int_0^t k(s)\,ds\right)\|\Delta u\|_2^2 \end{aligned} \tag{3.16}$$

for $t \in [0, T_m]$, which implies

$$I(t) > 0 \quad \text{for } t \in [0, T_m].$$

This contradicts (3.14). □

Remark 3.1 Let the conditions of Lemma 3.3 hold. From the result of Lemma 3.3 and the same argument of (3.15), we also find

$$\|\Delta u(t)\|_2^2 < \frac{2p_1 E(t)}{k_l(p_1-2)} \le \frac{2p_1 E(0)}{k_l(p_1-2)}. \tag{3.17}$$

From Lemma 3.2 and Lemma 3.3, we have the global existence result.

Theorem 3.1 (Global existence) *Assume that* (A_1), (A_2), (A_3), *and* (3.12) *hold. Then the solution* (u, y) *to problem* (2.7)–(2.12) *is global.*

Proof It suffices to show that $\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta\chi(u)\|_2^2 + \int_\Omega\int_0^1 |y(x,\rho,t)|^{m(x)}\,d\rho\,dx$ is bounded independent of t. From (3.7), (3.6), (3.13), (3.8), we have

$$\begin{aligned}&\min\left\{\frac{1}{4}, \frac{\xi\tau}{2}\right\}\left(\|u_t\|_2^2 + \|\Delta\chi(u)\|_2^2 + \int_\Omega\int_0^1 |y(x,\rho,t)|^{m(x)}\,d\rho\,dx\right) \\ &\quad\le \frac{1}{2}\|u_t\|_2^2 + \frac{1}{4}\|\Delta\chi(u)\|_2^2 + \frac{\xi\tau}{2}\int_\Omega\int_0^1 |y(x,\rho,t)|^{m(x)}\,d\rho\,dx \\ &\quad< E(t) \le E(0).\end{aligned}$$

From this and (3.17), we have

$$\|u_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta\chi(u)\|_2^2 + \int_0^1\int_\Omega |y(x,\rho,t)|^{m(x)}\,dx\,d\rho \le \left(\frac{1}{c_0} + \frac{2p_1}{k_l(p_1-2)}\right)E(0),$$

where $c_0 = \min\{\frac{1}{4}, \frac{\xi\tau}{2}\}$. □

4 General decay results

In this section, we derive general decay results for both cases when $m_1 \ge 2$ and when $1 < m_1 < 2$ by following the ideas in [20] and [23] with some necessary modification.

First, we let

$$k_\eta(t) = \eta k(t) - k'(t) \quad \text{and} \quad \mathcal{C}_\eta = \int_0^\infty \frac{k^2(s)}{k_\eta(s)}\,ds \quad \text{for } 0 < \eta < 1,$$

then, by the arguments of [14, 23], we have the following lemma.

Lemma 4.1 *Let* (A_3) *be satisfied. Then, for* $v \in L^2_{\mathrm{loc}}([0,\infty), L^2(\Omega))$, *it holds*

$$\int_\Omega \left(\int_0^t k(t-s)\big(v(t)-v(s)\big)\,ds \right)^2 dx \le C_\eta (k_\eta \Box v)(t). \tag{4.1}$$

Now, we define

$$L(t) = NE(t) + N_1\Phi(t) + N_2\Psi(t) + \Upsilon(t),$$

where $N > 0$, $N_i > 0$, $i = 1,2$,

$$\Phi(t) = \int_\Omega u u_t\,dx,$$
$$\Psi(t) = -\int_0^t k(t-s)\int_\Omega \big(u(t)-u(s)\big)u_t\,dx\,ds,$$

and

$$\Upsilon(t) = \tau \int_\Omega \int_0^1 e^{-\rho\tau}\big|y(x,\rho,t)\big|^{m(x)}\,d\rho\,dx.$$

Lemma 4.2 *Assume that* (A_1), (A_2), (A_3), *and* (3.12) *hold. Then* $L(t)$ *is equivalent to* $E(t)$.

Proof From (3.6), (3.13), and (3.7), we have

$$\begin{aligned}
&\big|L(t)-NE(t)\big| \\
&\quad\le \frac{N_1+N_2}{2}\|u_t\|_2^2 + \frac{N_1B_2^2}{2}\|\Delta u\|_2^2 + \frac{N_2B_2^2(1-k_l)}{2}(k\Box\Delta u) \\
&\qquad + \tau\int_\Omega\int_0^1 \big|y(x,\rho,t)\big|^{m(x)}\,d\rho\,dx \\
&\quad\le \frac{N_1+N_2}{2}\|u_t\|_2^2 + \frac{N_2B_2^2(1-k_l)}{2}(k\Box\Delta u) \\
&\qquad + \frac{N_1B_2^2p_1}{k_l(p_1-2)}\left\{J(t) - \frac{1}{p_1}I(t) - \frac{p_1-2}{2p_1}(k\Box\Delta u)\right\} + \tau\int_\Omega\int_0^1 \big|y(x,\rho,t)\big|^{m(x)}\,d\rho\,dx \\
&\quad\le \frac{N_1+N_2}{2}\|u_t\|_2^2 + \frac{N_2B_2^2(1-k_l)}{2}(k\Box\Delta u) + \frac{N_1B_2^2p_1}{k_l(p_1-2)}J(t) \\
&\qquad + \tau\int_\Omega\int_0^1 \big|y(x,\rho,t)\big|^{m(x)}\,d\rho\,dx \\
&\quad\le \frac{N_1+N_2}{2}\|u_t\|_2^2 + \left(\frac{2N_2B_2^2(1-k_l)p_1}{2(p_1-2)} + \frac{N_1B_2^2p_1}{k_l(p_1-2)}\right)J(t) \\
&\qquad + \tau\int_\Omega\int_0^1 \big|y(x,\rho,t)\big|^{m(x)}\,d\rho\,dx \\
&\quad\le \max\left\{N_1+N_2, \frac{2N_2B_2^2(1-k_l)p_1}{2(p_1-2)} + \frac{N_1B_2^2p_1}{k_l(p_1-2)}, \frac{2}{\xi}\right\}E(t).
\end{aligned}$$

Taking $N > \max\{N_1+N_2, \frac{2N_2B_2^2(1-k_l)p_1}{2(p_1-2)} + \frac{N_1B_2^2p_1}{k_l(p_1-2)}, \frac{2}{\xi}\}$, we finish the proof. □

Lemma 4.3 *The function* Υ *satisfies*

$$\Upsilon'(t) \le \int_\Omega |u_t|^{m(x)}\,dx - \tau e^{-\tau}\int_\Omega\int_0^1 \bigl|y(x,\rho,t)\bigr|^{m(x)}\,d\rho\,dx. \tag{4.2}$$

Proof Using (2.9) and $y(x,0,t)=u_t(x,t)$, we get

$$\begin{aligned}
\Upsilon'(t) &= \tau\int_\Omega\int_0^1 e^{-\rho\tau} m(x)\bigl|y(x,\rho,t)\bigr|^{m(x)-2} y(x,\rho,t)y_t(x,\rho,t)\,d\rho\,dx \\
&= -\int_\Omega\int_0^1 e^{-\rho\tau}\frac{\partial}{\partial\rho}\bigl|y(x,\rho,t)\bigr|^{m(x)}\,d\rho\,dx \\
&= -\int_\Omega e^{-\tau}\bigl|y(x,1,t)\bigr|^{m(x)}\,dx + \int_\Omega \bigl|y(x,0,t)\bigr|^{m(x)}\,dx \\
&\quad - \tau\int_\Omega\int_0^1 e^{-\rho\tau}\bigl|y(x,\rho,t)\bigr|^{m(x)}\,d\rho\,dx \\
&\le \int_\Omega |u_t|^{m(x)}\,dx - \tau e^{-\tau}\int_\Omega\int_0^1 \bigl|y(x,\rho,t)\bigr|^{m(x)}\,d\rho\,dx.
\end{aligned}$$
□

Lemma 4.4 *Let* $g(t)=\int_t^\infty k(s)\,ds$. *The following function*

$$\Lambda(t) = \int_0^t g(t-s)\bigl\|\Delta u(s)\bigr\|_2^2\,ds$$

satisfies

$$\Lambda'(t) \le 2(1-k_l)\|\Delta u\|_2^2 - \frac{1}{2}(k\Box\Delta u). \tag{4.3}$$

Proof Noting $g'(t)=-k(t)$ and using Young's inequality, we see

$$\begin{aligned}
\Lambda'(t) &= g(0)\|\Delta u\|_2^2 - \int_0^t k(t-s)\bigl\|\Delta u(s)\bigr\|_2^2\,ds \\
&= \biggl(\int_0^\infty k(s)\,ds\biggr)\|\Delta u\|_2^2 - (k\Box\Delta u) - \int_0^t k(t-s)\bigl\|\Delta u(t)\bigr\|_2^2\,ds \\
&\quad - 2\int_\Omega\int_0^t k(t-s)\Delta u(t)\bigl(\Delta u(s)-\Delta u(t)\bigr)\,ds\,dx \\
&\le \biggl(\int_t^\infty k(s)\,ds\biggr)\|\Delta u\|_2^2 - \frac{1}{2}(k\Box\Delta u) + 2\biggl(\int_0^t k(s)\,ds\biggr)\|\Delta u\|_2^2 \\
&\le 2(1-k_l)\|\Delta u\|_2^2 - \frac{1}{2}(k\Box\Delta u).
\end{aligned}$$
□

From here, c and C_i denote generic constants, $c_\delta>0$ denotes a generic constant depending on $\delta>0$, and $c_\delta(x) = \frac{m(x)-1}{\delta^{\frac{1}{m(x)-1}}(m(x))^{\frac{m(x)}{m(x)-1}}}$. We note that $c_\delta(x)$ is bounded on Ω for fixed $\delta>0$, that is, $|c_\delta(x)|\le c_\delta$ for all $x\in\Omega$.

4.1 General decay for the case $m_1 \ge 2$

In this subsection, we derive a general decay result for the case $m_1\ge 2$.

Lemma 4.5 *Assume that* (A_1), (A_2), (A_3), *and* (3.12) *hold. If* $m_1 \geq 2$, *then* Φ *satisfies*

$$\Phi'(t) \leq \|u_t\|_2^2 - \frac{k_l}{4}\|\Delta u\|_2^2 - \big\|\Delta\chi(u)\big\|_2^2 + \gamma\int_\Omega |u|^{p(x)}\,dx + \frac{\mathcal{C}_\eta}{2k_l}(k_\eta\Box\Delta u) + \alpha C_1\int_\Omega |u_t|^{m(x)}\,dx + |\beta|C_1\int_\Omega \big|y(x,1,t)\big|^{m(x)}\,dx. \tag{4.4}$$

Proof Using (2.7)–(2.12), we get

$$\begin{aligned}\Phi'(t) &= \|u_t\|_2^2 - \left(1-\int_0^t k(s)\,ds\right)\|\Delta u\|_2^2 - \big\|\Delta\chi(u)\big\|_2^2 + \gamma\int_\Omega |u|^{p(x)}\,dx \\ &\quad + \int_0^t k(t-s)\int_\Omega \big(\Delta u(s)-\Delta u(t)\big)\Delta u(t)\,dx\,ds \\ &\quad - \alpha\int_\Omega u|u_t|^{m(x)-2}u_t\,dx - \beta\int_\Omega u\big|y(x,1,t)\big|^{m(x)-2}y(x,1,t)\,dx.\end{aligned} \tag{4.5}$$

Using Young's inequality and (4.1), we have

$$\begin{aligned}&\int_0^t k(t-s)\int_\Omega \big(\Delta u(s)-\Delta u(t)\big)\Delta u(t)\,dx\,ds \\ &\quad \leq \frac{k_l}{2}\|\Delta u\|_2^2 + \frac{1}{2k_l}\int_\Omega\left(\int_0^t k(t-s)\big(\Delta u(s)-\Delta u(t)\big)\,ds\right)^2 dx \\ &\quad \leq \frac{k_l}{2}\|\Delta u\|_2^2 + \frac{\mathcal{C}_\eta}{2k_l}(k_\eta\Box\Delta u).\end{aligned} \tag{4.6}$$

From (4.5) and (4.6), one sees

$$\begin{aligned}\Phi'(t) &\leq \|u_t\|_2^2 - \frac{k_l}{2}\|\Delta u\|_2^2 - \big\|\Delta\chi(u)\big\|_2^2 + \gamma\int_\Omega |u|^{p(x)}\,dx + \frac{\mathcal{C}_\eta}{2k_l}(k_\eta\Box\Delta u) \\ &\quad \underbrace{-\alpha\int_\Omega u|u_t|^{m(x)-2}u_t\,dx}_{I_1}\ \underbrace{-\beta\int_\Omega u\big|y(x,1,t)\big|^{m(x)-2}y(x,1,t)\,dx}_{I_2}.\end{aligned} \tag{4.7}$$

Using Young's inequality with $\frac{1}{m(x)} + \frac{m(x)-1}{m(x)} = 1$, (3.1), and (3.17), we find

$$\begin{aligned}-I_1 &\leq \alpha\delta_1\int_\Omega |u|^{m(x)}\,dx + \alpha\int_\Omega c_{\delta_1}(x)|u_t|^{m(x)}\,dx \\ &\leq \alpha\delta_1 C_{E(0)}\|\Delta u\|_2^2 + \alpha\int_\Omega c_{\delta_1}(x)|u_t|^{m(x)}\,dx\end{aligned} \tag{4.8}$$

for $\delta_1 > 0$, where $C_{E(0)} = B_{m_1}^{m_1}\big(\frac{2p_1E(0)}{k_l(p_1-2)}\big)^{\frac{m_1-2}{2}} + B_{m_2}^{m_2}\big(\frac{2p_1E(0)}{k_l(p_1-2)}\big)^{\frac{m_2-2}{2}}$.

Similarly, using (2.5), we have

$$\begin{aligned}-I_2 &\leq |\beta|\delta_1 C_{E(0)}\|\Delta u\|_2^2 + |\beta|\int_\Omega c_{\delta_1}(x)\big|y(x,1,t)\big|^{m(x)}\,dx \\ &\leq \alpha\delta_1 C_{E(0)}\|\Delta u\|_2^2 + |\beta|\int_\Omega c_{\delta_1}(x)\big|y(x,1,t)\big|^{m(x)}\,dx.\end{aligned} \tag{4.9}$$

Combining (4.7), (4.8), (4.9) and taking $\delta_1 = \frac{k_l}{8\alpha C_{E(0)}}$, we obtain (4.4). □

Lemma 4.6 *Assume that* (A_1), (A_2), (A_3), *and* (3.12) *hold. If* $m_1 \geq 2$, *then* Ψ *satisfies*

$$\Psi'(t) \leq -\left(\int_0^t k(s)\,ds - \delta\right)\|u_t\|_2^2 + \delta C_3\|\Delta u\|_2^2 + \left(\frac{C_4(1+\mathcal{C}_\eta)}{\delta} + \mathcal{C}_\eta\right)(k_\eta\Box\Delta u)$$
$$+ \delta C_5(k\Box\Delta u) + \alpha\int_\Omega c_\delta(x)|u_t|^{m(x)}\,dx + |\beta|\int_\Omega c_\delta(x)\big|y(x,1,t)\big|^{m(x)}\,dx \tag{4.10}$$

for any $\delta > 0$.

Proof Using (2.7)–(2.12), we have

$$\begin{aligned}
\Psi'(t) &= -\left(\int_0^t k(s)\,ds\right)\|u_t\|_2^2 - \int_\Omega u_t\int_0^t k'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad + \left(1 - \int_0^t k(s)\,ds\right)\int_\Omega \Delta u\int_0^t k(t-s)\big(\Delta u(t)-\Delta u(s)\big)\,ds\,dx \\
&\quad + \int_\Omega\left(\int_0^t k(t-s)\big(\Delta u(t)-\Delta u(s)\big)\,ds\right)^2 dx \\
&\quad - \int_\Omega\big[u,\chi(u)\big]\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad - \gamma\int_\Omega |u|^{p(x)-2}u\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad + \alpha\int_\Omega |u_t|^{m(x)-2}u_t\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad + \beta\int_\Omega \big|y(x,1,t)\big|^{m(x)-2}y(x,1,t)\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&:= -\left(\int_0^t k(s)\,ds\right)\|u_t\|_2^2 + \sum_{i=1}^5 J_i \\
&\quad \underbrace{+ \alpha\int_\Omega |u_t|^{m(x)-2}u_t\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds\,dx}_{J_6} \\
&\quad \underbrace{+ \beta\int_\Omega \big|y(x,1,t)\big|^{m(x)-2}y(x,1,t)\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds\,dx}_{J_7}.
\end{aligned} \tag{4.11}$$

Using Young's inequality, $k' = \eta k - k_\eta$, (4.1) we get

$$\begin{aligned}
|J_1| &\leq \delta\|u_t\|_2^2 + \frac{1}{4\delta}\left\|\int_0^t k'(t-s)\big(u(t)-u(s)\big)\,ds\right\|_2^2 \\
&\leq \delta\|u_t\|_2^2 + \frac{1}{2\delta}\left(\left\|\int_0^t k_\eta(t-s)\big(u(t)-u(s)\big)\,ds\right\|_2^2 + \eta^2\left\|\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds\right\|_2^2\right) \\
&\leq \delta\|u_t\|_2^2 + \frac{1}{2\delta}\left\{\left(\int_0^t k_\eta(s)\,ds\right)(k_\eta\Box u) + \eta^2\mathcal{C}_\eta(k_\eta\Box u)\right\} \\
&\leq \delta\|u_t\|_2^2 + \frac{B_2^2(\eta(1-k_l)+k(0))}{2\delta}(k_\eta\Box\Delta u) + \frac{B_2^2\mathcal{C}_\eta}{2\delta}(k_\eta\Box\Delta u),
\end{aligned} \tag{4.12}$$

$$|J_2| \le \delta \|\Delta u\|_2^2 + \frac{\mathcal{C}_\eta}{4\delta}(k_\eta \Box \Delta u), \tag{4.13}$$

and

$$|J_3| \le \mathcal{C}_\eta (k_\eta \Box \Delta u). \tag{4.14}$$

Using (3.17), we infer

$$\begin{aligned}
|J_4| &\le c\|u\|_{H^2(\Omega)} \|\chi(u)\|_{W^{2,\infty}(\Omega)} \left\| \int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds \right\|_2 \\
&\le cB_2 \|\Delta u\|_2^3 \big(\mathcal{C}_\eta (k_\eta \Box \Delta u)\big)^{\frac{1}{2}} \\
&\le cB_2 \left(\frac{2p_1 E(0)}{k_l(p_1-2)} \right) \|\Delta u\|_2 \big(\mathcal{C}_\eta (k_\eta \Box \Delta u)\big)^{\frac{1}{2}} \\
&\le \delta \|\Delta u\|_2^2 + \frac{c}{4\delta} \mathcal{C}_\eta (k_\eta \Box \Delta u),
\end{aligned} \tag{4.15}$$

here we used the Karman bracket property (see p. 270 in [10])

$$\big\|[v_1, v_2]\big\| \le c\|v_1\|_{H^2(\Omega)} \|v_2\|_{W^{2,\infty}(\Omega)} \quad \text{for } v_1 \in H^2(\Omega), v_2 \in W^{2,\infty}(\Omega)$$

and

$$\big\|\chi(u)\big\|_{W^{2,\infty}(\Omega)} \le c\|u\|_{H^2(\Omega)}^2.$$

Using (3.1), (3.17), (4.1), we deduce

$$\begin{aligned}
|J_5| &\le \gamma\delta \int_\Omega |u|^{2(p(x)-1)}\,dx + \frac{\gamma}{4\delta} \int_\Omega \left(\int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds \right)^2 dx \\
&\le \gamma\delta \overline{C}_{E(0)} \|\Delta u\|_2^2 + \frac{\gamma B_2^2 \mathcal{C}_\eta}{4\delta} (k_\eta \Box \Delta u)),
\end{aligned} \tag{4.16}$$

where $\overline{C}_{E(0)} = B_{2(p_1-1)}^{2(p_1-1)} (\frac{2p_1 E(0)}{k_l(p_1-2)})^{\frac{2p_1-4}{2}} + B_{2(p_2-1)}^{2(p_2-1)} (\frac{2p_1 E(0)}{k_l(p_1-2)})^{\frac{2p_2-4}{2}}$.

Using the similar calculation of (4.8) and Hölder's inequality, we get

$$\begin{aligned}
|J_6| &\le \alpha\delta \int_\Omega \left| \int_0^t k(t-s)\big(u(t)-u(s)\big)\,ds \right|^{m(x)} dx + \alpha \int_\Omega c_\delta(x) |u_t|^{m(x)}\,dx \\
&\le \alpha\delta \int_\Omega \left(\int_0^t k(s)\,ds \right)^{m(x)-1} \left(\int_0^t k(t-s)\big|u(t)-u(s)\big|^{m(x)}\,ds \right) dx \\
&\quad + \alpha \int_\Omega c_\delta(x) |u_t|^{m(x)}\,dx \\
&\le \alpha\delta (1-k_l)^{m_1-1} \int_\Omega \left(\int_0^t k(t-s)\big|u(t)-u(s)\big|^{m(x)}\,ds \right) dx + \alpha \int_\Omega c_\delta(x)|u_t|^{m(x)}\,dx \\
&\le \alpha\delta (1-k_l)^{m_1-1} \hat{C}_{E(0)} \int_0^t k(t-s) \|\Delta u(t) - \Delta u(s)\|_2^2\,ds + \alpha \int_\Omega c_\delta(x) |u_t|^{m(x)}\,dx \\
&= \alpha\delta (1-k_l)^{m_1-1} \hat{C}_{E(0)} (k \Box \Delta u) + \alpha \int_\Omega c_\delta(x) |u_t|^{m(x)}\,dx,
\end{aligned} \tag{4.17}$$

where $\hat{C}_{E(0)} = B_{m_1}^{m_1} (\frac{8p_1 E(0)}{k_l(p_1-2)})^{\frac{m_1-2}{2}} + B_{m_2}^{m_2} (\frac{8p_1 E(0)}{k_l(p_1-2)})^{\frac{m_2-2}{2}}$.

Similarly, we also have

$$|J_7| \le |\beta|\delta(1-k_l)^{m_1-1}\hat{C}_{E(0)}(k\Box\Delta u) + |\beta|\int_\Omega c_\delta(x)\big|y(x,1,t)\big|^{m(x)}\,dx. \tag{4.18}$$

Applying the estimates of J_i to (4.11), we obtain (4.10). □

Lemma 4.7 *Assume that* (A_1), (A_2), (A_3), *and* (3.12) *hold. Moreover, we assume that*

$$B_{p_1}^{p_1}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_1-2}{2}} + B_{p_2}^{p_2}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_2-2}{2}} < \frac{k_l}{4\gamma}. \tag{4.19}$$

If $m_1 \ge 2$, *then there exists* $\lambda > 0$ *such that*

$$L'(t) \le -\lambda E(t) - 3(1-k_l)\|\Delta u\|_2^2 + \frac{1}{2}(k\Box\Delta u) \quad \textit{for } t \ge k^{-1}(\varepsilon). \tag{4.20}$$

Proof From (A_3), there exists $t_\varepsilon > 0$ with $k(t_\varepsilon) = \varepsilon$, that is, $t_\varepsilon = k^{-1}(\varepsilon)$. We put $\int_0^{t_\varepsilon} k(s)\,ds = k_\varepsilon$. Using (3.8), (4.4), (4.10), (4.2), and $k' = \eta k - k_\eta$, we have

$$\begin{aligned}
L'(t) \le{}& -\left\{N_2\left(\int_0^t k(s)\,ds - \delta\right) - N_1\right\}\|u_t\|_2^2 - \left\{\frac{N_1k_l}{4} - N_2\delta C_3\right\}\|\Delta u\|_2^2 \\
&- N_1\big\|\Delta\chi(u)\big\|_2^2 + N_1\gamma\int_\Omega |u|^{p(x)}\,dx + \left(\frac{N\eta}{2} + N_2\delta C_5\right)(k\Box\Delta u) \\
&- \left\{\frac{N}{4} - \frac{N_2C_4}{\delta} + \frac{N}{4} - \mathcal{C}_\eta\left(\frac{N_1}{2k_l} + \frac{N_2C_4}{\delta} + N_2\right)\right\}(k_\eta\Box\Delta u) \\
&- \int_\Omega\big(NC_0 - N_1\alpha C_1 - N_2\alpha c_\delta(x) - 1\big)|u_t|^{m(x)}\,dx \\
&- \int_\Omega\big(NC_0 - N_1|\beta|C_1 - N_2|\beta|c_\delta(x)\big)\big|y(x,1,t)\big|^{m(x)}\,dx \\
&- \tau e^{-\tau}\int_\Omega\int_0^1\big|y(x,\rho,t)\big|^{m(x)}\,d\rho\,dx \\
\le{}& -\lambda E(t) - \left\{N_2(k_\varepsilon - \delta) - N_1 - \frac{\lambda}{2}\right\}\|u_t\|_2^2 \\
&- \left\{\frac{N_1k_l}{4} - N_2\delta C_3 - \frac{\lambda}{2}\left(1 - \int_0^t k(s)\,ds\right)\right\}\|\Delta u\|_2^2 - \left(N_1 - \frac{\lambda}{4}\right)\big\|\Delta\chi(u)\big\|_2^2 \\
&+ \gamma\left(N_1 - \frac{\lambda}{p_2}\right)\int_\Omega |u|^{p(x)}\,dx + \left(\frac{N\eta}{2} + N_2\delta C_5 + \frac{\lambda}{2}\right)(k\Box\Delta u) \\
&- \left\{\frac{N}{4} - \frac{N_2C_4}{\delta} + \frac{N}{4} - \mathcal{C}_\eta\left(\frac{N_1}{2k_l} + \frac{N_2C_4}{\delta} + N_2\right)\right\}(k_\eta\Box\Delta u) \\
&- \int_\Omega\big(NC_0 - N_1\alpha C_1 - N_2\alpha c_\delta(x) - 1\big)|u_t|^{m(x)}\,dx \\
&- \int_\Omega\big(NC_0 - N_1|\beta|C_1 - N_2|\beta|c_\delta(x)\big)\big|y(x,1,t)\big|^{m(x)}\,dx \\
&- \left(\tau e^{-\tau} - \frac{\lambda\xi\tau}{2}\right)\int_\Omega\int_0^1\big|y(x,\rho,t)\big|^{m(x)}\,d\rho\,dx
\end{aligned} \tag{4.21}$$

for $\lambda > 0$ and $t \geq t_\varepsilon$. Using estimate (3.16) and taking $\delta = \frac{k_l}{4N_2C_5}$, we get

$$\begin{aligned}
L'(t) \leq{}& -\lambda E(t) - \left\{N_2k_\varepsilon - \frac{k_l}{4C_5} - N_1 - \frac{\lambda}{2}\right\}\|u_t\|_2^2 \\
&- \left[N_1\left\{\frac{k_l}{4} - \gamma\left(B_{p_1}^{p_1}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_1-2}{2}} + B_{p_2}^{p_2}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_2-2}{2}}\right)\right\}\right. \\
&\left. - \frac{k_lC_3}{4C_5} - \frac{\lambda}{2}\right]\|\Delta u\|_2^2 \\
&- \left(N_1 - \frac{\lambda}{4}\right)\|\Delta\chi(u)\|_2^2 + \left(\frac{N\eta}{2} + \frac{k_l}{4} + \frac{\lambda}{2}\right)(k\Box\Delta u) \\
&- \left\{\frac{N}{4} - \frac{4N_2^2C_4C_5}{k_l} + \frac{N}{4} - \mathcal{C}_\eta\left(\frac{N_1}{2k_l} + \frac{4N_2^2C_4C_5}{k_l} + N_2\right)\right\}(k_\eta\Box\Delta u) \\
&- (NC_0 - N_1\alpha C_1 - N_2\alpha c_\delta - 1)\int_\Omega |u_t|^{m(x)}\,dx \\
&- \left(NC_0 - N_1|\beta|C_1 - N_2|\beta|c_\delta\right)\int_\Omega \left|y(x,1,t)\right|^{m(x)}\,dx \\
&- \left(\tau e^{-\tau} - \frac{\lambda\xi\tau}{2}\right)\int_\Omega\int_0^1 \left|y(x,\rho,t)\right|^{m(x)}\,d\rho\,dx \quad \text{for } t \geq t_\varepsilon.
\end{aligned} \tag{4.22}$$

From (4.19), we know

$$\frac{k_l}{4} - \gamma\left(B_{p_1}^{p_1}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_1-2}{2}} + B_{p_2}^{p_2}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_2-2}{2}}\right) > 0.$$

Firstly, we take $N_1 > \frac{\lambda}{4}$ large enough to get

$$\begin{aligned}
&N_1\left\{\frac{k_l}{4} - \gamma\left(B_{p_1}^{p_1}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_1-2}{2}} + B_{p_2}^{p_2}\left(\frac{2p_1E(0)}{k_l(p_1-2)}\right)^{\frac{p_2-2}{2}}\right)\right\} - \frac{k_lC_3}{4C_5} \\
&\quad > 4(1-k_l),
\end{aligned} \tag{4.23}$$

and then choose $N_2 > 0$ satisfying

$$N_2k_\varepsilon - \frac{k_l}{4C_5} - N_1 > 1. \tag{4.24}$$

Noting $\frac{\eta k^2(s)}{k_\eta(s)} < k(s)$ and making use of the Lebesgue dominated convergence theorem, we have

$$\lim_{\eta\to 0^+}\eta\mathcal{C}_\eta = \lim_{\eta\to 0^+}\int_0^\infty \frac{\eta k^2(s)}{k_\eta(s)}\,ds = 0.$$

Thus, there exists $0 < \eta_0 < 1$ satisfying

$$\eta\mathcal{C}_\eta < \frac{1}{16(\frac{N_1}{2k_l} + \frac{4N_2^2C_4C_5}{k_l} + N_2)} \quad \text{for } \eta < \eta_0. \tag{4.25}$$

Secondly, we take $N > 0$ large enough again to get

$$\frac{1}{4N} < \eta_0, \qquad \frac{N}{4} - \frac{4N_2^2 C_4 C_5}{k_l} > 0, \tag{4.26}$$

$$NC_0 - N_1 \alpha C_1 - N_2 \alpha c_\delta - 1 > 0, \tag{4.27}$$

and

$$NC_0 - N_1 |\beta| C_1 - N_2 |\beta| c_\delta > 0. \tag{4.28}$$

Thirdly, selecting $\eta = \frac{1}{4N} < \eta_0$, we get

$$\frac{N\eta}{2} + \frac{k_l}{4} = \frac{1}{8} + \frac{k_l}{4} < \frac{3}{8} \tag{4.29}$$

and

$$\frac{N}{4} - C_\eta \left(\frac{N_1}{2k_l} + \frac{4N_2^2 C_4 C_5}{k_l} + N_2 \right) > \frac{N}{4} - \frac{1}{16\eta} = 0, \tag{4.30}$$

here we used (4.25). From (4.22), (4.23), (4.24), (4.26), (4.27), (4.28), (4.29), and (4.30), we get

$$\begin{aligned} L'(t) \le{} & -\lambda E(t) - \left\{ 1 - \frac{\lambda}{2} \right\} \|u_t\|_2^2 - \left\{ 4(1 - k_l) - \frac{\lambda}{2} \right\} \|\Delta u\|_2^2 \\ & + \left(\frac{3}{8} + \frac{\lambda}{2} \right) (k \square \Delta u) - \left(\tau e^{-\tau} - \frac{\lambda \xi \tau}{2} \right) \int_\Omega \int_0^1 \left| y(x, \rho, t) \right|^{m(x)} d\rho \, dx \end{aligned}$$

for $t \ge t_\varepsilon$. Finally, selecting $\lambda > 0$ satisfying

$$\lambda \le \min \left\{ 2(1 - k_l), \frac{1}{4}, \frac{2\tau e^{-\tau}}{\xi \tau} \right\},$$

we obtain (4.20). □

Lemma 4.8 *Assume that* (A_1), (A_2), (A_3), *and* (4.19) *hold and let* $m_1 \ge 2$. *Then*

$$0 < \int_0^\infty E(s) \, ds < \infty. \tag{4.31}$$

Proof From (4.20) and (4.3), we see

$$\big(L(t) + \Lambda(t) \big)' \le -\lambda E(t) \quad \text{for } t \ge t_\varepsilon, \tag{4.32}$$

and

$$0 < \int_{t_\varepsilon}^t E(s) \, ds \le -\frac{1}{\lambda} \int_{t_\varepsilon}^t \big(L'(s) + \Lambda'(s) \big) \, ds \le \frac{L(t_\varepsilon) + \Lambda(t_\varepsilon)}{\lambda} < \infty, \quad \forall t \ge t_\varepsilon,$$

which gives

$$0 < \int_0^\infty E(s)\,ds = \int_0^{t_\varepsilon} E(s)\,ds + \int_{t_\varepsilon}^\infty E(s)\,ds < \infty. \qquad \square$$

Theorem 4.1 *Assume that* (A_1), (A_2), (A_3), *and* (4.19) *hold and let* $m_1 \geq 2$. *Then there exist* $c_i, \omega_i > 0$, $i = 1, 2$, *such that, for* $t \geq k^{-1}(\varepsilon)$,

$$E(t) \leq c_1 \exp\left(-\omega_1 \int_{k^{-1}(\varepsilon)}^t \zeta(s)\,ds\right) \quad \text{in the case } K \text{ is linear} \tag{4.33}$$

and

$$E(t) \leq c_2 \tilde{K}^{-1}\left(\omega_2 \int_{k^{-1}(\varepsilon)}^t \zeta(s)\,ds\right) \quad \text{in the case } K \text{ is nonlinear},$$

where

$$\tilde{K}(s) = \int_s^\varepsilon \frac{1}{\tau K'(\tau)}\,d\tau. \tag{4.34}$$

Proof From Lemma 4.5, Lemma 4.6, and Lemma 4.7, the proof is similar to that of [23]. But, for the completeness, we give the proof. Since k and ζ are continuous in t, we have

$$a_1 \leq \zeta(t) K\big(k(t)\big) \leq a_2 \quad \text{for } t \in [0, t_\varepsilon]$$

for some $a_1, a_2 > 0$, and

$$k'(t) \leq -\zeta(t) K\big(k(t)\big) \leq -a_1 \leq -\frac{a_1}{k(0)} k(t) \quad \text{for } t \in [0, t_\varepsilon]. \tag{4.35}$$

From (4.20), (4.35), (3.8), we get

$$\begin{aligned} L'(t) &\leq -\lambda E(t) - \frac{k(0)}{2a_1}\big(k' \Box \Delta u\big) + \frac{1}{2} \int_{t_\varepsilon}^t k(s) \big\| \Delta u(t) - \Delta u(t-s) \big\|_2^2\,ds \\ &\leq -\lambda E(t) - \frac{k(0)}{a_1} E'(t) + \frac{1}{2} \int_{t_\varepsilon}^t k(s) \big\| \Delta u(t) - \Delta u(t-s) \big\|_2^2\,ds \quad \text{for } t \geq t_\varepsilon. \end{aligned} \tag{4.36}$$

Let

$$R(t) = L(t) + \frac{k(0)}{a_1} E(t),$$

then $R \sim E$ and

$$R'(t) \leq -\lambda E(t) + \frac{1}{2} \int_{t_\varepsilon}^t k(s) \big\| \Delta u(t) - \Delta u(t-s) \big\|_2^2\,ds \quad \text{for } t \geq t_\varepsilon. \tag{4.37}$$

Case 1: K is linear, that is, $K(s) = as$ *for some* $a > 0$. Put

$$\mathcal{R}_1(t) = \zeta(t) R(t) + \frac{1}{a} E(t).$$

From (4.37), (2.6), and (3.8), we have

$$\begin{aligned}\mathcal{R}_1'(t) &\le -\lambda\zeta(t)E(t) + \frac{1}{2}\int_{t_\varepsilon}^t \zeta(s)k(s)\big\|\Delta u(t) - \Delta u(t-s)\big\|_2^2\,ds + \frac{1}{a}E'(t) \\ &\le -\lambda\zeta(t)E(t) - \frac{1}{2a}\int_{t_\varepsilon}^t k'(s)\big\|\Delta u(t) - \Delta u(t-s)\big\|^2\,ds + \frac{1}{a}E'(t) \\ &\le -\lambda\zeta(t)E(t), \quad t \ge t_\varepsilon. \end{aligned} \tag{4.38}$$

This and the relation $\mathcal{R}_1(t) \sim E(t)$ prove (4.33).

Case 2: K is nonlinear. For $t \ge t_\varepsilon$, we put

$$\Gamma_1(t) = a_3\int_{t_\varepsilon}^t \big\|\Delta u(t) - \Delta u(t-s)\big\|_2^2\,ds$$

and

$$\Gamma_2(t) = -\int_{t_\varepsilon}^t k'(s)\big\|\Delta u(t) - \Delta u(t-s)\big\|_2^2\,ds.$$

From (3.17), (3.8), (4.31), we get

$$\begin{aligned}\int_{t_\varepsilon}^t \big\|\Delta u(t) - \Delta u(t-s)\big\|_2^2\,ds &\le 2\int_{t_\varepsilon}^t \big\|\Delta u(t)\big\|_2^2 + \big\|\Delta u(t-s)\big\|_2^2\,ds \\ &\le \frac{4p_1}{k_l(p_1-2)}\int_{t_\varepsilon}^t \big(E(t) + E(t-s)\big)\,ds \\ &\le \frac{4p_1}{k_l(p_1-2)}\left(\int_{t_\varepsilon}^t E(s)\,ds + \int_0^{t-t_\varepsilon} E(s)\right)ds) < \infty. \end{aligned} \tag{4.39}$$

Thus, there exists $0 < a_3 < 1$ satisfying

$$\Gamma_1(t) < 1 \quad \text{for } t \ge t_\varepsilon. \tag{4.40}$$

From (3.8), we know

$$\Gamma_2(t) \le -\big(k'\Box\Delta u\big)(t) \le -2E'(t). \tag{4.41}$$

Using (A_3), (4.40), the relation $\overline{K}(\varrho t) \le \varrho\overline{K}(t)$ for $0 \le \varrho \le 1$ and $t \in [0,\infty)$, and Jensen's inequality, we find

$$\begin{aligned}\Gamma_2(t) &= -\frac{1}{a_3\Gamma_1(t)}\int_{t_\varepsilon}^t \Gamma_1(t)k'(s)a_3\big\|\Delta u(t) - \Delta u(t-s)\big\|_2^2\,ds \\ &\ge \frac{1}{a_3\Gamma_1(t)}\int_{t_\varepsilon}^t \Gamma_1(t)\zeta(s)K\big(k(s)\big)a_3\big\|\Delta u(t) - \Delta u(t-s)\big\|_2^2\,ds \\ &\ge \frac{\zeta(t)}{a_3\Gamma_1(t)}\int_{t_\varepsilon}^t \overline{K}\big(\Gamma_1(t)k(s)\big)a_3\big\|\Delta u(t) - \Delta u(t-s)\big\|_2^2\,ds \\ &\ge \frac{\zeta(t)}{a_3}\overline{K}\left(a_3\int_{t_\varepsilon}^t k(s)\big\|\Delta u(t) - \Delta u(t-s)\big\|_2^2\,ds\right), \quad t \ge t_\varepsilon. \end{aligned} \tag{4.42}$$

So, we have

$$\int_{t_\varepsilon}^{t} k(s)\left\|\Delta u(t)-\Delta u(t-s)\right\|_2^2\,ds \le \frac{1}{a_3}\overline{K}^{-1}\left(\frac{a_3\Gamma_2(t)}{\zeta(t)}\right).$$

Applying this to (4.37), we get

$$R'(t) \le -\lambda E(t) + \frac{1}{2a_3}\overline{K}^{-1}\left(\frac{a_3\Gamma_2(t)}{\zeta(t)}\right) \quad \text{for } t \ge t_\varepsilon. \tag{4.43}$$

We know that the convex function $\overline{K}$ satisfies

$$st \le \overline{K}^*(s) + \overline{K}(t) \quad \text{for } s,t \ge 0 \tag{4.44}$$

and

$$\overline{K}^*(s) = s\left(\overline{K}'\right)^{-1}(s) - \overline{K}\left(\left(\overline{K}'\right)^{-1}(s)\right) \quad \text{for } s \ge 0, \tag{4.45}$$

where $\overline{K}^*$ is the conjugate function of $\overline{K}$.

Let $0 < \mu < \min\{\varepsilon, 2a_3\lambda E(0)\}$ and $\mathcal{E}(t) = \frac{E(t)}{E(0)}$. Using $\overline{K}'(s) > 0$, $\overline{K}''(s) > 0$, $E'(t) \le 0$, $\overline{K}(0) = \overline{K}'(0) = 0$, (4.43), (4.44), and (4.45), we infer

$$\begin{aligned}
\left(\overline{K}'\left(\mu\mathcal{E}(t)\right)R(t)\right)' &\le -\lambda\overline{K}'\left(\mu\mathcal{E}(t)\right)E(t) + \frac{1}{2a_3}\overline{K}'\left(\mu\mathcal{E}(t)\right)\overline{K}^{-1}\left(\frac{a_3\Gamma_2(t)}{\zeta(t)}\right)\\
&\le -\lambda\overline{K}'\left(\mu\mathcal{E}(t)\right)E(t) + \frac{1}{2a_3}\overline{K}^*\left(\overline{K}'\left(\mu\mathcal{E}(t)\right)\right) + \frac{\Gamma_2(t)}{2\zeta(t)}\\
&\le -\lambda\overline{K}'\left(\mu\mathcal{E}(t)\right)E(t) + \frac{\mu}{2a_3}\mathcal{E}(t)\overline{K}'\left(\mu\mathcal{E}(t)\right) + \frac{\Gamma_2(t)}{2\zeta(t)}\\
&= -a_4\mathcal{E}(t)K'\left(\mu\mathcal{E}(t)\right) + \frac{\Gamma_2(t)}{2\zeta(t)},
\end{aligned} \tag{4.46}$$

where $a_4 = \lambda E(0) - \frac{\mu}{2a_3} > 0$. Setting

$$\mathcal{R}_2(t) = \zeta(t)\overline{K}'\left(\mu\mathcal{E}(t)\right)R(t) + E(t),$$

from (4.46) and (4.41), we get

$$\mathcal{R}_2'(t) \le -a_4\zeta(t)\mathcal{E}(t)K'\left(\mu\mathcal{E}(t)\right) + \frac{\Gamma_2(t)}{2} + E'(t) \le -a_4\zeta(t)K_0\left(\mathcal{E}(t)\right) \quad \text{for } t \ge t_\varepsilon, \tag{4.47}$$

where $K_0(s) = sK'(\mu s)$. Since $\mathcal{R}_2(t) \sim E(t)$, there exist $a_5, a_6 > 0$ satisfying

$$a_5\mathcal{R}_2(t) \le E(t) \le a_6\mathcal{R}_2(t).$$

Finally, we let

$$\mathcal{L}(t) = \frac{a_5\mathcal{R}_2(t)}{E(0)}, \tag{4.48}$$

then

$$\mathcal{L}(t) \le \mathcal{E}(t) \le 1. \tag{4.49}$$

Since K_0 is an increasing function on $(0,1]$, from (4.48), (4.47), and (4.49), we deduce

$$\mathcal{L}'(t) \le -\omega_2\zeta(t)K_0\big(\mathcal{E}(t)\big) \le -\omega_2\zeta(t)K_0\big(\mathcal{L}(t)\big) \quad \text{for } t \ge t_\varepsilon,$$

where $\omega_2 = \frac{a_4a_5}{E(0)}$, and

$$\begin{aligned}\int_{t_\varepsilon}^t \omega_2\zeta(s)\,ds &\le -\int_{t_\varepsilon}^t \frac{\mathcal{L}'(s)}{K_0(\mathcal{L}(s))}\,ds = -\int_{t_\varepsilon}^t \frac{\mathcal{L}'(s)}{\mathcal{L}(s)K'(\mu\mathcal{L}(s))}\,ds = \int_{\mu\mathcal{L}(t)}^{\mu\mathcal{L}(t_\varepsilon)} \frac{1}{sK'(s)}\,ds \\ &\le \int_{\mu\mathcal{L}(t)}^{\varepsilon} \frac{1}{sK'(s)}\,ds = \tilde{K}\big(\mu\mathcal{L}(t)\big),\end{aligned}$$

here $\tilde{K}$ is the function given in (4.34). Because $\tilde{K}$ is strictly decreasing on $(0,\varepsilon]$, we obtain

$$\mathcal{L}(t) \le \frac{1}{\mu}\tilde{K}^{-1}\left(\omega_2\int_{t_\varepsilon}^t \zeta(s)\,ds\right) \quad \text{for } t \ge t_\varepsilon. \qquad \square$$

4.2 General decay for the case $1 < m_1 < 2$

In this subsection, we derive a general decay result for the case $1 < m_1 < 2$. We let

$$\Omega_1 = \big\{x \in \Omega : m(x) < 2\big\}, \qquad \Omega_2 = \big\{x \in \Omega : m(x) \ge 2\big\}$$

and

$$\Omega_i^- = \big\{x \in \Omega_i : \big|u_t(x,t)\big| < 1\big\}, \qquad \Omega_i^+ = \big\{x \in \Omega_i : \big|u_t(x,t)\big| \ge 1\big\}$$

for $i = 1, 2$.

Lemma 4.9 *Assume that* (A_1), (A_2), (A_3), *and* (3.12) *hold. If* $1 < m_1 < 2$, *then* Φ *satisfies*

$$\begin{aligned}\Phi'(t) \le{}& \|u_t\|_2^2 - \frac{k_l}{4}\|\Delta u\|_2^2 - \big\|\Delta\chi(u)\big\|_2^2 + \gamma\int_\Omega |u|^{p(x)}\,dx + \frac{\mathcal{C}_\eta}{2k_l}(k_\eta\Box\Delta u) \\ &+ \alpha C_6\left\{\int_\Omega |u_t|^{m(x)}\,dx + \left(\int_\Omega |u_t|^{m(x)}\,dx\right)^{m_1-1}\right\} \\ &+ |\beta|C_6\left\{\int_\Omega \big|y(x,1,t)\big|^{m(x)}\,dx + \left(\int_\Omega \big|y(x,1,t)\big|^{m(x)}\,dx\right)^{m_1-1}\right\}.\end{aligned} \tag{4.50}$$

Proof We re-estimate I_1 and I_2 in (4.7) for the case $1 < m_1 < 2$. Using Young's inequality, for $\delta_2 > 0$, we have

$$-\alpha\int_{\Omega_1} u|u_t|^{m(x)-2}u_t\,dx \le \alpha\delta_2\int_{\Omega_1} |u|^2\,dx + \frac{\alpha}{4\delta_2}\int_{\Omega_1} |u_t|^{2m(x)-2}\,dx. \tag{4.51}$$

Noting $2m_1 - 2 < 2m(x) - 2 < m(x) < 2$ for $x \in \Omega_1$ and using Hölder's inequality with $(2 - m_1) + (m_1 - 1) = 1$, we get

$$\begin{aligned}\int_{\Omega_1} |u_t|^{2m(x)-2}\,dx &\le \int_{\Omega_1^-} |u_t|^{2m_1-2}\,dx + \int_{\Omega_1^+} |u_t|^{m(x)}\,dx \\ &\le \left|\Omega_1^-\right|^{2-m_1}\left(\int_{\Omega_1^-} |u_t|^2\,dx\right)^{m_1-1} + \int_{\Omega_1^+} |u_t|^{m(x)}\,dx \\ &\le \left|\Omega_1^-\right|^{2-m_1}\left(\int_{\Omega_1^-} |u_t|^{m(x)}\,dx\right)^{m_1-1} + \int_{\Omega_1^+} |u_t|^{m(x)}\,dx \\ &\le c\left\{\left(\int_{\Omega} |u_t|^{m(x)}\,dx\right)^{m_1-1} + \int_{\Omega} |u_t|^{m(x)}\,dx\right\}. \end{aligned} \tag{4.52}$$

Applying (4.52) to (4.51), we see

$$\begin{aligned}&-\alpha\int_{\Omega_1} u|u_t|^{m(x)-2}u_t\,dx \\ &\quad\le \alpha\delta_2 B_2^2\|\Delta u\|_2^2 + \frac{\alpha c}{4\delta_2}\left\{\left(\int_{\Omega} |u_t|^{m(x)}\,dx\right)^{m_1-1} + \int_{\Omega} |u_t|^{m(x)}\,dx\right\}. \end{aligned} \tag{4.53}$$

As the estimates of (4.8), for $\delta_3 > 0$, we find

$$\begin{aligned}-\alpha\int_{\Omega_2} u|u_t|^{m(x)-2}u_t\,dx &\le \alpha\delta_3\int_{\Omega_2} |u|^{m(x)}\,dx + \alpha\int_{\Omega_2} c_{\delta_3}(x)|u_t|^{m(x)}\,dx \\ &\le \alpha\delta_3\tilde{C}_{E(0)}\|\Delta u\|_2^2 + \alpha\int_{\Omega} c_{\delta_3}(x)|u_t|^{m(x)}\,dx, \end{aligned} \tag{4.54}$$

where $\tilde{C}_{E(0)} = B_{m_-}^{m_-}(\frac{2p_1E(0)}{k_l(p_1-2)})^{\frac{m_--2}{2}} + B_{m_+}^{m_+}(\frac{2p_1E(0)}{k_l(p_1-2)})^{\frac{m_+-2}{2}}$, here

$$m_- = \operatorname*{ess\,inf}_{x\in\Omega_2} m(x) \ge 2 \quad \text{and} \quad m_+ = \operatorname*{ess\,sup}_{x\in\Omega_2} m(x) \ge 2.$$

Combining (4.53) and (4.54) and taking $\delta_2 = \frac{k_l}{16\alpha B_2^2}$ and $\delta_3 = \frac{k_l}{16\alpha\tilde{C}_{E(0)}}$, we have

$$-I_1 \le \frac{k_l}{8}\|\Delta u\|_2^2 + \alpha C_6\left\{\left(\int_{\Omega} |u_t|^{m(x)}\,dx\right)^{m_1-1} + \int_{\Omega} |u_t|^{m(x)}\,dx\right\}. \tag{4.55}$$

Similarly, we have

$$-I_2 \le \frac{k_l}{8}\|\Delta u\|_2^2 + |\beta|C_6\left\{\left(\int_{\Omega} \left|y(x,1,t)\right|^{m(x)}\,dx\right)^{m_1-1} + \int_{\Omega} \left|y(x,1,t)\right|^{m(x)}\,dx\right\}. \tag{4.56}$$

Adapting (4.55) and (4.56) to (4.7), we obtain (4.50). □

Lemma 4.10 *Assume that* (A_1), (A_2), (A_3), *and* (3.12) *hold. If* $1 < m_1 < 2$, *then* Ψ *satisfies*

$$\begin{aligned}\Psi'(t) \le{}& -\left(\int_0^t k(s)\,ds - \delta\right)\|u_t\|_2^2 + \delta C_3\|\Delta u\|_2^2 + \left(\frac{C_4(1+\mathcal{C}_\eta)}{\delta} + \mathcal{C}_\eta\right)(k_\eta\Box\Delta u) \\ &+ \delta C_7(k\Box\Delta u) + \frac{\alpha C_8}{\delta}\left\{c_\delta\int_{\Omega} |u_t|^{m(x)}\,dx + \left(\int_{\Omega} |u_t|^{m(x)}\,dx\right)^{m_1-1}\right\}\end{aligned}$$

$$+\frac{|\beta|C_8}{\delta}\left\{c_\delta\int_\Omega\bigl|y(x,1,t)\bigr|^{m(x)}\,dx+\left(\int_\Omega\bigl|y(x,1,t)\bigr|^{m(x)}\,dx\right)^{m_1-1}\right\} \tag{4.57}$$

for any $\delta > 0$.

Proof We re-estimate J_6 and J_7 in (4.11) for the case $1 < m_1 < 2$. Let $\delta > 0$. Using (4.52), we have

$$\begin{aligned}
&\alpha\int_{\Omega_1}|u_t|^{m(x)-2}u_t\int_0^t k(t-s)\bigl(u(t)-u(s)\bigr)\,ds\,dx\\
&\quad\le\alpha\delta\int_{\Omega_1}\left|\int_0^t k(t-s)\bigl(u(t)-u(s)\bigr)\,ds\right|^2dx+\frac{\alpha}{4\delta}\int_{\Omega_1}|u_t|^{2m(x)-2}\,dx\\
&\quad\le\alpha\delta(1-k_l)B_2^2(k\Box\Delta u)+\frac{\alpha c}{4\delta}\left\{\left(\int_\Omega|u_t|^{m(x)}\,dx\right)^{m_1-1}+\int_\Omega|u_t|^{m(x)}\,dx\right\}.
\end{aligned}\tag{4.58}$$

Since $m(x)\ge 2$ on Ω_2, we can apply the same argument of (4.17) on Ω_2 instead of Ω to obtain

$$\begin{aligned}
&\alpha\int_{\Omega_2}|u_t|^{m(x)-2}u_t\int_0^t k(t-s)\bigl(u(t)-u(s)\bigr)\,ds\,dx\\
&\quad\le\alpha\delta(1-k_l)^{m_1-1}\int_{\Omega_2}\left(\int_0^t k(t-s)\bigl|u(t)-u(s)\bigr|^{m(x)}\,ds\right)dx+\alpha\int_{\Omega_2}c_\delta(x)|u_t|^{m(x)}\,dx\\
&\quad\le\alpha\delta(1-k_l)^{m_1-1}\hat{C}_{E(0)}(k\Box\Delta u)+\frac{\alpha}{\delta}\int_\Omega\delta c_\delta(x)|u_t|^{m(x)}\,dx.
\end{aligned}\tag{4.59}$$

Combining (4.58) and (4.59) and noting that $\delta c_\delta(x)$ is bounded on Ω, we have

$$\begin{aligned}
|J_6|&\le\alpha\delta\bigl\{(1-k_l)B_2^2+(1-k_l)^{m_1-1}\hat{C}_{E(0)}\bigr\}(k\Box\Delta u)\\
&\quad+\frac{\alpha C_8}{\delta}\left\{\left(\int_\Omega|u_t|^{m(x)}\,dx\right)^{m_1-1}+c_\delta\int_\Omega|u_t|^{m(x)}\,dx\right\}.
\end{aligned}\tag{4.60}$$

Similarly, we find

$$\begin{aligned}
|J_7|&\le|\beta|\delta\bigl\{(1-k_l)B_2^2+(1-k_l)^{m_1-1}\hat{C}_{E(0)}\bigr\}(k\Box\Delta u)\\
&\quad+\frac{|\beta|C_8}{\delta}\left\{\left(\int_\Omega\bigl|y(x,1,t)\bigr|^{m(x)}\,dx\right)^{m_1-1}+c_\delta\int_\Omega\bigl|y(x,1,t)\bigr|^{m(x)}\,dx\right\}.
\end{aligned}\tag{4.61}$$

Substituting (4.12), (4.13), (4.14), (4.15), (4.16), (4.60), and (4.61) into (4.11), we obtain (4.57). □

Lemma 4.11 *Assume that* (A_1), (A_2), (A_3), *and* (4.19) *hold. If* $1 < m_1 < 2$, *there exists* $\lambda > 0$ *such that*

$$L'(t)\le-\lambda E(t)-3(1-k_l)\|\Delta u\|_2^2+\frac{1}{2}(k\Box\Delta u)+C_9\bigl(-E'(t)\bigr)^{m_1-1}\quad\textit{for } t\ge k^{-1}(\varepsilon).\tag{4.62}$$

Proof From (3.8), (4.2), (4.50), and (4.57), the proof is similar to that of (4.20) by replacing the constants C_1, C_5, and $c_\delta(x)$ by C_6, C_7, and $\frac{c_\delta C_8}{\delta}$, respectively, adding

$$\left(N_1\alpha C_6+\frac{N_2\alpha C_8}{\delta}\right)\left(\int_\Omega |u_t|^{m(x)}\,dx\right)^{m_1-1} + \left(N_1|\beta|C_6+\frac{N_2|\beta|C_8}{\delta}\right)\left(\int_\Omega \bigl|y(x,1,t)\bigr|^{m(x)}\,dx\right)^{m_1-1},$$

taking $\delta=\frac{k_l}{4N_2C_7}$ in (4.21), and using the relation

$$\left(\int_\Omega |u_t|^{m(x)}\,dx\right)^{m_1-1}+\left(\int_\Omega \bigl|y(x,1,t)\bigr|^{m(x)}\,dx\right)^{m_1-1}\le 2\left(-\frac{E'(t)}{C_0}\right)^{m_1-1},$$

which is seen from (3.8). □

Lemma 4.12 *Assume that* (A_1), (A_2), (A_3), *and* (4.19) *hold and let* $1<m_1<2$. *Then*

$$\int_{t_1}^{t_2} E(s)\,ds\le C_{10}(t_2-t_1)^{2-m_1}\quad \textit{for any } t_2\ge t_1\ge 0. \tag{4.63}$$

Proof By virtue of (4.62), estimate (4.32) is replaced by

$$\bigl(L(t)+\Lambda(t)\bigr)'\le -\lambda E(t)+C_9\bigl(-E'(t)\bigr)^{m_1-1}\quad \text{for } t\ge t_\varepsilon. \tag{4.64}$$

Using (3.8), (4.64), and Young's inequality with $(m_1-1)+(2-m_1)=1$, we observe

$$\begin{aligned}
&\bigl\{E^{\frac{2-m_1}{m_1-1}}(t)\bigl(L(t)+\Lambda(t)\bigr)+a_7E(t)\bigr\}'\\
&\quad\le -\lambda E^{\frac{1}{m_1-1}}(t)+C_9E^{\frac{2-m_1}{m_1-1}}(t)\bigl(-E'(t)\bigr)^{m_1-1}+a_7E'(t)\\
&\quad\le -\lambda E^{\frac{1}{m_1-1}}(t)+\frac{(2-m_1)\lambda}{2}E^{\frac{1}{m_1-1}}(t)+(m_1-1)C_9^{\frac{1}{m_1-1}}\left(\frac{\lambda}{2}\right)^{\frac{m_1-2}{m_1-1}}\bigl(-E'(t)\bigr)+a_7E'(t)\\
&\quad\le -\lambda E^{\frac{1}{m_1-1}}(t)+\frac{\lambda}{2}E^{\frac{1}{m_1-1}}(t)+C_9^{\frac{1}{m_1-1}}\left(\frac{\lambda}{2}\right)^{\frac{m_1-2}{m_1-1}}\bigl(-E'(t)\bigr)+a_7E'(t)\\
&\quad=-\frac{\lambda}{2}E^{\frac{1}{m_1-1}}(t),
\end{aligned}\tag{4.65}$$

where $a_7=C_9^{\frac{1}{m_1-1}}(\frac{\lambda}{2})^{\frac{m_1-2}{m_1-1}}$, which yields

$$\int_{t_\varepsilon}^{t} E^{\frac{1}{m_1-1}}(s)\,ds\le \frac{2}{\lambda}\bigl\{E^{\frac{2-m_1}{m_1-1}}(t_\varepsilon)\bigl(L(t_\varepsilon)+\Lambda(t_\varepsilon)\bigr)+a_7E(t_\varepsilon)\bigr\}:=a_8$$

for all $t\ge t_\varepsilon$. So, we get

$$0<\int_0^\infty E^{\frac{1}{m_1-1}}(s)\,ds\le \int_0^{t_\varepsilon} E^{\frac{1}{m_1-1}}(s)\,ds+a_8<\infty.$$

From this and Hölder inequality with $(2-m_1)+(m_1-1)=1$, we obtain

$$\int_{t_1}^{t_2} E(s)\,ds \le (t_2-t_1)^{2-m_1}\left(\int_{t_1}^{t_2} E^{\frac{1}{m_1-1}}(s)\,ds\right)^{m_1-1} \le C_{10}(t_2-t_1)^{2-m_1}$$

for any $t_2 \ge t_1 \ge 0$. □

Theorem 4.2 *Assume that* (A_1), (A_2), (A_3), *and* (4.19) *hold and let* $1<m_1<2$. *Then there exist* $c_i, \omega_i > 0$, $i=3,4$, *and* $t_0 > k^{-1}(\varepsilon)$ *satisfying*

(*i*) *if* K *is linear,*

$$E(t) \le c_3\left(\omega_3\int_{k^{-1}(\varepsilon)}^{t} \zeta(s)\,ds\right)^{\frac{1-m_1}{2-m_1}}, \quad t > k^{-1}(\varepsilon), \tag{4.66}$$

(*ii*) *if* K *is nonlinear,*

$$E(t) \le c_4\big(t-k^{-1}(\varepsilon)\big)^{2-m_1}\hat{K}^{-1}\left(\omega_4\left(\big(t-k^{-1}(\varepsilon)\big)^{\frac{2-m_1}{m_1-1}}\int_{t_0}^{t}\zeta(s)\,ds\right)^{-1}\right), \quad t>t_0,$$

where

$$\hat{K}(s) = s^{\frac{1}{m_1-1}}K'(s).$$

Proof Owing to (4.62), estimates (4.36) and (4.37) are replaced by

$$L'(t) \le -\lambda E(t) - \frac{k(0)}{a_1}E'(t) + \frac{1}{2}\int_{t_\varepsilon}^{t} k(s)\big\|\Delta u(t)-\Delta u(t-s)\big\|_2^2\,ds + C_9\big(-E'(t)\big)^{m_1-1}$$

and

$$R'(t) \le -\lambda E(t) + \frac{1}{2}\int_{t_\varepsilon}^{t} k(s)\big\|\Delta u(t)-\Delta u(t-s)\big\|_2^2\,ds + C_9\big(-E'(t)\big)^{m_1-1} \tag{4.67}$$

for $t \ge t_\varepsilon$, respectively.

Case 1: K is linear, that is, $K(s)=as$ *for some* $a>0$. Due to (4.67), estimate (4.38) is replaced by

$$\mathcal{R}_1'(t) \le -\lambda\zeta(t)E(t) + C_9\zeta(t)\big(-E'(t)\big)^{m_1-1}, \quad t \ge t_\varepsilon. \tag{4.68}$$

We set

$$\mathcal{R}_3(t) = E^{\frac{2-m_1}{m_1-1}}(t)\mathcal{R}_1(t) + a_9E(t),$$

where $a_9 = C_9^{\frac{1}{m_1-1}}(\frac{\lambda}{2})^{\frac{m_1-2}{m_1-1}}\zeta(0)$, which satisfies $\mathcal{R}_3(t)\sim E(t)$. Using (4.68) and the same argument of (4.65), we have

$$\begin{aligned}\mathcal{R}_3'(t) &\le -\lambda\zeta(t)E^{\frac{1}{m_1-1}}(t) + C_9\zeta(t)E^{\frac{2-m_1}{m_1-1}}(t)\big(-E'(t)\big)^{m_1-1} + a_9E'(t)\\ &\le -\frac{\lambda}{2}\zeta(t)E^{\frac{1}{m_1-1}}(t), \quad t\ge t_\varepsilon,\end{aligned}$$

which ensures (4.66).

Case 2: K is nonlinear. We let

$$\Gamma_3(t) = \frac{a_{10}}{(t-t_\varepsilon)^{2-m_1}} \int_{t_\varepsilon}^{t} \left\| \Delta u(t) - \Delta u(t-s) \right\|_2^2 \, ds, \quad t \ge t_\varepsilon.$$

Using (4.39) and (4.63), we get

$$\begin{aligned}
&\frac{1}{(t-t_\varepsilon)^{2-m_1}} \int_{t_\varepsilon}^{t} \left\| \Delta u(t) - \Delta u(t-s) \right\|_2^2 \, ds \\
&\quad \le \frac{4p_1}{k_l(p_1-2)(t-t_\varepsilon)^{2-m_1}} \left(\int_{t_\varepsilon}^{t} E(s)\, ds + \int_0^{t-t_\varepsilon} E(s) \right) ds) \\
&\quad \le \frac{8p_1 C_{10}}{k_l(p_1-2)} < \infty, \quad t \ge t_\varepsilon.
\end{aligned}$$

Thus, there exists $0 < a_{10} < 1$ satisfying

$$\Gamma_3(t) < 1 \quad \text{for } t \ge t_\varepsilon. \tag{4.69}$$

Using (4.69) and the same argument of (4.42), we can replace estimate (4.42) by

$$\begin{aligned}
\Gamma_2(t) &= -\frac{(t-t_\varepsilon)^{2-m_1}}{a_{10}\Gamma_3(t)} \int_{t_\varepsilon}^{t} \Gamma_3(t) k'(s) \frac{a_{10}\|\Delta u(t) - \Delta u(t-s)\|_2^2}{(t-t_\varepsilon)^{2-m_1}} \, ds \\
&\ge \frac{(t-t_\varepsilon)^{2-m_1}}{a_{10}\Gamma_3(t)} \int_{t_\varepsilon}^{t} \Gamma_3(t) \zeta(s) K\big(k(s)\big) \frac{a_{10}\|\Delta u(t) - \Delta u(t-s)\|_2^2}{(t-t_\varepsilon)^{2-m_1}} \, ds \\
&\ge \frac{(t-t_\varepsilon)^{2-m_1}\zeta(t)}{a_{10}\Gamma_3(t)} \int_{t_\varepsilon}^{t} \overline{K}\big(\Gamma_3(t) k(s)\big) \frac{a_{10}\|\Delta u(t) - \Delta u(t-s)\|_2^2}{(t-t_\varepsilon)^{2-m_1}} \, ds \\
&\ge \frac{(t-t_\varepsilon)^{2-m_1}\zeta(t)}{a_{10}} \overline{K}\left(\int_{t_\varepsilon}^{t} \frac{a_{10} k(s)\|\Delta u(t) - \Delta u(t-s)\|_2^2}{(t-t_\varepsilon)^{2-m_1}} \, ds \right), \quad t \ge t_\varepsilon,
\end{aligned}$$

which reads

$$\int_{t_\varepsilon}^{t} k(s) \left\| \Delta u(t) - \Delta u(t-s) \right\|_2^2 \, ds \le \frac{(t-t_\varepsilon)^{2-m_1}}{a_{10}} \overline{K}^{-1} \left(\frac{a_{10}\Gamma_2(t)}{(t-t_\varepsilon)^{2-m_1}\zeta(t)} \right).$$

From this and (4.67), we get

$$R'(t) \le -\lambda E(t) + \frac{(t-t_\varepsilon)^{2-m_1}}{2a_{10}} \overline{K}^{-1} \left(\frac{a_{10}\Gamma_2(t)}{(t-t_\varepsilon)^{2-m_1}\zeta(t)} \right) + C_9 \big(-E'(t)\big)^{m_1-1}, \quad t \ge t_\varepsilon. \tag{4.70}$$

Let $0 < \mu < \min\{\varepsilon, 2a_{10}\lambda E(0)\}$ and $\mathcal{E}(t) = \frac{E(t)}{E(0)}$. Using (4.70) and the same argument of (4.46), we obtain

$$\begin{aligned}
&\left(\overline{K}' \left(\frac{\mu \mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}} \right) R(t) \right)' \\
&\quad \le -\lambda \overline{K}' \left(\frac{\mu \mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}} \right) E(t) + C_9 \overline{K}' \left(\frac{\mu \mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}} \right) \big(-E'(t)\big)^{m_1-1} \\
&\qquad + \frac{(t-t_\varepsilon)^{2-m_1}}{2a_{10}} \overline{K}' \left(\frac{\mu \mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}} \right) \overline{K}^{-1} \left(\frac{a_{10}\Gamma_2(t)}{(t-t_\varepsilon)^{2-m_1}\zeta(t)} \right)
\end{aligned}$$

$$
\begin{aligned}
&\le -\lambda\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)E(t) + C_9\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)\bigl(-E'(t)\bigr)^{m_1-1} \\
&\quad + \frac{(t-t_\varepsilon)^{2-m_1}}{2a_{10}}\overline{K}^*\left(\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)\right) + \frac{\Gamma_2(t)}{2\zeta(t)} \\
&\le -a_{11}\mathcal{E}(t)\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right) + \frac{\Gamma_2(t)}{2\zeta(t)} + C_9\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)\bigl(-E'(t)\bigr)^{m_1-1}
\end{aligned}
\tag{4.71}
$$

for $t \ge t_\varepsilon$, where $a_{11} = \lambda E(0) - \frac{\mu}{2a_{10}}$. Letting

$$
\mathcal{R}_4(t) = \zeta(t)\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)R(t) + E(t),
$$

from (4.71) and (4.41), we get

$$
\begin{aligned}
\mathcal{R}_4'(t) &\le -\frac{a_{11}}{E(0)}\zeta(t)E(t)\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right) \\
&\quad + C_9\zeta(t)\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)\bigl(-E'(t)\bigr)^{m_1-1}
\end{aligned}
\tag{4.72}
$$

for $t \ge t_\varepsilon$. Define

$$
\mathcal{L}(t) = E^{\frac{2-m_1}{m_1-1}}(t)\mathcal{R}_4(t) + a_{12}E(t),
$$

where $a_{12} = C_9^{\frac{1}{m_1-1}}\zeta(0)(\frac{a_{11}}{2E(0)})^{\frac{m_1-2}{m_1-1}}\overline{K}'(\mu\mathcal{E}(0))$. Using (4.72) and the same argument of (4.65), we see

$$
\begin{aligned}
\mathcal{L}'(t) &\le -\frac{a_{11}}{E(0)}\zeta(t)E^{\frac{1}{m_1-1}}(t)\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right) \\
&\quad + C_9\zeta(t)E^{\frac{2-m_1}{m_1-1}}(t)\bigl(-E'(t)\bigr)^{m_1-1}\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right) + a_{12}E'(t) \\
&\le -\frac{a_{11}}{2E(0)}\zeta(t)E^{\frac{1}{m_1-1}}(t)\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right) \\
&\quad + C_9^{\frac{1}{m_1-1}}\left(\frac{a_{11}}{2E(0)}\right)^{\frac{m_1-2}{m_1-1}}\zeta(t)\bigl(-E'(t)\bigr)\overline{K}'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right) \\
&\quad + a_{12}E'(t), \quad t \ge t_\varepsilon.
\end{aligned}
\tag{4.73}
$$

Thanks to $\lim_{t\to\infty}\frac{1}{(t-t_\varepsilon)^{2-m_1}} = 0$, there exists $t_0 > t_\varepsilon$ such that

$$
\frac{1}{(t-t_\varepsilon)^{2-m_1}} < 1, \quad \forall t > t_0,
\tag{4.74}
$$

which ensures $\mathcal{R}_4(t) \sim E(t) \sim \mathcal{L}(t)$ for $t > t_0$. Moreover, from (4.73) and (4.74), we deduce

$$
\begin{aligned}
\mathcal{L}'(t) &\le -\frac{a_{11}}{2E(0)}\zeta(t)E^{\frac{1}{m_1-1}}(t)K'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right) \\
&\quad + C_9^{\frac{1}{m_1-1}}\zeta(0)\left(\frac{a_{11}}{2E(0)}\right)^{\frac{m_1-2}{m_1-1}}K'\bigl(\mu\mathcal{E}(0)\bigr)\bigl(-E'(t)\bigr) + a_{12}E'(t)
\end{aligned}
$$

$$= -\frac{a_{11}}{2E(0)}\zeta(t)E^{\frac{1}{m_1-1}}(t)K'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right), \quad t > t_0,$$

and hence

$$\int_{t_0}^{t}\zeta(s)E^{\frac{1}{m_1-1}}(s)K'\left(\frac{\mu\mathcal{E}(s)}{(s-t_\varepsilon)^{2-m_1}}\right)ds \le \frac{2E(0)}{a_{11}}\mathcal{L}(t_0), \quad t > t_0,$$

Since

$$\left\{E^{\frac{1}{m_1-1}}(t)K'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)\right\}' \le 0,$$
$$E^{\frac{1}{m_1-1}}(t)K'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)\int_{t_0}^{t}\zeta(s)\,ds \le \frac{2E(0)}{a_{11}}\mathcal{L}(t_0), \quad t > t_0,$$

multiplying this by $(\frac{\mu}{(t-t_\varepsilon)^{2-m_1}})^{\frac{1}{m_1-1}}$, we have

$$E^{\frac{1}{m_1-1}}(0)\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)^{\frac{1}{m_1-1}}K'\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right)\int_{t_0}^{t}\zeta(s)\,ds$$
$$\le \frac{2E(0)\mathcal{L}(t_0)}{a_{11}}\left(\frac{\mu}{(t-t_\varepsilon)^{2-m_1}}\right)^{\frac{1}{m_1-1}}$$

for $t > t_0$. So, letting $\hat{K}(s) = s^{\frac{1}{m_1-1}}K'(s)$, we have

$$E^{\frac{1}{m_1-1}}(0)\hat{K}\left(\frac{\mu\mathcal{E}(t)}{(t-t_\varepsilon)^{2-m_1}}\right) \le \frac{2E(0)}{a_{11}}\mathcal{L}(t_0)\left(\frac{\mu}{(t-t_\varepsilon)^{2-m_1}}\right)^{\frac{1}{m_1-1}}\left(\int_{t_0}^{t}\zeta(s)\,ds\right)^{-1}$$

for $t > t_0$, which gives

$$\mathcal{E}(t) \le \frac{(t-t_\varepsilon)^{2-m_1}}{\mu}\hat{K}^{-1}\left(\omega_4\left((t-t_\varepsilon)^{\frac{2-m_1}{m_1-1}}\int_{t_0}^{t}\zeta(s)\,ds\right)^{-1}\right)$$

for $t > t_0$ and some $\omega_4 > 0$. □

5 Conclusion

In this paper, we considered a viscoelastic von Karman equation with damping, source, and time delay terms of variable exponent type. Under assumptions (A_1), (A_2), (A_3), and (3.12), we showed that the local solution of problem (2.7)–(2.11) is global. Moreover, we established very general decay results of the solution for both cases $1 < \operatorname{ess\,inf}_{x\in\Omega} m(x) < 2$ and $\operatorname{ess\,inf}_{x\in\Omega} m(x) \ge 2$ by giving additional condition (4.19) on initial data.

Acknowledgements
The author is grateful to the anonymous referees for their careful reading and valuable comments.

Funding
This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2020R1I1A3066250).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 12 October 2021 Accepted: 21 March 2022 Published online: 08 April 2022

References

1. Aboulaich, R., Meskine, D., Souissi, A.: New diffusion models in image processing. Comput. Math. Appl. **56**, 874–882 (2008)
2. Antontsev, S.: Wave equation with $p(x,t)$-Laplacian and damping term: existence and blow-up. J. Differ. Equ. Appl. **3**, 503–525 (2011)
3. Antontsev, S., Shmarev, S.: A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions. Nonlinear Anal. **60**, 515–545 (2005)
4. Belhannache, F., Algharabli, M.M., Messaoudi, S.A.: Asymptotic stability for a viscoelastic equation with nonlinear damping and very general type of relaxation functions. J. Dyn. Control Syst. **26**(1), 45–67 (2020)
5. Cavalcanti, M.M., Cavalcanti, A.D.D., Lasiecka, I., Wang, X.: Existence and sharp decay rate estimates for a von Karman system with long memory. Nonlinear Anal., Real World Appl. **22**, 289–306 (2015)
6. Chueshov, I., Lasiecka, I.: Von Karman Evolution Equations: Well-Posedness and Long-Time Dynamics. Springer, New York (2010)
7. Datko, R.: Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. SIAM J. Control Optim. **26**, 697–713 (1988)
8. Edmunds, D., Rakosnik, J.: Sobolev embeddings with variable exponent. Stud. Math. **143**, 267–293 (2000)
9. Fan, X., Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. **263**, 424–446 (2001)
10. Favini, A., Horn, M.A., Lasiecka, I., Tataru, D.: Global existence, uniqueness and regularity of solutions to a von Karman system with nonlinear boundary dissipation. Differ. Integral Equ. **6**, 267–294 (1996) [Addendum to this paper Differ. Integral Equ. **10**, 197–200 (1997)]
11. Feng, B.: Global well-posedness and stability for a viscoelastic plate equation with a time delay. Math. Probl. Eng. **2015**, Article ID 585021 (2015)
12. Ghegal, S., Hamchi, I., Messaoudi, S.: Global existence and stability of a nonlinear wave equation with variable-exponent nonlinearities. Appl. Anal. **99**(8), 1333–1343 (2020)
13. Ha, T.G., Park, S.-H.: Existence and decay results for a von Karman equation with variable exponent nonlinearities. Math. Methods Appl. Sci. **44**, 9475–9486 (2021)
14. Jin, K.-P., Liang, J., Xiao, T.-J.: Coupled second order evolution equations with fading memory: optimal energy decay rate. J. Differ. Equ. **257**, 1501–1528 (2014)
15. Kang, J.-R.: Existence and blow-up of solutions for von Karman equations with time delay and variable exponents. Appl. Math. Comput. **371**, Article ID 124917 (2020)
16. Kirane, M., Said-Houari, B.: Existence and asymptotic stability of a viscoelastic wave equation with a delay. Z. Angew. Math. Phys. **62**, 1065–1082 (2011)
17. Komornik, V.: Exact Controllability and Stabilization: The Multiplier Method. Wiley, New York (1994)
18. Lars, D., Harjulehto, P., Hasto, P., Ruzicka, M.: Lebesque and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics, vol. 2017. Springer, Berlin (2011)
19. Lian, S., Gao, W., Cao, C., Yuan, H.: Study of the solutions to a model porous medium equation with variable exponent of nonlinearity. J. Math. Anal. Appl. **342**, 27–38 (2008)
20. Messaoudi, S.A., Al-Gharabli, M.M., Al-Mahdi, A.M.: On the decay of solutions of a viscoelastic wave equation with variable sources. Math. Methods Appl. Sci. https://doi.org/10.1002/mma.7141
21. Messaoudi, S.A., Talahmeh, A.A., Al-Smail, J.H.: Nonlinear damped wave equation: existence and blow-up. Comput. Math. Appl. **74**, 3024–3041 (2017)
22. Munoz Rivera, J.E., Menzala, G.P.: Decay rates of solutions of a von Karman system for viscoelastic plates with memory. Q. Appl. Math. **LVII**(1), 181–200 (1999)
23. Mustafa, M.I.: Optimal decay rates for the viscoelastic wave equation. Math. Methods Appl. Sci. **41**, 192–204 (2018)
24. Nicaise, S., Pignotti, C.: Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim. **45**, 1561–1585 (2006)
25. Park, J.Y., Park, S.-H., Kang, Y.H.: General decay for a von Karman equation of memory type with acoustic boundary conditions. Z. Angew. Math. Phys. **63**, 813–823 (2012)
26. Park, S.-H.: A general decay result for a von Karman equation with memory and acoustic boundary conditions. J. Appl. Anal. Comput. https://doi.org/10.11948/20200460
27. Park, S.-H., Kang, J.-R.: Blow-up of solutions for a viscoelastic wave equation with variable exponents. Math. Methods Appl. Sci. **42**, 2083–2097 (2019)
28. Park, S.H.: Arbitrary decay rates of energy for a von Karman equation of memory type. Comput. Math. Appl. **70**(8), 1878–1886 (2015)

29. Rajagopal, K., Ružička, M.: Mathematical modeling of electrorheological materials. Contin. Mech. Thermodyn. **13**, 59–78 (2001)
30. Raposo, C.A., Santos, M.L.: General decay to a von Karman system with memory. Nonlinear Anal. **74**, 937–945 (2011)
31. Shangerganesh, L., Gurusamy, A., Balachandran, K.: Weak solutions for nonlinear parabolic equations with variable exponents. Commun. Math. **25**, 55–70 (2017)
32. Suh, I.H., Bien, Z.: Use of time delay action in the controller design. IEEE Trans. Autom. Control **25**, 600–603 (1980)
33. Zhikov, V.V.: Solvability of the three-dimensional thermistor problem. Proc. Steklov Inst. Math. **261**, 98–111 (2008)