Zhi and Zhan *Boundary Value Problems* (2022) 2022:27
https://doi.org/10.1186/s13661-022-01608-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

The partial boundary value conditions of nonlinear degenerate parabolic equation

Yuan Zhi[1] and Huashui Zhan[2*]

*Correspondence:
huashuizhan@163.com
[2]School of Mathematics and Statistics, Xiamen University of Technology, Xiamen 361024, China
Full list of author information is available at the end of the article

Abstract

The stability of the solutions to a parabolic equation

$$\frac{\partial u}{\partial t} = \Delta A(u) + \sum_{i=1}^{N} b_i(x,t)D_i u - c(x,t)u - f(x,t)$$

with homogeneous boundary condition is considered. Since the set $\{s : A'(s) = a(s) = 0\}$ may have an interior point, the equation is with strong degeneracy and the Dirichlet boundary value condition is overdetermined generally. How to find a partial boundary value condition to match up with the equation is studied in this paper. By choosing a suitable test function, the stability of entropy solutions is obtained by Kruzkov bi-variables method.

MSC: 35L65; 35L85; 35K55

Keywords: Parabolic equation; *n*-dimensional cube; Entropy solution; Kruzkov bi-variables method; Submanifold

1 The boundary condition

We consider the equation

$$\frac{\partial u}{\partial t} = \Delta A(u) + \sum_{i=1}^{N} b_i(x,t)D_i u - c(x,t)u - f(x,t), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1}$$

and assume that

$$A(u) = \int_0^u a(s)\,ds, \quad a(s) \geq 0, a(0) = 0, \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is an appropriately smooth bounded domain, $D_i = \frac{\partial}{\partial x_i}$, $b_i(x,t) \in C^1(\overline{Q}_T)$, $c(x,t), f(x,t) \in C(\overline{Q}_T)$. Equation (1) has a widely applied background, for example, the reaction diffusion problem [10] and the spread of an epidemic disease in heterogeneous environments.

For the Cauchy problem of equation (1), the paper [2] by Vol'pert and Hudjaev was the first one to study its solvability. Since then there have been many papers to study its well-posedness ceaselessly, one can refer to book [18] and references [1–6, 8, 9, 11, 13–20], and [25, 26].

If we want to consider the initial boundary value problem of equation (1), the initial value condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{3}$$

is always necessary. But the Dirichlet homogeneous boundary value condition

$$u(x,t) = 0, \quad (x,t) \in \Sigma_T = \partial\Omega \times (0,T), \tag{4}$$

may be overdetermined generally. In [21, 22, 24], a version of equation (1)

$$\frac{\partial u}{\partial t} = \Delta A(u) + \sum_{i=1}^{N} \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{5}$$

was studied. Instead of it, a partial boundary value

$$u(x,t) = 0, \quad (x,t) \in \Sigma_1 \times (0,T), \tag{6}$$

is enough, where $\Sigma_1 \subset \partial\Omega$ is a relative open subset. One can refer to [21, 22, 24] for details, in which the equation $\Sigma_1 \subset \partial\Omega$ was depicted out in some special ways. Such a fact was found firstly in [19], in which the non-Newtonian fluid equation

$$\begin{aligned} &\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) - \sum_{i=1}^{N} b_i(x) D_i u + c(x,t)u = f(x,t), \\ &\quad (x,t) \in Q_T = \Omega \times (0,T), \end{aligned} \tag{7}$$

was considered. Here $p > 1$, $D_i = \frac{\partial}{\partial x_i}$, $0 \le a(x) \in C(\overline{\Omega})$, $b_i(x) \in C^1(\overline{\Omega})$, $c(x,t)$ and $f(x,t)$ are continuous functions on $\overline{Q}_T$.

However, in [21] and [22], because there are two parameters including in Σ_1, the expression Σ_1 seems very complicated and hard to be verified, and the stability of entropy solutions is proved under the assumptions

$$|\triangle d| \le c, \quad \frac{1}{\lambda}\int_{\Omega_\lambda} dx\,dt \le c. \tag{8}$$

Here $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $\Omega_\lambda = \{x \in \Omega, d(x) < \lambda\}$, λ is small enough, while [24] considered the case of Ω being unbounded and satisfying some harsh terms. Moreover, all partial boundary value conditions appearing in [19, 21, 22] and [24] are with the form as (6).

The dedications of this paper lie in that, for any given bounded domain Ω, due to the fact that the coefficient $b_i(x,t)$ depends on the time variable t, we find that, unlike (6), the partial boundary value condition matching up with equation (1) must be of the following form:

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \subset \partial\Omega \times (0,T), \tag{9}$$

where Σ_p is just a submanifold of $\partial\Omega \times (0,T)$ and it cannot be a cylinder as $\Sigma_1 \times (0,T)$. By choosing some technical test functions, the stability of entropy solutions is established by Kruzkov's bi-variables method.

2 The definition and the main results

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)\,d\tau, \qquad h_\eta(s) = \frac{2}{\eta}\left(1 - \frac{|s|}{\eta}\right)_+.$$

Obviously, $h_\eta(s) \in C(\mathbb{R})$ and

$$\lim_{\eta\to 0} S_\eta(s) = \operatorname{sgn} s, \qquad \lim_{\eta\to 0} s S_\eta'(s) = 0. \tag{10}$$

Let

$$A_\eta(u,k) = \int_k^u a(s) S_\eta(s-k)\,ds, \qquad I_\eta(u-k) = \int_0^{u-k} S_\eta(s)\,ds.$$

Define that $u \in \mathrm{BV}(Q_T)$ if and only if $u \in L^1_{\mathrm{loc}}(Q_T)$ and

$$\int_0^T \int_\Omega \left|u(x_1+h_1,\dots,x_N+h_N,t+h_{N+1}) - u(x,t)\right|dx\,dt \le C|h|,$$

where $h = (h_1,h_2,\dots,h_N,h_{N+1})$. This is equivalent to that the generalized derivatives of every function in $\mathrm{BV}(\Omega)$ are regular measures on Ω. Under the norm

$$\|f\|_{\mathrm{BV}} = \|f\|_{L^1} + \int_\Omega |Df|,$$

$\mathrm{BV}(\Omega)$ is a Banach space.

A basic property of BV function is that (see [17, 18]): if $f \in \mathrm{BV}(Q_T)$, then there exists a sequence $\{f_n\} \subset C^\infty(Q_T)$ such that

$$\lim_{n\to\infty} \iint_{Q_T} |f_n - f|\,dx\,dt = 0,$$
$$\lim_{n\to\infty} \iint_{Q_T} |\nabla f_n|\,dx\,dt = \iint_{Q_T} |\nabla f|.$$

So, we can define the trace of the functions in a BV space as in a Sobolev space i.e. the trace of $f(x) \in \mathrm{BV}(Q_T)$ on the boundary $\partial\Omega$ is defined as the limit of a sequence $f_n(x)$ as follows:

$$f(x)|_{x\in\partial\Omega} = \lim_{n\to\infty} f_n(x)|_{x\in\partial\Omega}. \tag{11}$$

Then it is well known that the BV function space is the weakest space such that the trace of $u \in \mathrm{BV}(Q_T)$ can be defined as (11) and the integration by parts can be used. Also, one can refer to [7] for the definition of the trace of $u \in \mathrm{BV}(Q_T)$ on the boundary value in another way.

Definition 1 A function u is said to be the entropy solution of equation (1) with the initial value condition (3) and with the boundary value condition (9) if

1. u satisfies

$$u \in \mathrm{BV}(Q_T) \cap L^\infty(Q_T),\ \frac{\partial}{\partial x_i}\int_0^u \sqrt{a(s)}\,ds \in L^2(Q_T).$$

2. For any $\varphi \in C_0^2(Q_T)$, $\varphi \geq 0$, for any $k \in \mathbb{R}$, for any small $\eta > 0$, u satisfies

$$\begin{aligned}
&\iint_{Q_T}\left[I_\eta(u-k)\varphi_t - \sum_{i=1}^N b_i(x,t)I_\eta(u-k)\varphi_{x_i} + A_\eta(u,k)\Delta\varphi\right]dx\,dt \\
&\quad - \iint_{Q_T} S_\eta'(u-k)\left|\nabla\int_0^u \sqrt{a(s)}\,ds\right|^2 \varphi\,dx\,dt \\
&\quad - \iint_{Q_T}\left[\left(c(x,t) + \sum_{i=1}^N b_{ix_i}(x,t)\right)u + f(x,t)\right]\varphi S_\eta(u-k)\,dx\,dt \\
&\geq 0.
\end{aligned} \tag{12}$$

3. The partial homogeneous boundary value condition (9) is true in the sense of trace.
4. The initial value condition (3) is true in the sense that

$$\lim_{t\to 0}\int_\Omega \left|u(x,t) - u_0(x)\right|dx = 0. \tag{13}$$

If $a(0) = 0$, $b_i(x,t) \in C^1(\overline{Q}_T)$, $c(x,t)$ and $f(x,t)$ are bounded functions, the existence of the entropy solution in the sense of Definition 1 can be proved by a similar way as that in [21, 26], we omit the details here.

In this paper, we study the stability of the entropy solutions of equation (1) without condition (8). In order to display the method used in our paper, the unite n-dimensional cube

$$D_1 = \{(x_1, x_2, \dots, x_N) : 0 < x_i < 1, i = 1, 2, \dots, N\}$$

is considered firstly. By choosing special test functions, we can prove the following theorem.

Theorem 2 *Let $u(x,t)$, $v(x,t)$ be solutions of equation (1) with the initial values $u_0(x)$, $v_0(x) \in L^\infty(D_1)$, respectively, and with the same partial boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Sigma_p. \tag{14}$$

Then

$$\int_{D_1}\left|u(x,t) - v(x,t)\right|dx \leq \int_{D_1}|u_0 - v_0|\,dx. \tag{15}$$

Here, if

$$b_i(x,t) \geq 0, \quad i = 1, 2, \dots, N, \tag{16}$$

then $\Sigma_p = \emptyset$. *While if*

$$b_i(x,t) \leq 0, \quad i = 1, 2, \dots, N, \tag{17}$$

is true, then

$$\Sigma_p = \left\{ (x,t) \in \partial D_1 \times (0,T) : \sum_{i=1}^{N} b_i(x,t) < 0 \right\}. \tag{18}$$

Secondly, we generalize Theorem 2 to a general bounded domain Ω.

The main result of this paper is the following stability theorem.

Theorem 3 *Suppose that* Ω *is a bounded smooth domain in* $\mathbb{R}^N$, *and when* x *is near to the boundary* $\partial\Omega$, *the distance function* $\rho(x)$ *is a* C^2 *function,* $A(s)$ *is a Lipschitz function, and* $a(0) = 0$. *Let* $u(x,t)$ *and* $v(x,t)$ *be the solutions of equation* (1) *with the initial values* $u_0(x) \in L^\infty(\Omega)$ *and* $v_0(x) \in L^\infty(\Omega)$, *respectively, and with the same partial boundary value condition*

$$\gamma u = \gamma v = 0, \quad (x,t) \in \Sigma_p \subset \partial\Omega \times (0,T).$$

Then we have

$$\int_\Omega \left| u(x,t) - v(x,t) \right| dx \leq \int_\Omega \left| u_0(x) - v_0(x) \right| dx. \tag{19}$$

Here,

$$\Sigma_p = \left\{ (x,t) \in \partial\Omega \times (0,T) : \sum_{i=1}^{N} b_i(x,t) n_i(x) < 0 \right\}, \tag{20}$$

where $\vec{n} = \{n_i\}$ $(i = 1, 2, \dots, N)$ *is the outer normal vector of* Ω.

We give a simple comment. For a linear degenerate parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x)\nabla u\big) - \sum_{i=1}^{N} b_i(x) D_i u + c(x,t)u = f(x,t), \tag{21}$$

where $a(x)$, $b(x)$, $c(x,t)$, and $f(x,t)$ are continuous functions, if

$$a(x) = 0, \quad x \in \partial\Omega; \qquad a(x) > 0, \quad x \in \Omega,$$

which implies equation (21) is only degenerate on the boundary $\partial\Omega$, according to the Fichera–Oleinik theory [12], the optimal boundary value condition matching up with equation (21) is

$$u(x,t) = 0, \quad (x,t) \in \Sigma_1 \times [0,T)$$

with

$$\Sigma_1 = \left\{ x \in \partial\Omega : \sum_{i=1}^{N} b_i(x) n_i(x) < 0 \right\}. \tag{22}$$

For the nonlinear degenerate parabolic equation (1), the most important characteristic lies in that the set $\{s \in \mathbb{R} : a(s) = 0\}$ may have interior points, and so it is a strongly degenerate parabolic equation. In addition, when the Dirichlet boundary value condition (4) is imposed, $a(0) = 0$ exactly implies that equation (1) is degenerate on the boundary $\Sigma_T = \partial\Omega \times (0, T)$. On the other hand, when a partial boundary value condition (9) is imposed, we only know that (1) is degenerate on the boundary Σ_p, while on $\Sigma_T \setminus \Sigma_p$, whether equation (1) is degenerate or not is uncertain. To the best knowledge of the author, this is the first paper to study the stability of entropy solutions to equation (1) when the partial boundary value condition is imposed on a submanifold $\Sigma_p \subset \Sigma_T$.

3 An important inequality

In this section, we use the Kruzkov bi-variables method to deduce an important inequality. Such a method was used in [18, 21, 26] and many other references. We begin with some basic denotations. For $u \in \mathrm{BV}(Q_T)$, we denote by that Γ_u is the set of all jump points, ν is the normal of Γ_u at $X = (x, t)$, $u^+(X)$ and $u^-(X)$ are the approximate limits of u at $X \in \Gamma_u$ with respect to $(\nu, Y - X) > 0$ and $(\nu, Y - X) < 0$, respectively. For a continuous function $f(u)$, the composite mean value of f is defined as

$$\widehat{f}(u) = \int_0^1 f\bigl(\tau u^+ + (1-\tau)u^-\bigr)\, d\tau.$$

When $f(s) \in C^1(\mathbb{R})$, $u \in BV(Q_T)$, by [17, 18], we know $f(u) \in \mathrm{BV}(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u) \frac{\partial u}{\partial x_i}, \quad i = 1, 2, \dots, N, N+1,$$

where $x_{N+1} = t$.

Just as that in [23, 26], we have the following lemma.

Lemma 4 *Let u be an entropy solution of* (1). *Then*

$$a(s) = 0, \quad s \in I\bigl(u^+(x,t), u^-(x,t)\bigr) \quad a.e.\ on\ \Gamma^u, \tag{23}$$

where $I(\alpha, \beta)$ denotes the closed interval with endpoints α and β, and (23) *is in the sense of Hausdorff measure $H_N(\Gamma^u)$.*

Let $u(x, t)$, $v(x, t)$ be two entropy solutions of equation (1.1) with the initial value conditions

$$u(x, 0) = u_0(x)$$

and

$$v(x, 0) = v_0(x),$$

respectively.

From Definition 1, for any $\varphi \in C_0^2(Q_T)$, we have

$$\begin{aligned}
&\iint_{Q_T}\left[I_\eta(u-v)\varphi_t - \sum_{i=1}^N b_i(x,t)I_\eta(u-v)\varphi_{x_i} + A_\eta(u,v)\Delta\varphi\right]dx\,dt \\
&\quad - \iint_{Q_T} S'_\eta(u-v)\left|\nabla\int_0^u \sqrt{a(s)}\,ds\right|^2 \varphi\,dx\,dt \\
&\quad - \iint_{Q_T}\left[\left(c(x,t)+\sum_{i=1}^N b_{ix_i}(x,t)\right)u + f(x,t)\right]\varphi S_\eta(u-v)\,dx\,dt \\
&\geq 0. \qquad (24)
\end{aligned}$$

$$\begin{aligned}
&\iint_{Q_T}\left[I_\eta(v-u)\varphi_\tau - \sum_{i=1}^N b_i(y,\tau)I_\eta(v-u)\varphi_{y_i} + A_\eta(v,u)\Delta\varphi\right]dy\,d\tau \\
&\quad - \iint_{Q_T} S'_\eta(v-u)\left|\nabla\int_0^v \sqrt{a(s)}\,ds\right|^2 \varphi\,dy\,d\tau \\
&\quad - \iint_{Q_T}\left[\left(c(y,\tau)+\sum_{i=1}^N b_{iy_i}(y,\tau)\right)v + f(y,\tau)\right]\varphi S_\eta(v-u)\,dy\,d\tau \\
&\geq 0. \qquad (25)
\end{aligned}$$

Let ω_h be the mollifier which is defined as

$$\omega_h(s) = \frac{1}{h}\omega\left(\frac{s}{h}\right), \omega(s) \in C_0^\infty(\mathbb{R}), \omega(s) \geq 0, \omega(s) = 0 \quad \text{if } |s| > 1, \int_{-\infty}^{\infty}\omega(s)\,ds = 1.$$

Define $\psi(x,t,y,\tau) = \phi(x,t)j_h(x-y,t-\tau)$, where $\phi(x,t) \geq 0$, $\phi(x,t) \in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau) = \omega_h(t-\tau)\prod_{i=1}^N \omega_h(x_i-y_i).$$

Now, we choose $\varphi = \psi$ in (24) and (25), then we have

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\left[I_\eta(u-v)(\psi_t+\psi_\tau) + A_\eta(u,v)\Delta_x\psi + A_\eta(v,u)\Delta_y\psi\right]dx\,dt\,dy\,d\tau \\
&\quad - \sum_{i=1}^N \iint_{Q_T}\iint_{Q_T}\left(b_i(x,t)\psi_{x_i} + b_i(y,\tau)\psi_{y_i}\right)I_\eta(u-v)\,dx\,dt\,dy\,d\tau \\
&\quad - \iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\left(\left|\nabla\int_0^u\sqrt{a(s)}\,ds\right|^2 + \left|\nabla\int_0^v\sqrt{a(s)}\,ds\right|^2\right)\psi\,dx\,dt\,dy\,d\tau \\
&\quad - \iint_{Q_T}\left[\left(c(x,t)+\sum_{i=1}^N b_{ix_i}(x,t)\right)u + f(x,t)\right]\psi S_\eta(u-v)\,dx\,dt \\
&\quad - \iint_{Q_T}\iint_{Q_T}\left[\left(c(y,\tau)+\sum_{i=1}^N b_{iy_i}(y,\tau)\right)v + f(y,\tau)\right]\psi S_\eta(v-u)\,dx\,dt \\
&\geq 0. \qquad (26)
\end{aligned}$$

Since

$$\frac{\partial j_h}{\partial t}+\frac{\partial j_h}{\partial \tau}=0,\qquad \frac{\partial j_h}{\partial x_i}+\frac{\partial j_h}{\partial y_i}=0,\quad i=1,\dots,N;$$
$$\frac{\partial \psi}{\partial t}+\frac{\partial \psi}{\partial \tau}=\frac{\partial \phi}{\partial t}j_h,\qquad \frac{\partial \psi}{\partial x_i}+\frac{\partial \psi}{\partial y_i}=\frac{\partial \phi}{\partial x_i}j_h,$$

we have

$$\begin{aligned}&\lim_{h\to 0}\lim_{\eta\to 0}\iint_{Q_T}\iint_{Q_T}\big(b_i(x,t)\psi_{x_i}+b_i(y,\tau)\psi_{y_i}\big)I_\eta(u-v)\,dx\,dt\,dy\,d\tau\\ &\quad=\lim_{h\to 0}\iint_{Q_T}\iint_{Q_T}\big(b_i(x,t)+b_i(y,\tau)\big)|u-v|\phi_{x_i}j_h\,dx\,dt\,dy\,d\tau\\ &\quad=2\iint_{Q_T}b_i(x,t)|u-v|\phi_{x_i}\,dx\,dt.\end{aligned}\tag{27}$$

Meanwhile, we have

$$\begin{aligned}&\iint_{Q_T}\iint_{Q_T}\big[A_\eta(u,v)\Delta_x\psi+A_\eta(v,u)\Delta_y\psi\big]\,dx\,dt\,dy\,d\tau\\ &\quad=\iint_{Q_T}\iint_{Q_T}\big\{A_\eta(u,v)(\Delta_x\phi j_h+2\phi_{x_i}j_{hx_i}+\phi\Delta j_h)+A_\eta(v,u)\phi\Delta_y j_h\big\}\,dx\,dt\,dy\,d\tau\\ &\quad=\iint_{Q_T}\iint_{Q_T}\big\{A_\eta(u,v)\Delta_x\phi j_h+A_\eta(u,v)\phi_{x_i}j_{hx_i}+A_\eta(v,u)\phi_{x_i}j_{hy_i}\big\}\,dx\,dt\,dy\,d\tau\\ &\qquad-\iint_{Q_T}\iint_{Q_T}\bigg\{\widehat{a(u)S_\eta(u}-v)\frac{\partial u}{\partial x_i}\\ &\qquad-\int_u^v\widehat{a(s)S_\eta'(s}-v)\,ds\frac{\partial u}{\partial x_i})\phi j_{hx_i}\bigg\}\,dx\,dt\,dy\,d\tau,\end{aligned}\tag{28}$$

where

$$\widehat{a(u)S_\eta(u}-v)=\int_0^1 a\big(su^++(1-s)u^-\big)S_\eta\big(su^++(1-s)u^--v\big)\,ds,$$
$$\int_u^v\widehat{a(s)S_\eta'(s}-v)\,ds=\int_0^1\int_{su^++(1-s)u^-}^{v}a(\sigma)S_\eta\big(\sigma-su^+-(1-s)u^-\big)d\sigma\,ds.$$

By that

$$\begin{aligned}&\iint_{Q_T}\iint_{Q_T}S_\eta'(u-v)\bigg(\bigg|\nabla_x\int_0^u\sqrt{a(s)}\,ds\bigg|^2+\bigg|\nabla_y\int_0^v\sqrt{a(s)}\,ds\bigg|^2\bigg)\psi\,dx\,dt\,dy\,d\tau\\ &\quad=\iint_{Q_T}\iint_{Q_T}S_\eta'(u-v)\bigg(\bigg|\nabla_x\int_0^u\sqrt{a(s)}\,ds\bigg|-\bigg|\nabla_y\int_0^v\sqrt{a(s)}\,ds\bigg|\bigg)^2\psi\,dx\,dt\,dy\,d\tau\\ &\qquad+2\iint_{Q_T}\iint_{Q_T}S_\eta'(u-v)\nabla_x\int_0^u\sqrt{a(s)}\,ds\cdot\nabla_y\int_0^v\sqrt{a(s)}\,ds\psi\,dx\,dt\,dy\,d\tau,\end{aligned}\tag{29}$$

and using Lemma 4, we obtain the facts

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\nabla_x\nabla_y\int_v^u\sqrt{a(\delta)}\int_\delta^v\sqrt{a(\sigma)}S'_\eta(\sigma-\delta)\,d\sigma\,d\delta\psi\,dx\,dt\,dy\,d\tau\\
&=\iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1\sqrt{a\bigl(su^++(1-s)u^-\bigr)}\sqrt{a(\sigma v^++(1-\sigma)v^-}\\
&\quad\times S'_\eta\bigl[\sigma v^++(1-\sigma)v^--su^+-(1-s)u^-\bigr]\,ds\,d\sigma\,\nabla_xu\nabla_yv\,dx\,dt\,dy\,d\tau\\
&=\iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1 S'_\eta\bigl[\sigma v^++(1-\sigma)v^--su^+-(1-s)u^-\bigr]\,ds\,d\sigma\\
&\quad\times\widehat{\sqrt{a(u)}}\nabla_xu\widehat{\sqrt{a(v)}}\nabla_yv\,dx\,dt\,dy\,d\tau\\
&=\iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1 S'_\eta(v-u)\nabla_x\int_0^u\sqrt{a(s)}\,ds\nabla_y\int_0^v\sqrt{a(s)}\,ds\,dx\,dt\,dy\,d\tau
\end{aligned}\tag{30}$$

and

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\nabla_x\nabla_y\int_v^u\sqrt{a(\delta)}\int_\delta^v\sqrt{a(\sigma)}S'_\eta(\sigma-\delta)\,d\sigma\,d\delta\psi\,dx\,dt\,dy\,d\tau\\
&=\iint_{Q_T}\iint_{Q_T}\int_0^1\sqrt{a\bigl(su^++(1-s)u^-\bigr)}\\
&\quad\times\int_{su^++(1-s)u^-}^v\sqrt{a(\sigma)}S'_\eta\bigl(\sigma-su^+-(1-s)u^-\bigr)\,d\sigma\,ds\frac{\partial u}{\partial x_i}j_{hx_i}\phi\,dx\,dt\,dy\,d\tau.
\end{aligned}\tag{31}$$

Then we have

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\biggl(a(u)\widehat{S_\eta(u}-v)\frac{\partial u}{\partial x_i}-\widehat{\int_u^v a(s)S'_\eta(s-u)\,ds}\frac{\partial u}{\partial x_i}\biggr)j_{hx_i}\phi\,dx\,dt\,dy\,d\tau\\
&\quad+2\iint_{Q_T}\iint_{Q_T}S'_\eta(u-v)\nabla_x\int_0^u\sqrt{a(s)}\,ds\cdot\nabla_y\int_0^v\sqrt{a(s)}\,ds\psi\,dx\,dt\,dy\,d\tau\\
&=\iint_{Q_T}\iint_{Q_T}\biggl[\int_0^1 a\bigl(su^++(1-s)u^-\bigr)S_\eta\bigl(su^++(1-s)u^--v\bigr)\,ds\\
&\quad-\int_0^1\int_{su^++(1-s)u^-}^v a(\sigma)S'_\eta\bigl(\sigma-su^+-(1-s)u^-\bigr)\,d\sigma\,ds\\
&\quad+2\int_0^1\sqrt{a\bigl(su^++(1-s)u^-\bigr)}\int_{su^++(1-s)u^-}^v\sqrt{a(\sigma)}S'_\eta\bigl(\sigma-su^+-(1-s)u^-\bigr)\,d\sigma\,ds\biggr]\\
&\quad\times\frac{\partial u}{\partial x_i}j_{hx_i}\phi\,dx\,dt\,dy\,d\tau\\
&=-\iint_{Q_T}\iint_{Q_T}\int_0^1\int_{su^++(1-s)u^-}^v\Bigl[\sqrt{a(\sigma)}-\sqrt{a\bigl(su^++(1-s)u^-\bigr)}\Bigr]^2\\
&\quad\times S'_\eta\bigl(\sigma-su^+-(1-s)u^-\bigr)\,d\sigma\,ds\frac{\partial u}{\partial x_i}j_{hx_i}\phi\,dx\,dt\,dy\,d\tau\\
&\to 0
\end{aligned}\tag{32}$$

as $\eta\to 0$.

Once more,

$$\lim_{\eta\to 0} A_\eta(u,v) = \lim_{\eta\to 0} A_\eta(v,u) = \operatorname{sgn}(u-v)\big[A(u)-A(v)\big],$$

we have

$$\lim_{\eta\to 0}\big[A_\eta(u,v)\phi_{x_i} j_{hx_i} + A_\eta(u,v)\phi_{y_i} j_{hy_i}\big] = 0. \tag{33}$$

Also, we clearly have

$$\begin{aligned}
&-\lim_{h\to 0}\lim_{\eta\to 0}\iint_{Q_T}\iint_{Q_T}\left[\left(c(x,t)+\sum_{i=1}^{N} b_{ix_i}(x,t)\right)u+f(x,t)\right]\psi S_\eta(u-v)\,dx\,dt\,dy\,d\tau\\
&\quad-\lim_{h\to 0}\lim_{\eta\to 0}\iint_{Q_T}\iint_{Q_T}\left[\left(c(y,\tau)+\sum_{i=1}^{N} b_{iy_i}(y,\tau)\right)v+f(y,\tau)\right]\psi S_\eta(v-u)\,dx\,dt\,dy\,d\tau\\
&=-\lim_{h\to 0}\iint_{Q_T}\iint_{Q_T}\left[\left(c(x,t)+\sum_{i=1}^{N} b_{ix_i}(x,t)\right)u+f(x,t)\right]\psi\operatorname{sgn}(u-v)\,dx\,dt\,dy\,d\tau\\
&\quad-\lim_{h\to 0}\iint_{Q_T}\iint_{Q_T}\left[\left(c(y,\tau)+\sum_{i=1}^{N} b_{iy_i}(y,\tau)\right)v+f(y,\tau)\right]\psi\operatorname{sgn}(v-u)\,dx\,dt\,dy\,d\tau\\
&=-\iint_{Q_T}\left(c(x,t)+\sum_{i=1}^{N} b_{ix_i}(x,t)\right)\phi|u-v|\,dx\,dt.
\end{aligned}\tag{34}$$

Letting $\eta\to 0$, $h\to 0$ in (26) and combining (27)–(34), we get

$$\begin{aligned}
&\iint_{Q_T}\Bigg[\big|u(x,t)-v(x,t)\big|\phi_t-2|u-v|\sum_{i=1}^{N} b_i(x,t)\phi_{x_i}\\
&\quad-\left(c(x,t)+\sum_{i=1}^{N} b_{ix_i}(x,t)\right)\phi|u-v|+\big|A(u)-A(v)\big|\Delta\phi\Bigg]\,dx\,dt\\
&\geq 0.
\end{aligned}\tag{35}$$

This is the most important inequality to prove Theorem 2 and Theorem 3.

4 Proof of Theorem 2

The proof of Theorem 2 Let

$$d_i(x_i)=\begin{cases} x_i, & \text{if } 0\leq x_i\leq \frac{1}{2},\\ 1-x_i, & \text{if } \frac{1}{2}<x_i\leq 1.\end{cases}$$

For small enough λ, we set

$$\varphi_{i\lambda}(x_i)=\begin{cases} \sin\frac{1}{\lambda}(d_i(x_i)), & \text{if } 0\leq d_i(x_i)\leq \frac{\pi\lambda}{2},\\ 1, & \text{if } d_i(x)\geq \frac{\pi\lambda}{2}.\end{cases}$$

Let $0 \le \eta(t) \in C_0^2(t)$ and

$$\phi(x,t) = \eta(t)\prod_{j=1}^{N}\varphi_{j\lambda}(x_j).$$

Then

$$\begin{aligned}\partial_{x_i}\phi(x,t) &= \eta(t)\partial_{x_i}\varphi_{i\lambda}(x_i)\prod_{j=1,j\neq i}^{N}\varphi_{j\lambda}(x_j)\\ &= \eta(t)\frac{1}{\lambda}\cos\frac{1}{\lambda}\big(d_i(x_i)\big)d_{ix_i}(x_i)\prod_{j=1,j\neq i}^{N}\varphi_{j\lambda}(x_j), \quad 0 \le d_i(x_i) \le \frac{\pi\lambda}{2}. \qquad (36)\end{aligned}$$

$$\begin{aligned}\triangle\phi(x,t) &= \frac{1}{\lambda}\eta(t)\prod_{j=1,j\neq i}^{N}\varphi_{j\lambda}(x_j)\left[-\frac{1}{\lambda}\sin\frac{1}{\lambda}\big(d_i(x_i)\big)d_{ix_i}^2 + \frac{1}{\lambda}\cos\frac{1}{\lambda}\big(d_i(x_i)\big)\triangle d_i(x_i)\right]\\ &= -\frac{1}{\lambda^2}\eta(t)\prod_{j=1,j\neq i}^{N}\varphi_{j\lambda}(x_j)\sin\frac{1}{\lambda}\big(d_i(x_i)\big)d_{ix_i}^2, \quad 0 \le d_i(x_i) \le \frac{\pi\lambda}{2}. \qquad (37)\end{aligned}$$

When λ is small enough, $0 \le d_i(x_i) \le \frac{\pi\lambda}{2} < \frac{1}{2}$,

$$d_{ix_i}(x_i) = 1.$$

Then (36) yields

$$\begin{aligned}&-2|u-v|\sum_{i=1}^{N}b_i(x,t)\phi_{x_i}(x,t)\\ &\quad = -2|u-v|\eta(t)\frac{1}{\lambda}\cos\frac{1}{\lambda}\sum_{i=1}^{N}d_i(x_i)b_i(x,t)\prod_{j=1,j\neq i}^{N}\varphi_{j\lambda}(x_j). \qquad (38)\end{aligned}$$

We now substitute these formulas into (35), if $b_i(x,t) \ge 0$, by (38), we have

$$\begin{aligned}&\iint_{Q_T}\left[\big|u(x,t)-v(x,t)\big|\phi_t - \left(c(x,t) + \sum_{i=1}^{N}b_{ix_i}(x,t)\right)\phi|u-v|\right]dx\,dt\\ &\quad \ge 0. \qquad (39)\end{aligned}$$

Accordingly, we have

$$\iint_{Q_T}\big[\big|u(x,t)-v(x,t)\big|\eta_t + c|u-v|\big]\,dx\,dt \ge 0. \qquad (40)$$

If $b_i(x,t) \le 0$, we have

$$\begin{aligned}
&\iint_{Q_T} \left[\left|u(x,t) - v(x,t)\right| \phi_t - \left(c(x,t) + \sum_{i=1}^{N} b_{ix_i}(x,t) \right) \phi |u - v| \right] dx\,dt \\
&\quad - 2 \int_0^T \int_{\Omega_\lambda} \eta(t) \frac{1}{\lambda} \sum_{i=1}^{N} b_i(x,t) \big| |u - v| \,dx\,dt \\
&\quad \ge 0, \qquad (41)
\end{aligned}$$

where $\Omega_\lambda = \{(x \in \Omega : d_i(x_i) < \frac{\lambda\pi}{2}\}$. According to the definition of the trace of BV functions (see [7]), when $x \in \Sigma_{1t} = \{\partial\Omega : \sum_{i=1}^{N} b_i(x,t) < 0\}$, $u(x,t) = v(x,t) = 0$, let $\lambda \to 0$ in (41). We also have (40).

Let $0 < s < \tau < T$, and

$$\eta(t) = \int_{\tau-t}^{s-t} \alpha_\varepsilon(\sigma)\,d\sigma, \quad \varepsilon < \min\{\tau, T - s\},$$

where $\alpha_\varepsilon(t)$ is the kernel of mollifier with $\alpha_\varepsilon(t) = 0$ for $t \notin (-\varepsilon, \varepsilon)$. Then

$$\int_0^T \big[\alpha_\varepsilon(t - s) - \alpha_\varepsilon(t - \tau)\big] |u - v|_{L^1(\Omega)}\,dt + c\int_\tau^s \int_\Omega |u - v|\,dx\,dt \ge 0.$$

Let $\varepsilon \to 0$. Then

$$\big|u(x,\tau) - v(x,\tau)\big|_{L^1(\Omega)} \le \big|u(x,s) - v(x,s)\big|_{L^1(\Omega)} + c\int_\tau^s \int_\Omega |u - v|\,dx\,dt.$$

Let $s \to 0$, then the desired result follows by Gronwall's inequality. $\square$

Thus, by the Kruzkov bi-variables method, we have proved Theorem 2.

5 Proof of Theorem 3

Proof of Theorem 3 Let $u(x,t)$ and $v(x,t)$ be two entropy solutions of equation (1) with the initial values

$$u(x,0) = u_0(x) \quad \text{and} \quad v(x,0) = v_0(x).$$

Recalling (35), for any $\phi(x,t) \in C_0^2(Q_T)$, we have

$$\begin{aligned}
&\iint_{Q_T} \left[\left|u(x,t) - v(x,t)\right| \phi_t - 2|u - v| \sum_{i=1}^{N} b_i(x,t)\phi_{x_i} \right. \\
&\quad \left. - \left(c(x,t) + \sum_{i=1}^{N} b_{ix_i}(x,t) \right) \phi |u - v| + \left|A(u) - A(v)\right| \Delta\phi \right] dx\,dt \\
&\quad \ge 0. \qquad (42)
\end{aligned}$$

Since when x is near to the boundary $\partial\Omega$, the distance function $\rho(x)$ is a C^2 function, we can define

$$\phi(x,t) = \eta(t)\varphi_\lambda(x), \tag{43}$$

where $0 \le \eta(t) \in C_0^2(t)$ and

$$\varphi_\lambda(x) = \begin{cases} \sin\frac{\rho(x)}{\lambda}, & \text{if } 0 \le \rho(x) < \frac{\pi\lambda}{2}, \\ 1, & \text{if } \rho(x) \ge \frac{\pi\lambda}{2}. \end{cases}$$

Then we have

$$\begin{aligned} \partial_{x_i}\phi(x,t) &= \eta(t)\partial_{x_i}\varphi_\lambda(x) \\ &= \eta(t)\frac{1}{\lambda}\cos\frac{\rho(x)}{\lambda}\rho_{x_i}(x) \end{aligned} \tag{44}$$

and

$$\begin{aligned} \triangle\phi(x,t) &= \frac{1}{\lambda}\eta(t)\left[-\frac{1}{\lambda}\sin\frac{\rho(x)}{\lambda}\sum_{i=1}^{N}\rho_{x_i}^2 + \cos\frac{\rho(x)}{\lambda}\triangle\rho(x)\right] \\ &= -\frac{1}{\lambda^2}\eta(t)\sin\frac{\rho(x)}{\lambda}\sum_{i=1}^{N}\rho_{x_i}^2 + \frac{1}{\lambda}\eta(t)\cos\frac{\rho(x)}{\lambda}\triangle\rho(x), \end{aligned} \tag{45}$$

where $0 \le \rho(x) < \frac{\pi\lambda}{2}$ for small λ.

We define $\Omega_\lambda = \{x \in \Omega : \rho(x) < \frac{\pi\lambda}{2}\}$. Since

$$\lim_{\lambda\to 0}\int_0^{\frac{\lambda\pi}{2}} \frac{1}{\lambda^2}\sin\frac{\rho(x)}{\lambda}\,d\rho = \infty,$$

$$\lim_{\lambda\to 0}\int_0^{\frac{\lambda\pi}{2}} \frac{1}{\lambda}\cos\frac{\rho(x)}{\lambda}|\triangle\rho|\,d\rho < \infty,$$

we have

$$\begin{aligned} &\int_\Omega \bigl|A(u) - A(v)\bigr|\Delta\phi\,dx \\ &= \int_{\Omega_\lambda}\left[\frac{1}{\lambda}\eta(t)\cos\frac{\rho(x)}{\lambda}\triangle\rho - \frac{1}{\lambda^2}\eta(t)\sin\frac{\rho(x)}{\lambda}\right]\bigl|A(u) - A(v)\bigr|\,dx \\ &\le \int_{\Omega_\lambda}\left[\frac{1}{\lambda}\eta(t)\cos\frac{\rho(x)}{\lambda}|\triangle\rho| - \frac{1}{\lambda^2}\eta(t)\sin\frac{\rho(x)}{\lambda}\bigl|A(u) - A(v)\bigr|\right]dx \\ &< 0. \end{aligned} \tag{46}$$

Notice that

$$2|u - v|\sum_{i=1}^{N} b_i(x,t) \le c_i|u - v|.$$

We denote

$$\Omega_{\lambda 1} = \left\{ x \in \Omega_\lambda : -\sum_{i=1}^{N} b_i(x,t)\rho_{x_i} > 0 \right\}$$

and derive that

$$\begin{aligned}
&-2\sum_{i=1}^{N} \lim_{\lambda\to 0} \int_{\Omega_\lambda} |u-v| \sum_{i=1}^{N} b_i(x,t)\phi_{x_i}\,dx \\
&\quad = -2\sum_{i=1}^{N} \lim_{\lambda\to 0} \int_{\Omega_\lambda} \eta(t)|u-v| \sum_{i=1}^{N} b_i(x,t)\frac{1}{\lambda}\cos\frac{\rho(x)}{\lambda}\rho_{x_i}\,dx \\
&\quad \le 2\eta(t) \lim_{\lambda\to 0} \frac{1}{\lambda} \int_{\Omega_\lambda} |u-v| \left(-\sum_{i=1}^{N} b_i(x,t)\rho_{x_i} \right) dx \\
&\quad \le 2\eta(t) \lim_{\lambda\to 0} \frac{1}{\lambda} \int_{\Omega_{\lambda 1}} |u-v| \left(-\sum_{i=1}^{N} b_i(x,t)\rho_{x_i} \right) dx \\
&\quad = -2\int_{\Sigma_p} |u-v| \sum_{i=1}^{N} b_i(x,t)n_i(x)\,d\Sigma \\
&\quad = 0,
\end{aligned} \tag{47}$$

where $\vec{n} = \{n_i\}$ $(i = 1,2,\ldots,N)$ is the outer normal vector of Ω.

At the same time, we have

$$\begin{aligned}
&-\iint_{Q_T} \left(c(x,t) + \sum_{i=1}^{N} b_{ix_i}(x,t) \right) \phi|u-v| + \left|A(u)-A(v)\right|\Delta\phi]\,dx\,dt \\
&\quad \le -\iint_{Q_T} \left(c(x,t) + \sum_{i=1}^{N} b_{ix_i}(x,t) \right) \phi|u-v|\,dx\,dt \\
&\quad \le c\iint_{Q_T} \phi|u-v|\,dx\,dt.
\end{aligned} \tag{48}$$

Processing in an analogous manner as we did in the discussion of (40), letting $\lambda \to 0$ in (42), we arrive at the desired result. □

Thus, we have proved Theorem 3 by the Kruzkov bi-variables method. One can see that the partial boundary value condition is imposed on a submanifold $\Sigma_p \subset \partial\Omega \times (0,T)$. Such a conclusion reflects how the time variable t affects the well-posedness problem of a degenerate parabolic equation. By the way, the assumption that, when x is near to the boundary $\partial\Omega$, the distance function $\rho(x)$ is a C^2 function can be weakened as follows: there is a subdomain $\Omega_\lambda = \{x \in \Omega : \rho(x) < \lambda\}$, $\rho(x)$ is an almost everywhere C^2 function on Ω_λ, and $\int_{\Omega_\lambda} |\Delta\rho|\,dx \le c$.

Acknowledgements
The authors would like to thank everyone for their kind help!

Funding
The paper is supported by NSFC-52171308, China.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Author details
[1]School of Sciences, Jimei University, Xiamen 361021, China. [2]School of Mathematics and Statistics, Xiamen University of Technology, Xiamen 361024, China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 February 2022 Accepted: 5 April 2022 Published online: 22 April 2022

References
1. Bendahamane, M., Karlsen, K.H.: Reharmonized entropy solutions for quasilinear anisotropic degenerate parabolic equations. SIAM J. Math. Anal. **36**(2), 405–422 (2004)
2. Brezis, H., Crandall, M.G.: Uniqueness of solutions of the initial value problem for $u_t - \Delta\varphi(u) = 0$. J. Math. Pures Appl. **58**, 564–587 (1979)
3. Carrillo, J.: Entropy solutions for nonlinear degenerate problems. Arch. Ration. Mech. Anal. **147**, 269–361 (1999)
4. Chen, G.Q., DiBenedetto, E.: Stability of entropy solutions to Cauchy problem for a class of nonlinear hyperbolic-parabolic equations. SIAM J. Math. Anal. **33**(4), 751–762 (2001)
5. Chen, G.Q., Perthame, B.: Well-posedness for non-isotropic degenerate parabolic-hyperbolic equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **20**(4), 645–668 (2003)
6. Cockburn, B., Gripenberg, G.: Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations. J. Differ. Equ. **151**, 231–251 (1999)
7. Enrico, G.: Minimal Surfaces and Functions of Bounded Variation. Birkhäuser, Bosten (1984)
8. Evans, L.C.: Weak convergence methods for nonlinear partial differential equations. Conference Board of the Mathematical Sciences, Regional Conferences Series in Mathematics Number 74 (1998)
9. Karlsen, K.H., Risebro, N.H.: On the uniqueness of entropy solutions of nonlinear degenerate parabolic equations with rough coefficient. Discrete Contain. Dye. Sys. **9**(5), 1081–1104 (2003)
10. Kobayasi, K., Ohwa, H.: Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle. J. Differ. Equ. **252**, 137–167 (2012)
11. Kružkov, S.N.: First order quasilinear equations in several independent variables. Math. USSR Sb. **10**, 217–243 (1970)
12. Oleinik, O.A., Radkevic, E.V.: Second order differential equations with nonnegative characteristic form Rhode Island: American Mathematical Society, Plenum Press, New York (1973)
13. Oleinik, O.A., Samokhin, V.N.: Mathematical Models in Boundary Layer Theorem. Chapman and Hall/CRC, London (1999)
14. Volpert, A.I.: BV space and quasilinear equations. Mat. Sb. **73**, 255–302 (1967)
15. Volpert, A.I., Hudjaev, S.I.: On the problem for quasilinear degenerate parabolic equations of second order. Mat. Sb. **3**, 374–396 (1967) (Russian)
16. Volpert, A.I., Hudjaev, S.I.: Analysis of class of discontinuous functions and the equations of mathematical physics. Izda. Nauka Moskwa (1975) (Russian)
17. Wu, Z., Yin, J.: Some properties of functions in BV_x and their applications to the uniqueness of solutions for degenerate quasilinear parabolic equations. Northeast. Math. J. **5**(4), 395–422 (1989)
18. Wu, Z., Zhao, J., Yin, J., Li, H.: Nonlinear Diffusion Equations. Word Scientific Publishing, Singapore (2001)
19. Yin, J., Wang, C.: Evolutionary weighted p-Laplacian with boundary degeneracy. J. Differ. Equ. **237**, 421–445 (2007)
20. Zhan, H.: The study of the Cauchy problem of a second order quasilinear degenerate parabolic equation and the parallelism of a Riemannian manifold. Doctor Thesis, Xiamen University (2004)
21. Zhan, H.: On the stability of the equation with a partial boundary value condition. Bound. Value Probl. **2018**, 30 (2018)
22. Zhan, H., Feng, Z.: Stability of hyperbolic-parabolic mixed type equations with partial boundary condition. J. Differ. Equ. **264**(12), 7384–7411 (2018)
23. Zhan, H., Feng, Z.: Partial boundary value condition for a nonlinear degenerate parabolic equation. J. Differ. Equ. **267**(5), 2874–2890 (2019)
24. Zhan, H., Li, Y.: The entropy solution inequality method of a reaction-diffusion equation on an unbounded domain. J. Inequal. Appl. **2019**, 3 (2019)
25. Zhao, J.: Uniqueness of solutions of quasilinear degenerate parabolic equations. Northeast. Math. J. **1**(2), 153–165 (1985)
26. Zhao, J., Zhan, H.: Uniqueness and stability of solution for Cauchy problem of degenerate quasilinear parabolic equations. Sci. China Ser. A **48**, 583–593 (2005)