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An inhomogeneous perturbation for a class of nonlinear scalar field equations

Xin Sun[1], Yu Duan[1] and Jiu Liu[2,3*]

*Correspondence: jiuliu2011@163.com
[2]School of Mathematics and Statistics, Qiannan Normal University for Nationalities, Duyun, Guizhou 558000, People's Republic of China
[3]Key Laboratory of Complex Systems and Intelligent Computing of Qiannan, Duyun, Guizhou 558000, People's Republic of China
Full list of author information is available at the end of the article

Abstract
This paper deals with a class of nonlinear scalar field equations with an inhomogeneous perturbation. Two positive solutions were obtained using the variational methods.

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Keywords: Nonlinear scalar field equations; Inhomogeneous perturbation; Variational methods; Positive solution

1 Introduction and main result

In this paper, we consider the following nonlinear scalar field equations with an inhomogeneous perturbation

$$-\Delta u = g(u) + h(x), \quad u \in H^1(\mathbb{R}^N), N \geq 3. \tag{1.1}$$

Actually, g satisfies the Berestycki–Lions conditions:

(g_1) $g \in C(\mathbb{R}, \mathbb{R})$;
(g_2) $-\infty < \liminf_{s\to 0^+} \frac{g(s)}{s} \leq \limsup_{s\to 0^+} \frac{g(s)}{s} = -m < 0$;
(g_3) $\lim_{s\to +\infty} \frac{g(s)}{s^{2^*-1}} = 0$, where $2^* = \frac{2N}{N-2}$;
(g_4) there exists $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(\tau)\,d\tau > 0$;

and h satisfies

(h_1) there exists $p \in [\frac{2N}{N+2}, 2]$ such that $h \in L^p(\mathbb{R}^N)$;
(h_2) h is nonnegative and $h \not\equiv 0$;
(h_3) h is radially symmetric;
(h_4) $(\nabla h, x) \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$, where $(\cdot,\cdot)$ denotes scalar product in $\mathbb{R}^N$.

When $h \equiv 0$, Eq. (1.1) reduces to the following nonlinear scalar field equations

$$-\Delta u = g(u), \quad u \in H^1(\mathbb{R}^N), N \geq 3. \tag{1.2}$$

Equation (1.2) possesses strong physical background as introduced in [1, 6] and has been extensively studied, for example, in [3, 5, 10, 12]. Especially in [3], Berestycki and Lions gave nearly optimal conditions known as the Berestycki–Lions conditions.

In [2, 8, 9], the authors studied Eq. (1.2) with a homogeneous perturbation and obtained a positive solution using the variational method. In this paper, we consider the effect of an inhomogeneous perturbation. In other words, we investigate the existence of positive solutions of Eq. (1.1). With regard to Eq. (1.1), there are some results, for example, [4, 14, 15]. Compared with those results, in the present paper, the nonlinearity g is almost optimal.

Set $|\cdot|_s = (\int_{\mathbb{R}^N} |\cdot|^s \,dx)^{\frac{1}{s}}$. Using the variational method, we get the following

Theorem 1.1 *Suppose that* (g_1)–(g_3) *and* (h_1)–(h_3) *hold. Then there exists* $\Lambda > 0$ *such that when* $|h|_p < \Lambda$, *Eq.* (1.1) *has a positive solution. If we add* (g_4) *and* (h_4), *then when* $|h|_p < \Lambda$, *Eq.* (1.1) *has another positive solution.*

We have something to say about the perturbation h. The assumptions (h_1) and (h_2) are necessary, and (h_3) is to overcome the lack of compactness. Moreover, to prove the second positive solution, we need to use the Pohožaev identity, and then (h_4) seems appropriate.

Set $f(s) = g(s) + ms$, then Eq. (1.1) equals to the following equation

$$-\Delta u + mu = f(u) + h(x), \quad u \in H^1\left(\mathbb{R}^N\right), N \geq 3. \tag{1.3}$$

where f satisfies

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$;

(f_2) $-\infty < \liminf_{s\to 0^+} \frac{f(s)}{s} \leq \limsup_{s\to 0^+} \frac{f(s)}{s} = 0$;

(f_3) $\lim_{s\to+\infty} \frac{f(s)}{s^{2^*-1}} = 0$;

(f_4) there exists $\zeta > 0$ such that $F(\zeta) := \int_0^\zeta f(\tau)\,d\tau > \frac{1}{2}m\zeta^2$.

We only need to prove the following

Theorem 1.2 *Suppose that* (f_1)–(f_3) *and* (h_1)–(h_3) *hold. Then there exists* $\Lambda > 0$ *such that when* $|h|_p < \Lambda$, *Eq.* (1.3) *has a positive solution. If we add* (f_4) *and* (h_4), *then when* $|h|_p < \Lambda$, *Eq.* (1.3) *has another positive solution.*

Remark 1.3 (i) f can be sign-changing. (ii) There exist some functions that satisfy (h_1)–(h_4). For example,

$$h_1(x) = \frac{\Lambda}{2\sqrt{\omega_N}(1 + |x|^N)}, \qquad h_2(x) = \frac{\Lambda e^{-\frac{|x|}{2}}}{\sqrt{N\omega_N[1 + (N + 1)!]}},$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. By computing, we have $h_i \in L^2(\mathbb{R}^N)$, $(\nabla h_i, x) \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$ and $|h_i|_2 < \Lambda$, $i = 1, 2$.

The rest of the paper is organized as follows: In Sect. 2, we introduce some preliminaries. In Sect. 3, we give the proof of the first positive solution. Section 4 is devoted to obtaining the second positive solution.

2 Preliminaries

From now on, $C, C_1, C_2, \ldots,$ denotes various positive constant, $u^\pm = \max\{\pm u, 0\}$ and $(H, \|\cdot\|)$ is a Hilbert space, where

$$H = \left\{u \in H^1\left(\mathbb{R}^N\right) : u(x) = u\left(|x|\right)\right\}, \qquad \|\cdot\| = \left[\int_{\mathbb{R}^N} \left(|\nabla\cdot|^2 + m|\cdot|^2\right) dx\right]^{\frac{1}{2}}.$$

To ensure the positivity of solutions and for simplicity, we always take $f(s) = 0$ for all $s \leq 0$. As is well known, the solutions of Eq. (1.3) correspond to the critical points of the following energy functional

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u)\,dx - \int_{\mathbb{R}^N} h(x)u\,dx.$$

By Principle of symmetric criticality [13], we know that if u is a critical point of I restricted to H, then u is a critical point of I in $H^1(\mathbb{R}^N)$. Set

$$F_1(s) = \int_0^s f^+(t)\,dt \quad \text{and} \quad F_2(s) = \int_0^s f^-(t)\,dt,$$

then $F_1(s) \geq 0$, $F_2(s) \geq 0$, $F(s) = F_1(s) - F_2(s)$ for all $s \in \mathbb{R}$,

$$I(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_2(u)\,dx - \int_{\mathbb{R}^N} F_1(u)\,dx - \int_{\mathbb{R}^N} h(x)u\,dx$$

and by (f_1)–(f_3), we have

$$\lim_{s\to 0^+} \frac{f^+(s)}{s} = \lim_{s\to +\infty} \frac{f^\pm(s)}{s^{2^*-1}} = 0. \tag{2.1}$$

3 The first positive solution of Eq. (1.3)

In this section, we prove that Eq. (1.3) has a local minimal solution.

Lemma 3.1 *Suppose that (f_1)–(f_3) and (h_1) hold. Then there exist $\rho > 0$, $\Lambda > 0$, $\alpha > 0$ such that when $|h|_p < \Lambda$, $I(u) \geq \alpha$ for all $\|u\| = \rho$.*

Proof From (2.1), it follows that

$$F_1(s) \leq \frac{m}{4}|s|^2 + C_1|s|^{2^*} \quad \text{for all } s \in \mathbb{R}.$$

Combining with Hölder's inequality and Sobolev's inequality, we get

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{m}{4}\int_{\mathbb{R}^N} u^2\,dx - C_1\int_{\mathbb{R}^N} |u|^{2^*}\,dx - |h|_p|u|_{\frac{p}{p-1}} \\ &\geq \bigl(C_2\|u\| - C_3\|u\|^{2^*-1} - C_4|h|_p\bigr)\|u\|. \end{aligned}$$

Define $k(t) = C_2 t - C_3 t^{2^*-1}$ for $t > 0$, then there exists $\rho > 0$ such that $k(t)$ is increasing in $[0,\rho]$, $k(t)$ is decreasing in $[\rho,+\infty)$, and $k(\rho) = \max_{t>0} k(t)$. Hence when $|h|_p < \Lambda := \frac{k(\rho)}{C_4}$, we have $I(u) \geq \alpha := [k(\rho) - C_4|h|_p]\rho$ for all $\|u\| = \rho$. □

Define $\overline{B}_\rho = \{u \in H : \|u\| \leq \rho\}$ and $\mathfrak{m} = \inf_{u\in\overline{B}_\rho} I(u)$, then we have

Lemma 3.2 *Suppose that (f_1)–(f_3) and (h_1)–(h_2) hold. Then $\mathfrak{m} \in (-\infty, 0)$.*

Proof It follows from (f_1)–(f_2) that there exist $M > 0$ and $\theta > 0$ such that

$$F(s) \geq -Ms^2 \quad \text{for all } s \in [0,\theta).$$

By (h_2), there exist $L \in (0,\theta)$ and $\varphi \in H$ such that $\int_{\mathbb{R}^N} h\varphi\,dx > 0$ and $0 \leq \varphi(x) \leq L$ for all $x \in \mathbb{R}^N$. Then we have

$$\lim_{t\to 0^+} \frac{I(t\varphi)}{t} \leq \lim_{t\to 0^+} \left[\frac{t\|\varphi\|^2}{2} + Mt\int_{\mathbb{R}^N} \varphi^2\,dx - \int_{\mathbb{R}^N} h\varphi\,dx\right] = -\int_{\mathbb{R}^N} h\varphi\,dx < 0,$$

which implies that there exists $t_0 > 0$ such that $\|t_0\varphi\| \leq \rho$ and $I(t_0\varphi) < 0$. Hence $\mathfrak{m} < 0$. It is obvious that $\mathfrak{m} > -\infty$. □

Lemma 3.3 *Suppose that* (f_1)–(f_3) *and* (h_1)–(h_3) *hold. Then* $\mathfrak{m}$ *is achieved.*

Proof By the definition of $\mathfrak{m}$, there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\| \leq \rho$ and $I(u_n) = \mathfrak{m} + o(1)$. Then there exists $u \in H$ such that up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H, \\ u_n \to u, & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N), 2 < s < 2^*, \\ u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

The weakly lower semicontinuity of the norm infers

$$\|u\| \leq \liminf_{n\to\infty} \|u_n\|. \tag{3.2}$$

Thus $\|u\| \leq \rho$. Fatou's lemma [11] and Strauss's compactness lemma [3] yield

$$\int_{\mathbb{R}^N} F_2(u)\,dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} F_2(u_n)\,dx \tag{3.3}$$

and

$$\int_{\mathbb{R}^N} F_1(u_n)\,dx = \int_{\mathbb{R}^N} F_1(u)\,dx + o(1). \tag{3.4}$$

Since (h_1) holds,

$$\int_{\mathbb{R}^N} hu_n\,dx = \int_{\mathbb{R}^N} hu\,dx + o(1). \tag{3.5}$$

By (3.2)–(3.5), we get

$$\begin{aligned} \mathfrak{m} &= \liminf_{n\to\infty} I(u_n) \\ &\geq \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_2(u)\,dx - \int_{\mathbb{R}^N} F_1(u)\,dx - \int_{\mathbb{R}^N} h(x)u\,dx \\ &= I(u) \\ &\geq \mathfrak{m}. \end{aligned}$$

Hence $I(u) = \mathfrak{m}$. □

The proof of the first positive solution From Lemma 3.3, there exists $u \in H$ such that $\|u\| \leq \rho$ and $I(u) = \mathfrak{m}$. Lemma 3.1 infers $\|u\| < \rho$. Thus for any $v \in H$,

$$\langle I'(u), v\rangle = \lim_{t\to 0^+} \frac{I(u+tv) - I(u)}{t} \geq 0$$

and

$$\langle I'(u), v\rangle = \lim_{t\to 0^-} \frac{I(u+tv) - I(u)}{t} \leq 0,$$

which imply $I'(u) = 0$. By $\langle I'(u), u^-\rangle = 0$, we know $u^- = 0$. The strong maximum principle deduces $u > 0$ in $\mathbb{R}^N$. □

4 The second positive solution of Eq. (1.3)

In this section, we prove that Eq. (1.3) has another positive solution. In order to obtain a bounded Palais–Smale sequence, we use the following Jeanjean's theorem [7].

Theorem 4.1 *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_\mu\}_{\mu\in J}$ of C^1-functionals on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,$$

where $B(u) \geq 0$ for all $u \in X$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to +\infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\mu = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\mu\big(\gamma(t)\big) > \max\big\{\Phi_\mu(v_1), \Phi_\mu(v_2)\big\},$$

where

$$\Gamma = \big\{\gamma \in C\big([0,1], X\big) : \gamma(0) = v_1, \gamma(1) = v_2\big\}.$$

Then for almost every $\mu \in J$, there is a sequence $\{u_n\} \subset X$ such that

(i) *$\{u_n\}$ is bounded in X,*
(ii) *$\Phi_\mu(u_n) \to c_\mu$ and*
(iii) *$\Phi'_\mu(u_n) \to 0$ in the dual X^* of X.*

Moreover, the map $\mu \to c_\mu$ is non-increasing and continuous from the left.

From [3], we know that Eq. (1.2) has a positive ground state solution $\omega \in H$ and

$$\int_{\mathbb{R}^N} F_1(\omega)\,dx - \int_{\mathbb{R}^N} F_2(\omega)\,dx - \frac{m}{2}\int_{\mathbb{R}^N} \omega^2\,dx > 0.$$

Then there exists $\delta \in (0,1)$ such that

$$\delta\int_{\mathbb{R}^N} F_1(\omega)\,dx - \int_{\mathbb{R}^N} F_2(\omega)\,dx - \frac{m}{2}\int_{\mathbb{R}^N} \omega^2\,dx > 0. \tag{4.1}$$

In Theorem 4.1, we set

$$X = H, \qquad \|\cdot\|_X = \|\cdot\|, \qquad \Phi_\mu = I_\mu, \qquad J = [\delta, 1], \qquad B(u) = \int_{\mathbb{R}^N} F_1(u)\,dx$$

and

$$A(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_2(u)\,dx - \int_{\mathbb{R}^N} hu\,dx.$$

By Hölder's inequality and Sobolev's inequality, we have

$$\frac{A(u)}{\|u\|} \geq \frac{1}{2}\|u\| - \int_{\mathbb{R}^N} h\frac{u}{\|u\|}\,dx \geq \frac{1}{2}\|u\| - C|h|_p \to +\infty \quad \text{as } \|u\| \to +\infty,$$

which implies $A(u) \to +\infty$ as $\|u\| \to +\infty$. Note that

$$I_\mu(u) = A(u) - \mu B(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} F_2(u)\,dx - \mu\int_{\mathbb{R}^N} F_1(u)\,dx - \int_{\mathbb{R}^N} hu\,dx.$$

In the following, I_1 will always replace I. The next lemma is to verify the assumptions of Theorem 4.1.

Lemma 4.2 *Suppose that* (f_1)–(f_4) *and* (h_1)–(h_2) *hold. Then when* $|h|_p < \Lambda$, *there exist* $v_1, v_2 \in E$ *such that for any* $\mu \in J$, $c_\mu \geq \alpha > \max\{I_\mu(v_1), I_\mu(v_2)\}$, *where* Λ, α *are from Lemma* 3.1.

Proof From Lemma 3.1, it follows that for any $\mu \in J$, $I_\mu(u) \geq I_1(u) \geq \alpha$ for all $\|u\| = \rho$. Define

$$\omega_t(x) = \begin{cases} 0, & t = 0, \\ \omega(t^{-1}x), & t > 0, \end{cases}$$

where ω satisfies (4.1). For any $\mu \in J$, one has

$$I_\mu(\omega_t) \leq \frac{t^{N-2}}{2}\int_{\mathbb{R}^N} |\nabla\omega|^2\,dx - t^N\int_{\mathbb{R}^N}\left[\delta F_1(\omega) - F_2(\omega) - \frac{m}{2}\omega^2\right]dx.$$

Note that

$$\|\omega_t\|^2 = t^{N-2}\int_{\mathbb{R}^N} |\nabla\omega|^2\,dx + mt^N\int_{\mathbb{R}^N} |\omega|^2\,dx.$$

Thus, there exists $t_0 > 0$ such that $\|\omega_{t_0}\| > \rho$ and $I_\mu(\omega_{t_0}) < 0$. Set $v_1 = 0$, $v_2 = \omega_{t_0}$. Hence for any $\gamma \in \Gamma$, $\max_{t\in[0,1]} I_\mu(\gamma(t)) \geq \alpha > 0$. Consequently, $c_\mu \geq \alpha > 0 = \max\{I_\mu(v_1), I_\mu(v_2)\}$. □

From Theorem 4.1, we know that for almost every $\mu \in J$, there is a sequence $\{u_n\} \subset H$ such that

$$\begin{aligned} &\text{(i)} \quad \{u_n\} \quad \text{is bounded in } H, \\ &\text{(ii)} \quad I_\mu(u_n) \to c_\mu, \\ &\text{(iii)} \quad I'_\mu(u_n) \to 0 \quad \text{in the dual } H^* \text{ of } H. \end{aligned} \tag{4.2}$$

Moreover, the map $\mu \to c_\mu$ is non-increasing and continuous from the left.

Lemma 4.3 *Fix $\mu \in J$. Suppose that (f_1)–(f_4) and (h_1)–(h_3) hold. Assume that $\{u_n\} \subset H$ satisfies (4.2). Then there exists a positive function $u \in H$ such that $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$.*

Proof Since $\{u_n\} \subset H$ satisfies (4.2), there exists $u \in H$ such that up to a subsequence, (3.1)–(3.5) hold, and Fatou's lemma [11] and Strauss's compactness lemma [3] yield

$$\int_{\mathbb{R}^N} f^-(u)u\,dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} f^-(u_n)u_n\,dx \tag{4.3}$$

and

$$\int_{\mathbb{R}^N} f^+(u_n)u_n\,dx = \int_{\mathbb{R}^N} f^+(u)u\,dx + o(1). \tag{4.4}$$

Obviously, $I'_\mu(u) = 0$. That is

$$\|u\|^2 - \mu \int_{\mathbb{R}^N} f^+(u)u\,dx + \int_{\mathbb{R}^N} f^-(u)u\,dx - \int_{\mathbb{R}^N} hu\,dx = 0. \tag{4.5}$$

Note that

$$\begin{aligned} 0 &= \liminf_{n\to\infty} \langle I'_\mu(u_n), u_n\rangle \\ &\ge \liminf_{n\to\infty} \|u_n\|^2 - \limsup_{n\to\infty}\left[\mu \int_{\mathbb{R}^N} f^+(u_n)u_n\,dx - \int_{\mathbb{R}^N} f^-(u_n)u_n\,dx + \int_{\mathbb{R}^N} hu_n\,dx\right]. \end{aligned}$$

Using (3.1)–(3.5) and (4.3)–(4.5), we obtain

$$\begin{aligned} \|u\|^2 &\le \liminf_{n\to\infty} \|u_n\|^2 \\ &\le \limsup_{n\to\infty}\left[\mu \int_{\mathbb{R}^N} f^+(u_n)u_n\,dx - \int_{\mathbb{R}^N} f^-(u_n)u_n\,dx + \int_{\mathbb{R}^N} hu_n\,dx\right] \\ &\le \mu \int_{\mathbb{R}^N} f^+(u)u\,dx - \int_{\mathbb{R}^N} f^-(u)u\,dx + \int_{\mathbb{R}^N} hu\,dx \\ &= \|u\|^2. \end{aligned}$$

It is easy to know that $\|u_n\| \to \|u\|$. Combining with (3.1), we get $u_n \to u$ in H. Therefore, $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$. By $\langle I'_\mu(u), u^-\rangle = 0$, we know $u^- = 0$. The strong maximum principle deduces $u > 0$ in $\mathbb{R}^N$. □

The proof of the second positive solution We choose $\mu_n \in J$ and $\mu_n \nearrow 1$. Lemma 4.3 implies that there exists a positive sequence $\{u_{\mu_n}\} \subset H$ such that $I_{\mu_n}(u_{\mu_n}) = c_{\mu_n}$ and $I'_{\mu_n}(u_{\mu_n}) = 0$. Note that (h_4) holds. Then we have

$$\begin{aligned} 0 &= \langle I'_{\mu_n}(u_{\mu_n}), u_{\mu_n}\rangle \\ &= \|u_{\mu_n}\|^2 + \int_{\mathbb{R}^N} f^-(u_{\mu_n})u_{\mu_n}\,dx - \mu_n \int_{\mathbb{R}^N} f^+(u_{\mu_n})u_{\mu_n}\,dx - \int_{\mathbb{R}^N} h(x)u_{\mu_n}\,dx \end{aligned} \tag{4.6}$$

and the following Pohožaev identity

$$0 = P_{\mu_n}(u_{\mu_n})$$

$$:= \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^2\,dx + \frac{Nm}{2}\int_{\mathbb{R}^N}u_{\mu_n}^2\,dx + N\int_{\mathbb{R}^N}F_2(u_{\mu_n})\,dx$$
$$- N\mu_n\int_{\mathbb{R}^N}F_1(u_{\mu_n})\,dx - N\int_{\mathbb{R}^N}hu_{\mu_n}\,dx - \int_{\mathbb{R}^N}(\nabla h, x)u_{\mu_n}\,dx.$$

Combining with Hölder's inequality and Sobolev's inequality, we get

$$\begin{aligned} c_\delta &\geq c_{\mu_n} \\ &= I_{\mu_n}(u_{\mu_n}) - \frac{1}{N}P_{\mu_n}(u_{\mu_n}) \\ &= \frac{1}{N}\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^2\,dx + \frac{1}{N}\int_{\mathbb{R}^N}(\nabla h, x)u_{\mu_n}\,dx \\ &\geq \frac{1}{N}\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^2\,dx - C\big|(\nabla h, x)\big|_{\frac{2N}{N+2}}\left(\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^2\,dx\right)^{\frac{1}{2}}. \end{aligned}$$

So

$$\int_{\mathbb{R}^N}|\nabla u_{\mu_n}|^2\,dx \leq C \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

By (2.1), (4.6), (4.7), Hölder's inequality, and Sobolev's inequality, we yield

$$\begin{aligned} m\int_{\mathbb{R}^N}u_{\mu_n}^2\,dx &\leq \|u_{\mu_n}\|^2 + \int_{\mathbb{R}^N}f^-(u_{\mu_n})u_{\mu_n}\,dx \\ &= \mu_n\int_{\mathbb{R}^N}f^+(u_{\mu_n})u_{\mu_n}\,dx + \int_{\mathbb{R}^N}h(x)u_{\mu_n}\,dx \\ &\leq \frac{m}{2}\int_{\mathbb{R}^N}u_{\mu_n}^2\,dx + C_1\int_{\mathbb{R}^N}u_{\mu_n}^{2^*}\,dx + C_2|h|_p\|u_{\mu_n}\| \\ &\leq \frac{m}{2}\int_{\mathbb{R}^N}u_{\mu_n}^2\,dx + C_3 + C_2|h|_p\left(C + m\int_{\mathbb{R}^N}u_{\mu_n}^2\,dx\right)^{\frac{1}{2}}. \end{aligned}$$

So

$$\int_{\mathbb{R}^N}u_{\mu_n}^2\,dx \leq C \quad \text{for all } n \in \mathbb{N}.$$

Hence $\{u_{\mu_n}\}$ is bounded in H. Note that $\mu_n \nearrow 1$,

$$I_1(u_{\mu_n}) = I_{\mu_n}(u_{\mu_n}) + (\mu_n - 1)\int_{\mathbb{R}^N}F_1(u_{\mu_n})\,dx = c_{\mu_n} + (\mu_n - 1)\int_{\mathbb{R}^N}F_1(u_{\mu_n})\,dx$$

and

$$\big\|I_1'(u_{\mu_n})\big\|_* = \big\|I_1'(u_{\mu_n}) - I_{\mu_n}'(u_{\mu_n})\big\|_* = \sup_{\|v\|=1}\left|(\mu_n - 1)\int_{\mathbb{R}^N}f^+(u_{\mu_n})v\,dx\right|,$$

where $\|\cdot\|_*$ denotes the norm in H^*. Therefore, $I_1(u_{\mu_n}) \to c_1$ and $\|I_1'(u_{\mu_n})\|_* \to 0$. According to Lemma 4.2 and Lemma 4.3, we get that there exists a positive function $u \in H$ such that $I_1(u) = c_1 > 0$ and $I_1'(u) = 0$. □

By Sect. 3 and Sect. 4, we complete the proof of Theorem 1.2.

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Authors' contributions
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Author details
[1]College of Science, Guizhou University of Engineering Science, Bijie, Guizhou 551700, People's Republic of China. [2]School of Mathematics and Statistics, Qiannan Normal University for Nationalities, Duyun, Guizhou 558000, People's Republic of China. [3]Key Laboratory of Complex Systems and Intelligent Computing of Qiannan, Duyun, Guizhou 558000, People's Republic of China.

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