Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Mixed Cauchy problem with lateral boundary condition for noncharacteristic degenerate hyperbolic equations

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Abstract

In this paper, in a cylindrical domain $D = \Omega \times (0, T)$ with $\Omega \subset R^n$, we consider a mixed Cauchy problem with a potential lateral boundary condition for the following noncharacteristic degenerated equation

$$Lu = u_{tt} - k(t)\Delta_x u(x,t) = f(x,t),$$

where $k(t) \geq 0$. As in the case for strictly hyperbolic equations, we first establish that $u \in W_2^1(D)$ and $u \in W_2^2(D)$ under the assumptions $\| \frac{f}{k} \|_{L_2(\Omega)}(t) < \infty$ and $\| \frac{\mathrm{grad}_x f}{k} \|_{L_2(\Omega)}(t) < \infty$ for every $t \in [0, T]$, respectively.

Keywords: Mixed Cauchy boundary value problem; Hyperbolic equation; Newton potential

1 Introduction

A number of studies have been devoted to the mixed Cauchy problem for noncharacteristically degenerate second-order hyperbolic equations, starting from the work of M.L. Krasnov [1]. Later these works were generalized for general degenerate higher-order equations by D.T. Dzhuraev [2], V.N. Vragov [3], and A.I. Kozhanov [4]. The study of boundary value problems for an equation of the mixed type, started by F.G. Tricomi [5], led to the study of new boundary value problems for hyperbolic equations in the characteristic cone, first investigated in the works of S. Gellerstedt [6], A.V. Bitsadze [7] A.M. Nakhushev [8], and T.S. Kal'menov [9–11]. In recent years, the well-posedness of the Cauchy problem for the wave equation with strongly singular coefficients has been investigated by M. Ruzhansky, N. Tokmagambetov [12]. More complete bibliography may be found in the monographs of M.M. Smirnov [13], I.E. Egorov, S.G. Pyatkov, S.V. Popov [14], E.V. Radkevich, O.A. Olejnik [15] and M. Ruzhansky, M. Sadybekov, D. Suragan [16].

In the study of the mixed Cauchy problem in a cylindrical domain, the lateral boundary conditions are usually local boundary conditions of the Dirichlet type or periodic boundary conditions.

In [17], the boundary condition for the Newton (volume) potential was found, which is a new integro-differential self-adjoint boundary condition for the Laplace equation. In this paper, we study the mixed Cauchy problem for one class of noncharacteristic degenerate hyperbolic equations using this boundary condition. Unlike other works devoted to this topic, where solutions of the mixed Cauchy problem with different lateral boundary conditions of the problems under consideration are obtained in weighted spaces; in this paper, all solutions of the mixed Cauchy problems under consideration are obtained in classical Sobolev spaces.

Note that in [17], the Newton potential (volume potential) is given by a self-adjoint integral operator

$$u(x) = \int_{\Omega} \varepsilon(x,\xi)\rho(\xi)\,d\xi, \tag{1}$$

where $\rho(\xi) \in L_2(\Omega)$ and $\varepsilon(x,\xi)$ is a fundamental solution of the Laplace equation

$$-\Delta_x \varepsilon(x,\xi) = \delta(x-\xi), \qquad \varepsilon(x,\xi) = \varepsilon(\xi,x), \tag{2}$$

the function $u \in W_2^2(\Omega)$ satisfies the equation

$$-\Delta_x u = \rho(x) \tag{3}$$

and the lateral boundary condition

$$-\frac{u(x)}{2} + \int_{\partial\Omega} \left(\frac{\partial \varepsilon}{\partial n_\xi}(x,\xi) \cdot u(\xi) - \varepsilon(x,\xi) \cdot \frac{\partial u}{\partial n_\xi}(\xi) \right) d\xi = 0. \tag{4}$$

Conversely, if $u \in W_2^2(\Omega)$ satisfies equation (3) and boundary condition (4), then $u(x)$ coincides with the Newton potential (1).

The aim of this paper is to study the mixed Cauchy problem with condition (4).

2 Preliminaries

Let $\Omega \subset R^n$ be a finite domain with smooth boundary $\partial\Omega \subset C^2$, $D = \Omega \times [0,T]$ a cylindrical domain. In D, we consider the following mixed Cauchy problem.

Find a solution of the following equation

$$Lu = u_{tt} - k(t)\Delta_x u + b(t)\frac{\partial u}{\partial t} + a(t)u = f(x,t), \tag{5}$$

which satisfies the initial conditions

$$u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \tag{6}$$

and the lateral boundary condition

$$N[u] \equiv -\frac{u(x,t)}{2} + \int_{\partial\Omega} \left(\frac{\partial \varepsilon}{\partial n_\xi}(x,\xi) \cdot u(\xi,t) - \varepsilon(x,\xi) \cdot \frac{\partial u}{\partial n_\xi}(\xi,t) \right) d\xi = 0, \tag{7}$$

$$0 < t < T, x \in \partial\Omega,$$

where $k \in C^{1+\alpha}[0,T]$, $0 < \alpha < 1$, $k(t) > 0$, $t > 0$, $k(0) = 0$, $k'(t) \geq 0$, and $\varepsilon(x,\xi)$ is the fundamental solution of the Laplace equation (2).

The eigenfunctions of the Newton potential satisfies the following equation

$$-\Delta e_m(x) = \lambda_m e_m(x), \tag{8}$$

and the boundary condition

$$N[e_m] \equiv -\frac{e_m(x)}{2} + \int_{\partial\Omega} \left(\frac{\partial \varepsilon}{\partial n_\xi}(x,\xi) e_m(\xi) - \varepsilon(x,\xi) \frac{\partial e_m}{\partial n_\xi}(\xi) \right) d\xi = 0, \quad x \in \partial\Omega. \tag{9}$$

According to relations (1)–(4), the set of eigenfunctions $\{e_m(x)\}$ of self-adjoint boundary problem (8)–(9) forms a complete orthonormal system in $L_2(\Omega)$.

For $a(t) \equiv b(t) \equiv 0$, two-dimensional equation (5) is the Chaplygin equation, which is applied to model the supersonic flow of liquid and gas.

In what follows, the boundary condition (7) will be called a potential boundary condition. Although the boundary condition of problem (5)–(7) is cumbersome, the Green function of this problem coincides with the fundamental solution $\varepsilon(x,\xi)$ of the Laplace equation, which means that the Green function is given explicitly in an arbitrary domain.

As in the case for strictly hyperbolic equations, we first show that $u \in W_2^1(D)$ and $u \in W_2^2(D)$ under the assumptions $\|\frac{f}{k}\|_{L_2(\Omega)}(t) < \infty, t \in [0,T]$ and $\frac{a}{k} \in C^{1+\alpha}(\bar{D})$, $\frac{b}{k} \in C^{1+\alpha}(\bar{D})$, respectively.

3 Mixed Cauchy problem with the condition $a(t) \equiv b(t) \equiv 0$

Let us consider the problems (5)–(7) in the case $a(t) \equiv b(t) \equiv 0$. Let

$$Lu = u_{tt} - k(t)\Delta_x u = f(x,t), \quad k(t) > 0, t > 0, k(0) = 0, k'(t) \geq 0, \tag{10}$$

$$u|_{t=0} = 0, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \tag{11}$$

$$N[u] \equiv -\frac{u(x,t)}{2} + \int_{\partial\Omega} \left(\frac{\partial \varepsilon}{\partial n_\xi}(x,\xi) \cdot u(\xi,t) - \varepsilon(x,\xi) \cdot \frac{\partial u}{\partial n_\xi}(\xi,t) \right) d\xi = 0, \tag{12}$$

$$(x,t) \in \partial\Omega \times [0,T].$$

Due to the complexity of potential boundary condition (12), to establish an apriori estimates for problem (10)–(12), we will use the spectral decomposition method.

Let $\{e_m(x)\}$ be a complete orthonormal system of eigenvectors of problem (8)–(9).

The solution of (10)–(12) can be written in the form

$$u(x,t) = \sum_{|m|=1}^{\infty} u_m(t) e_m(x), \quad e_m = e_{m_1 m_2 \ldots m_n}, \tag{13}$$

$$f(x,t) = \sum_{|m|=1}^{\infty} f_m(t) e_m(x), \tag{14}$$

where

$$u_m(t) = \int_\Omega u(x,t) e_m(x)\, dx, \qquad f_m(t) = \int_\Omega f(x,t) e_m(x)\, dx.$$

Substituting (13) and (14) into equation (10), for $u_m(t)$, we get the following one-dimensional Cauchy problem:

$$\frac{d^2u_m}{dt^2} + \lambda_m k(t)u_m(t) = f_m(t), \tag{15}$$

$$u_m(0) = 0, \qquad \frac{d}{dt}u_m(0) = 0. \tag{16}$$

Lemma 3.1 *All solutions $u_m \in W_2^2(0,T)$ of the Cauchy problem (15)–(16) satisfy the inequality*

$$\frac{1}{k(t)}\left|\frac{\partial u_m}{\partial t}\right|^2(t) + \lambda_m u_m^2(t) + \int_0^t \frac{k(\eta)}{k^2(\eta)}\left|\frac{\partial u_m}{\partial \eta}\right|^2(\eta)\,d\eta \le d_1 \int_0^t \left|\frac{f_m(\eta)}{k(\eta)}\right|^2 d\eta, \tag{17}$$

where d_1 is some positive constant independent of f.

Proof Since $u_m(0) = 0$ and $\frac{du_m}{dt}(0) = 0$, we obtain

$$\frac{du_m}{dt}(t) = \int_0^t \frac{d^2u_m}{d\eta^2}(\eta)\,d\eta \quad \text{and} \quad u_m(t) = \int_0^t \frac{du_m}{d\eta}\,d\eta. \tag{18}$$

Integrating both sides of (15) from 0 to t and using equality (18), we get

$$\begin{aligned}\int_0^t \frac{d^2u_m}{d\eta^2}(\eta)\,d\eta + \lambda_m \int_0^t k(\eta)u_m(\eta)\,d\eta &= \frac{du_m}{dt} + \lambda_m \int_0^t k(\eta)(t-\eta)\frac{du_m}{d\eta}\,d\eta \\ &= \int_0^t k(\eta)\frac{f_m(\eta)}{k(\eta)}\,d\eta.\end{aligned} \tag{19}$$

Assuming that $\frac{f_m}{k} \in L_2(0,T)$, integral equation (19) is a Volterra integral equation. Therefore, since $k' \ge 0$, the inequalities

$$\int_0^t k(\eta)\left|\frac{f(\eta)}{k(\eta)}\right| d\eta \le \sup_{0\le\xi\le t} k(\xi) \int_0^t \left|\frac{f(\eta)}{k(\eta)}\right| d\eta = k(t)\int_0^t \left|\frac{f(\eta)}{k(\eta)}\right| d\eta$$

hold, and we can see that

$$\left|\frac{\partial u_m}{\partial t}\right|(t) \le C_m k(t) \int_0^t \left|\frac{f_m(\eta)}{k(\eta)}\right| d\eta, \tag{20}$$

where C_m depends on λ_m.

Now we obtain the necessary apriori estimates for problem (15)–(16) and rewrite this problem in the form

$$\frac{1}{k(t)}\frac{d^2u_m}{dt^2}(t) + \lambda_m u_m(t) = \frac{f_m(t)}{k(t)}, \tag{21}$$

$$u_m(0) = u_m'(0) = 0.$$

Let us calculate the inner product of (21) and $\frac{du_m(t)}{dt}$ in $L_2(0,T)$

$$\begin{aligned}
&\int_0^t \frac{1}{2}k(\eta)\frac{d^2u_m}{d\eta^2}(\eta)\frac{du_m}{d\eta}\,d\eta + \lambda_m\int_0^t u_m(\eta)\frac{du_m}{d\eta}\,d\eta \\
&\quad = \frac{1}{2}\int_0^t \frac{1}{k(\eta)}\frac{d}{d\eta}\left(\frac{du_m}{d\eta}\right)^2 d\eta + \frac{\lambda_m}{2}\int_0^t \frac{du_m^2}{d\eta}(\eta)\,d\eta \\
&\quad = \frac{1}{2}\frac{1}{k(t)}\left(\frac{du_m}{dt}\right)^2(t) - \frac{1}{2}\lim_{t\to 0}\frac{1}{k(t)}\left(\frac{du_m}{dt}\right)^2(t) \\
&\qquad - \frac{1}{2}\int_0^t \left(\frac{d}{d\eta}\frac{1}{k(\eta)}\right)\left(\frac{du_m}{d\eta}\right)^2 d\eta + \frac{\lambda_m}{2}u_m^2(t) \\
&\quad = \int_0^t \frac{f_m(\eta)}{k(\eta)}\frac{du_m(\eta)}{d\eta}\,d\eta.
\end{aligned}$$

By inequality (20), we obtain

$$\lim_{t\to 0}\frac{1}{k(t)}\left(\frac{du_m}{dt}\right)^2(t) = 0.$$

From the equality $-\frac{d}{d\eta}\frac{1}{k(\eta)} = \frac{k'(\eta)}{k^2(\eta)}$ and

$$\int_0^t \frac{f_m(\eta)}{k(\eta)}\frac{du_m(\eta)}{d\eta}\,d\eta \le \frac{\varepsilon}{2}\int_0^t \left|\frac{du_m(\eta)}{d\eta}\right|^2 d\eta + \frac{2}{\varepsilon}\int_0^t \left|\frac{f_m(\eta)}{k(\eta)}\right|^2 d\eta, \tag{22}$$

where ε is a positive real number, it is easy to verify that

$$\int_0^t \left|\frac{\partial u}{\partial\eta}\right|^2 d\eta \le t\cdot \sup_{0\le\eta\le t}\left|\frac{\partial u_m}{\partial\eta}\right|^2. \tag{23}$$

By (22)–(23), it is easy to check that

$$\begin{aligned}
&\frac{1}{2}\sup_{0\le\eta\le t}\frac{1}{k(\eta)}\left|\frac{du}{d\eta}\right|^2 + \frac{1}{2}\lambda_m u_m^2(t) + \frac{1}{2}\int_0^t \frac{k'}{k^2}\left|\frac{du_m}{d\eta}\right|^2 d\eta \\
&\qquad\qquad \le \frac{\varepsilon}{2}\int_0^t \left|\frac{du_m(\eta)}{d\eta}\right|^2 d\eta + \frac{2}{\varepsilon}\int_0^t \left|\frac{f_m(\eta)}{k(\eta)}\right|^2 d\eta \\
&\qquad\qquad \le \frac{\varepsilon}{2}\cdot t\cdot \sup_{0\le\eta\le t}\left|\frac{\partial u_m}{\partial\eta}\right|^2 + \frac{2}{\varepsilon}\int_0^t \left|\frac{f_m(\eta)}{k(\eta)}\right|^2 d\eta.
\end{aligned}$$

Therefore,

$$\frac{1}{2}\sup_{0\le\eta\le t}\left|\frac{\partial u_m}{\partial\eta}\right|^2\frac{1-\varepsilon k(\eta)t}{k(\eta)} + \frac{\lambda_m}{2}u_m^2(t) + \frac{1}{2}\int_0^t \frac{k'(\eta)}{k^2(\eta)}\left|\frac{\partial u_m}{\partial\eta}\right|^2(\eta)\,d\eta \le \frac{2}{\varepsilon}\int_0^t \left|\frac{f_m(\eta)}{k(\eta)}\right|^2 d\eta.$$

Since $k(t)$ is bounded in $[0,T]$, for small ε, we have $1-\varepsilon k(t)\cdot t > \delta$. Therefore, from the above inequality, we get

$$\frac{1}{2}\frac{1}{k(t)}\left(\frac{\partial u_m}{\partial t}\right)^2(t) + \frac{\lambda_m}{2\delta_1}u_m^2(t) + \frac{1}{2\delta_1}\int_0^t \frac{k'(\eta)}{k^2(\eta)}\left|\frac{\partial u_m}{\partial\eta}\right|^2(\eta)\,d\eta \le \frac{2}{\varepsilon\delta_1}\int_0^t \left|\frac{f_m(\eta)}{k(\eta)}\right|^2 d\eta,$$

which is equivalent to

$$\frac{1}{k(t)}\left|\frac{\partial u_m}{\partial t}\right|^2(t)+\lambda_m u_m^2(t)+\int_0^t \frac{k(\eta)}{k^2(\eta)}\left|\frac{\partial u_m}{\partial \eta}\right|^2(\eta)\,d\eta \le \frac{4}{\varepsilon\cdot\delta_1}\cdot\int_0^t \frac{f_m^2(\eta)}{k^2(\eta)}\,d\eta. \tag{24}$$

This completes the proof. □

Remark 3.1 Note that to prove the main inequality (24), we have used inequality (23).

Lemma 3.2 *Let* $k \in C^{1+\alpha}[0,T]$, $1>\alpha>0$, $k(t)>0$, $t>0$, $t>0$, $k(0)=0$, $k'(t)\ge 0$, $\frac{f_m}{k}\in L_2(0,T)$. *Then all solutions* $u_m \in W_2^2(0,T)$ *of the mixed Cauchy problem* (16)–(19) *satisfy the inequality*

$$\begin{aligned}&\frac{1}{k(t)}\left(\sqrt{\lambda_m}\frac{du_m}{dt}\right)^2(t)+\bigl(\lambda_m u_m(t)\bigr)^2+\int_0^t\left(\sqrt{\lambda_m}\frac{du_m}{dt}(t)\right)^2 d\eta\\&\quad\le d_2\left[\int_0^t\left(\frac{\sqrt{\lambda_m}f_m(\eta)}{k(\eta)}\right)^2 d\eta+\frac{f_m^2(t)}{k^2(t)}\right].\end{aligned} \tag{25}$$

Proof Multiplying both sides of (17) in Lemma 3.1 by λ_m, we get

$$\begin{aligned}&\frac{1}{k(t)}\left(\sqrt{\lambda_m}\left(\frac{d^2u_m}{dt^2}\right)\right)^2+\lambda_m^2u_m^2(t)+\int_0^t\frac{k'(\eta)}{k^2(\eta)}\left(\sqrt{\lambda_m}\left(\frac{du_m}{d\eta}\right)\right)^2(\eta)\,d\eta\\&\quad\le d_3\int_0^t\left(\frac{\sqrt{\lambda_m}f_m(\eta)}{k(\eta)}\right)^2 d\eta.\end{aligned} \tag{26}$$

Hence, we obtain $\lambda_m^2u_m^2(t)\le d_3\int_0^t(\frac{\sqrt{\lambda_m}f_m(\eta)}{k(\eta)})^2\,d\eta$.

From (16) and the above inequalities for

$$\frac{1}{k(t)}\frac{\partial^2 u_m}{\partial t^2}(t)=-\lambda_m u_m(t)+\frac{f_m(t)}{k(t)},$$

we have

$$\left|\frac{1}{k(t)}\frac{\partial^2 u_m}{\partial t^2}\right|^2\le 2\left[\lambda_m^2u_m^2(t)+\frac{f_m^2(t)}{k^2(t)}\right]\le 2\left[d_3\int_0^t\left(\frac{\sqrt{\lambda_m}f_m(\eta)}{k(\eta)}\right)^2 d\eta+\frac{f_m^2(t)}{k^2(t)}\right]. \tag{27}$$

By (26) and (27), it is easy to see

$$\begin{aligned}&\frac{1}{k(t)}\left(\sqrt{\lambda_m}\frac{d^2u_m}{dt^2}\right)^2+\bigl(\lambda_m u_m(t)\bigr)^2+\int_0^t\left(\sqrt{\lambda_m}\frac{du_m}{dt}\right)^2 d\eta\\&\quad\le d_2\left[\int_0^t\left(\frac{\sqrt{\lambda_m}f_m(\eta)}{k(\eta)}\right)^2 d\eta+\frac{f_m^2(t)}{k^2(t)}\right],\end{aligned}$$

which finishes the proof. □

4 General case

Now we will consider *the mixed Cauchy problem.*

Let $\Omega \subset R^n$ be a finite domain with smooth boundary $\partial\Omega \subset C^2$, $D = \Omega \times [0, T]$ a cylindrical domain. Find a solution of the following equation in D

$$\frac{\partial^2 u}{\partial t^2} - k(t)\Delta_x u + b(t)\frac{\partial u}{\partial t} + a(t)u = f(x,t) \tag{28}$$

that satisfies the initial conditions

$$u|_{t=0} = 0 \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \tag{29}$$

and the potential lateral boundary condition

$$N[u] \equiv -\frac{u(x,t)}{2} + \int_{\partial\Omega}\left(\frac{\partial\varepsilon}{\partial\eta_\xi}(x,\xi)u(\xi,t) - \varepsilon(x,\xi)\frac{\partial u}{\partial\eta_\xi}(\xi,t)\right)d\xi = 0. \tag{30}$$

As in the case of Sect. 2, the solution of (28)–(30) has the form

$$u(x,t) = \sum_{|m|=1}^{\infty} u_m(t)e_m(x), \tag{31}$$

$$f(x,t) = \sum_{|m|=1}^{\infty} f_m(t)e_m(x), \tag{32}$$

where $\{e_m(x)\}$ is the complete orthonormal system of functions of the following spectral problem

$$-\Delta_x e_m(x) = \lambda_m e_m(x),$$
$$N[e_m]|_{x\in\partial\Omega} \equiv 0,$$
$$f_m(t) = \int_\Omega f(x,t)e_m(x)\,dx, \qquad u_m(t) = \int_\Omega u(x,t)e_m(x)\,dx.$$

Substituting (31)–(32) into (28), we obtain the following Cauchy problem

$$\frac{d^2u_m}{dt^2}(t) + k(t)\lambda_m u_m(t) + b(t)\frac{du_m}{dt}(t) + a(t)u_m(t) = f_m(t), \tag{33}$$

$$u_m(0) = 0, \qquad u'_m(0) = 0. \tag{34}$$

Due to initial conditions (34), it is easy to verify

$$\frac{du_m}{dt}(t) = \int_0^t \frac{\partial^2 u_m}{\partial\eta^2}(\eta)\,d\eta, \qquad u_m(t) = \int_0^t \frac{\partial u_m}{\partial\eta}(\eta)\,d\eta.$$

Using these relations as in (20), we prove the following lemma.

Lemma 4.1 *Let the following conditions be satisfied*: $k \in C^{1+\alpha}[0,T]$, $1 > \alpha > 0$, $k(t) > 0$, $t > 0$, $k(0) = 0$, $k'(t) \geq 0$, $\frac{a}{k} \in C^{1+\alpha}[0,T]$, $\frac{b}{k} \in C^{1+\alpha}[0,T]$ *and* $\frac{f_m}{k} \in L_2[0,T]$. *Then solution* $u_m \in W_2^2(0,T)$ *to problem* (33)–(34) *satisfy the following inequality*

$$\left|\frac{du_m}{dt}\right|(t) \leq d_4 \cdot \left|k(t)\right|\left|\int_0^t \frac{f_m(\eta)}{k(\eta)}\,d\eta\right|.$$

Let conditions $b(t) \geq 0$, $a(t) \geq 0$, $\frac{\partial}{\partial t}\frac{a(t)}{k(t)} \leq 0$ and all the conditions of Lemma 4.1 be satisfied. Then, the regular solution $u \in W_2^2(0,T)$ of the Cauchy problem (33)–(34) satisfies the following inequality

$$\begin{aligned}
&\frac{1}{k(t)}\left(\frac{du_m}{dt}\right)^2(t) + \lambda_m u_m^2(t) + \frac{a(t)}{k(t)}u_m^2(t) + \int_0^t \frac{b(\eta)}{k(\eta)}\left(\frac{du_m}{d\eta}\right)^2 d\eta \\
&\quad + \frac{1}{2}\int_0^t \left[\left(\frac{k'(\eta)}{k^2\eta}\frac{du_m}{d\eta}\right)^2 - \frac{\partial}{\partial\eta}\frac{a(\eta)}{k(\eta)}\right]u_m^2(\eta)\,d\eta \leq d_5\int_0^t \left|\frac{f(\eta)}{k(\eta)}\right|^2 d\eta.
\end{aligned} \tag{35}$$

Multiplying both sides of equation (33) by λ_m, we have

$$\begin{aligned}
&\frac{\lambda_m}{k(t)}\left(\frac{d^2u_m}{dt^2}\right)^2(t) + \lambda_m^2 u_m^2(t) + \lambda_m\frac{a(t)}{k(t)}u_m^2(t) \\
&\quad + \lambda_m\int_0^t \left(\frac{k'(\eta)}{k^2(\eta)} + \frac{b(\eta)}{k(\eta)} - \frac{\partial}{\partial\eta}\left(\frac{a(\eta)}{k(\eta)}\right)\right)u_m^2(\eta)\,d\eta \\
&\quad \leq d_6\left[\int_0^t \left(\frac{\sqrt{\lambda_m}f_m(\eta)}{k(\eta)}\right)^2 d\eta + \left(\frac{f_m(\eta)}{k_m(\eta)}\right)^2\right].
\end{aligned} \tag{36}$$

By the Parseval equality, we rewrite (36) in terms of the space x, t.

Let $g \in L_2(D)$, then

$$g(x,t) = \sum_{|m|=1}^{\infty} g_m(t)e_m(x),$$

$$\|g(x,t)\|_{l_2(\Omega)}^2 = \sum_{|m|=1}^{\infty} |g_m(t)|^2 < \infty, \quad t \in [0,T],$$

where $\{e_m(x)\}$ is the complete orthonormal system of eigenfunctions of the Newton (volume) potential corresponding to the eigenvalue λ_m.

Let $\alpha > 0$, by $(-\Delta_x)^\alpha$, we will denote the operator acting on $g(x,t)$ by the formula

$$(-\Delta_x)^\alpha g = \sum_{|m|=1}^{\infty} g_m(t)\lambda_m^\alpha e_m(x), \tag{37}$$

$$\|(-\Delta_x)^\alpha g\|_{L_2(\Omega)}^2(t) = \sum_{|m|=1}^{\infty} |g_m(t)\lambda_m^\alpha|^2 < \infty, \quad t \in [0,T]. \tag{38}$$

Since (37)–(38), inequality (35) can be rewritten as

$$\begin{aligned}
&\left\|\frac{1}{\sqrt{k(t)}}\left(\frac{\partial u}{\partial t}\right)\right\|_{L_2(\Omega)}^2(t) + \|(-\Delta_x)^{\frac{1}{2}}u\|_{L_2(\Omega)}^2(t) \\
&\quad + \int_0^t \left\|\left(\frac{k'(\eta)}{k^2(\eta)}\right)^{\frac{1}{2}}\left(\frac{\partial u}{\partial t}\right)\right\|_{L_2(\Omega)}^2(\eta)\,d\eta \leq d_7\left\|\frac{f}{k}\right\|_{L_2(\Omega)}^2(t).
\end{aligned}$$

Since (31) and (32), from (36), it follows

Theorem 4.1 *Let $k \in C^{1+\alpha}[0,T]$, $1 > \alpha > 0$, $k(t) > 0$, $k(0) = 0$, $k'(t) \geq 0$. If $\|\frac{f}{k}\|_{L_2(\Omega)}(t) < \infty$ and $\|\frac{(-\Delta_x)^{\frac{1}{2}} f}{k}\|_{L_2(\Omega)}(t) < \infty$ for all $t \in [0,T]$, the solution $u \in W^2_{2,k}(D)$ to the mixed Cauchy problem (10)–(12) satisfies the inequality*

$$\begin{aligned} \|u\|^2{}_{W^2_{2,k}(D)} &= \left\| \frac{1}{\sqrt{k(t)}} \frac{\partial^2 u}{\partial t^2} \right\|^2_{L_2(\Omega)}(t) + \|\Delta_x u\|^2{}_{L_2(\Omega)}(t) \\ &\quad + \int_0^t \left\| (-\Delta_x)^{\frac{1}{2}} \frac{\partial u}{\partial \eta} \right\|^2_{L_2(\Omega)}(\eta)\, d\eta + \|u\|^2{}_{L_2(\Omega)}(t) \\ &\leq d_8 \left(\left\| \frac{f}{k} \right\|^2_{L_2(\Omega)}(t) + \left\| \frac{(-\Delta_x)^{\frac{1}{2}} f}{k} \right\|^2_{L_2(\Omega)}(t) \right), \quad t \in [0,T]. \end{aligned} \tag{39}$$

From (39), it follows that $u \in W^2_{2,k}(D) \subset W^2_2(D)$.

Let us prove the existence of the solution of the mixed Cauchy problem (10)–(12). To do this, we will consider the regularized mixed Cauchy problem:

$$L_\varepsilon u = \frac{1}{k(t)+\varepsilon} \frac{\partial^2 u_\varepsilon}{\partial t^2} - \Delta_x u_\varepsilon = \frac{f(x,t)}{k(t)+\varepsilon} \tag{40}$$

$$u_\varepsilon|_{t=0} = \left.\frac{\partial u_\varepsilon}{\partial t}\right|_{t=0} = 0, \qquad N[u_\varepsilon] \equiv 0, \tag{41}$$

where $\varepsilon > 0$ is an arbitrary positive number.

Since (31) and (32), using the spectral decomposition of $u_\varepsilon(x,t)$ and $f(x,t)$ by $e_m(x)$, from (40)–(41), we obtain

$$\begin{aligned} &\frac{1}{k(t)+\varepsilon} \frac{\partial^2 u_{\varepsilon m}}{\partial t^2} + \lambda_m u_{\varepsilon m} = \frac{f_m(t)}{k(t)+\varepsilon} \\ &u_{\varepsilon m}|_{t=0} = 0, \qquad \left.\frac{\partial u_{\varepsilon m}}{\partial t}\right|_{t=0} = 0. \end{aligned}$$

Due to the properties of the solutions of the Cauchy problem, if $f_m \in L_2(0,T)$, then its solution is $u_m \in W^2_2[0,T]$. Similarly to inequalities (24) and (25), we verify the following inequalities

$$\begin{aligned} &\frac{1}{k(t)+\varepsilon} \left(\frac{\partial u_{\varepsilon m}}{\partial t} \right)^2 (t) + \lambda_m u^2_{\varepsilon m}(t) + \int_0^t \frac{k'}{k(t)+\varepsilon} \left(\frac{\partial u_{\varepsilon m}}{\partial \eta} \right)^2 d\eta \\ &\quad \leq d_9 \int_0^t \left| \frac{f_m(\eta)}{k(\eta)+\varepsilon} \right|^2 d\eta, \end{aligned} \tag{42}$$

$$\begin{aligned} &\left(\left(\frac{1}{k(t)+\varepsilon} \right)^{\frac{1}{2}} \frac{\partial^2 u_{\varepsilon m}}{\partial t^2}(t) \right)^2 + \left(\lambda_m u_{\varepsilon m}(t) \right)^2 + \int_0^t \frac{k'}{k(t)+\varepsilon} \lambda_m^{\frac{1}{2}} \left(\frac{\partial u_{\varepsilon m}}{\partial \eta} \right)^2 d\eta \\ &\quad \leq d_{10} \left[\int_0^t \left(\frac{f_m(\eta)}{k(\eta)} \right)^2 d\eta + \left(\frac{\sqrt{\lambda_m} f_m(\eta)}{k(\eta)+\varepsilon} \right)^2 d\eta \right]. \end{aligned} \tag{43}$$

Using the spectral decomposition of the functions $u_\varepsilon(x,t)$ and $f(x,t)$ in the terms of $e_m(x)$ from (42) when $\varepsilon \to 0$, we get $u_\varepsilon \longrightarrow u \in W_2^1(D)$ and

$$\begin{aligned}\|u\|^2_{W^1_{2,k}(D)} &= \left\|\frac{1}{\sqrt{k(t)}}\frac{\partial u}{\partial t}\right\|^2_{L_2(\Omega)}(t) + \left\|(-\Delta_x)^{\frac{1}{2}}u\right\|^2_{L_2(\Omega)}(t) \\ &\quad + \int_0^t \left\|\sqrt{\frac{k'}{k}}\frac{\partial u}{\partial \eta}\right\|^2_{L_2(\Omega)}(\eta)\,d\eta + \|u\|^2_{L_2(\Omega)}(t) \\ &\le d_{11}\int_0^t \left\|\frac{f}{k}\right\|^2_{L_2(\Omega)}(\eta)\,d\eta.\end{aligned}$$

From (43), it also follows

Theorem 4.2 *Let $k \in C^{1+\alpha}[0,T]$, $1 > \alpha > 0$, $k(t) > 0$, $t > 0$, $k(0) = 0$, $k'(t) \ge 0$. If $\|\frac{f}{k}\|_{L_2(\Omega)}(t) < \infty$ and $\|\frac{(-\Delta_x)^{\frac{1}{2}}f}{k}\|_{L_2(\Omega)}(t)$, $t \in [0,T]$, then there exists a unique solution $u \in W^2_{2,k}(D)$ of the mixed Cauchy problem (10)–(12) that satisfies the inequality*

$$\begin{aligned}\|u\|^2{}_{W^2_{2,k}(D)} &= \left\|\frac{1}{\sqrt{k(t)}}\frac{\partial^2 u}{\partial t^2}\right\|^2_{L_2(\Omega)}(t) + \|\Delta_x u\|^2_{L_2(\Omega)}(t) \\ &\quad + \int_0^t \left\|\sqrt{\frac{k'}{k}}(-\Delta_x)^{\frac{1}{2}}\frac{\partial u}{\partial \eta}\right\|^2_{L_2(\Omega)}(\eta)\,d\eta + \|u\|^2_{L_2(\Omega)}(t) \\ &\le \int_0^t \left\|\frac{f}{k}\right\|^2_{L_2(\Omega)}(\eta)\,d\eta + \int_0^t \left\|\frac{(-\Delta_x)^{\frac{1}{2}}f}{k}\right\|^2_{L_2(\Omega)}(\eta)\,d\eta.\end{aligned}$$

Corollary 4.1 *Note that the weighted Sobolev space $W^2_{2,k}(D)$ is a subspace of the classical space $W_2^2(D)$. As in the case of strictly hyperbolic equations, we have first established that $u \in W_2^1(D)$ and $u \in W_2^2(D)$ under the condition $\|\frac{f}{k}\|_{L_2(\Omega)}(t) < \infty$ and $\|\frac{\mathrm{grad}_x f}{k}\|_{L_2(\Omega)}(t) < \infty$ for all $t \in [0,T]$, respectively.*

Using the inequalities (35)–(36), the unique solvability of the mixed Cauchy problem (10)–(12) for the general equation $\frac{\partial^2 u}{\partial t^2} - k(t)\Delta_x u + b(t)\frac{\partial u}{\partial t} + a(t)u = f(x,t)$ is established in exactly the same way.

Acknowledgements
The authors thank Professor M. Sadybekov for a very careful reading of the manuscript and useful comments.

Funding
The work was supported by the grant (No. AP09260126) of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan.

Abbreviations
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors completed the main study, carried out the results of this article, and drafted the paper. Both authors read and approved the final version of the manuscript.

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Received: 24 September 2021 Accepted: 3 May 2022 Published online: 13 May 2022

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