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On dynamic inequalities in two independent variables on time scales and their applications for boundary value problems

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Abstract
Our work is based on the multiple inequalities illustrated by Boudeliou and Khalaf in 2015. With the help of the Leibniz integral rule on time scales, we generalize a number of those inequalities to a general time scale. Besides that, in order to obtain some new inequalities as special cases, we also extend our inequalities to discrete, quantum, and continuous calculus. These inequalities may be of use in the analysis of some kinds of partial dynamic equations on time scales and their applications in environmental phenomena, physical and engineering sciences described by partial differential equations.

MSC: 26D10; 26D15; 34N05; 26E70

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1 Introduction
In 2015, Boudeliou and Khalaf [15] proved the following inequalities.

Theorem 1.1 *Let $u, f, \phi \in C(\Omega, \mathbb{R}_+)$ and $a \in C(\Omega, \mathbb{R}_+)$ be nondecreasing with respect to $(x,y) \in I_1 \times I_2$; let $\theta \in C^1(I_1, I_1)$, $\vartheta \in C^1(I_2, I_2)$ be nondecreasing with $\theta(x) \le x$ on I_1, $\vartheta(y) \le y$ on I_2. Let $\phi_1, \phi_2 \in C(\Omega, \mathbb{R}_+)$. Further, let $\psi, \omega, \eta \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\{\psi, \omega, \eta\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty} \psi(u) = +\infty$.*

(A_1) If u satisfies

$$\psi\big(u(x,y)\big) \le a(x,y) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t)\Big[f(s,t)\omega\big(u(s,t)\big) + \int_0^s \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\,d\tau\Big]\,dt\,ds$$

for $(x,y) \in \Omega$, then

$$u(x,y) \le \psi^{-1}\Big\{G^{-1}\Big(p(x,y) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t) f(s,t)\,dt\,ds\Big)\Big\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where G is defined by* (2.3) *and*

$$p(x,y) = G\big(a(x,y)\big) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t)\left(\int_0^s \phi_2(\tau,t)\,d\tau\right) dt\,ds$$

and $(x_1,y_1) \in \Omega$ *is chosen so that* $(p(x,y) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t) f(s,t)\,dt\,ds) \in \mathrm{Dom}(G^{-1})$.

(A_2) *If* $u(x,y)$ *satisfies*

$$\psi\big(u(x,y)\big) \le a(x,y) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t)\bigg[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big) + \int_0^s \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\,d\tau\bigg]\,dt\,ds$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \psi^{-1}\left\{G^{-1}\left(F^{-1}\left[F\big(p(x,y)\big) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t) f(s,t)\,dt\,ds\right]\right)\right\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where G and p are as in* (A_1), *and*

$$F(v) = \int_{v_0}^{v} \frac{ds}{\eta(\psi^{-1}(G^{-1}(s)))}, \quad v \ge v_0 > 0, \qquad F(+\infty) = +\infty$$

and $(x_1,y_1) \in \Omega$ *is chosen so that* $[F(p(x,y)) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t) f(s,t)\,dt\,ds] \in \mathrm{Dom}(F^{-1})$.

(A_3) *If* $u(x,y)$ *satisfies*

$$\psi\big(u(x,y)\big) \le a(x,y) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t)\bigg[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big) + \int_0^s \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\eta\big(u(\tau,t)\big)\,d\tau\bigg]\,dt\,ds$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \psi^{-1}\left\{G^{-1}\left(F^{-1}\left[p_0(x,y) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t) f(s,t)\,dt\,ds\right]\right)\right\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where*

$$p_0(x,y) = F\Big(G\big(a(x,y)\big)\Big) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t)\left(\int_0^s \phi_2(\tau,t)\,d\tau\right) dt\,ds$$

and $(x_1,y_1) \in \Omega$ *is chosen so that* $[p_0(x,y) + \int_0^{\theta(x)} \int_0^{\vartheta(y)} \phi_1(s,t) f(s,t)\,dt\,ds] \in \mathrm{Dom}(F^{-1})$.

Hilger in his PhD thesis [26] was the first to accomplish the unification and extension of differential equations, difference equations, q-difference equations, and so on to the encompassing theory of dynamic equations on time scales.

Throughout this work a knowledge and understanding of time scales and time-scale notation is assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [11, 13].

Over several decades Gronwall–Bellman-type inequalities, which have many applications in stability and oscillation theory, have attracted many researchers, and several refinements and extensions have been done to the previous results. For example, Yuzhen Mi [32] applied his results to study a boundary value problem of differential equations with impulsive terms. Also, we refer the reader to the works [1, 3, 4, 8, 18–20, 24, 34, 35, 40], see also [2, 5–7, 9, 10, 16, 17, 22, 27–30, 33, 36, 37].

Before we arrive at the main results in the next section, we need the following theorems and essential relations on some time scales such as $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$. Note that:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \quad \mu(t) = 0, \quad \psi^{\Delta}(t) = \psi'(t), \quad \int_a^b \psi(t)\Delta t = \int_a^b \psi(t)\,dt. \tag{1.1}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} &\sigma(t) = t+1, \qquad \mu(t) = 1, \qquad \psi^{\Delta}(t) = \psi(t+1) - \psi(t), \\ &\int_a^b \psi(t)\Delta t = \sum_{t=a}^{b-1} \psi(t). \end{aligned} \tag{1.2}$$

(iii) If $\mathbb{T} = h\mathbb{Z}$, then

$$\begin{aligned} &\sigma(t) = t+h, \qquad \mu(t) = h, \qquad \psi^{\Delta}(t) = \frac{\psi(t+h) - \psi(t)}{h}, \\ &\int_a^b \psi(t)\Delta t = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} \psi(th)h. \end{aligned} \tag{1.3}$$

(iv) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then

$$\begin{aligned} &\sigma(t) = qt, \qquad \mu(t) = (q-1)t, \qquad \psi^{\Delta}(t) = \frac{\psi(qt) - \psi(t)}{(q-1)t}, \\ &\int_a^b \psi(t)\Delta t = (q-1) \sum_{t=(\log_q a)}^{(\log_q b)-1} \psi\bigl(q^t\bigr)q^t. \end{aligned} \tag{1.4}$$

Theorem 1.2 *If f is Δ-integrable on $[a,b]$, then so is $|f|$, and*

$$\left| \int_a^b f(t)\Delta t \right| \le \int_a^b \bigl|f(t)\bigr|\Delta t.$$

Theorem 1.3 (Chain rule on time scales [12]) *Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^{\kappa}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists $c \in [t, \sigma(t)]_{\mathbb{R}}$ with*

$$(f \circ g)^{\Delta}(t) = f'\bigl(g(c)\bigr)g^{\Delta}(t). \tag{1.5}$$

Theorem 1.4 (see [14]) *Let $t_0 \in \mathbb{T}^{\kappa}$ and $k : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ be continuous at (t,t), where $t > t_0$ and $t \in \mathbb{T}^{\kappa}$. Assume that $k^{\Delta}(t,\cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. If for any $\varepsilon > 0$ there exists a* *neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\left|\left[k\big(\sigma(t),\tau\big) - k(s,\tau)\right] - k^{\Delta}(t,\tau)\left[\sigma(t) - s\right]\right| \le \varepsilon\left|\sigma(t) - s\right|, \quad \forall s \in U.$$

If k^{Δ} denotes the derivative of k with respect to the first variable, then

$$f(t) = \int_{t_0}^{t} k(t,\tau)\Delta\tau$$

yields

$$f^{\Delta}(t) = \int_{t_0}^{t} k^{\Delta}(t,\tau)\Delta\tau + k\big(\sigma(t),t\big).$$

Theorem 1.5 ([21, Leibniz rule on time scales]) *In the following, by $f^{\Delta}(t,s)$ we mean the delta derivative of $f(t,s)$ with respect to t. Similarly, $f^{\nabla}(t,s)$ is understood. If f, f^{Δ}, and f^{∇} are continuous and $u,h : \mathbb{T} \to \mathbb{T}$ are delta differentiable functions, then the following formulas hold $\forall t \in \mathbb{T}^{\kappa}$:*

(i) $[\int_{u(t)}^{h(t)} f(t,s)\Delta s]^{\Delta} = \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t)f(\sigma(t),h(t)) - u^{\Delta}(t)f(\sigma(t),u(t))$;
(ii) $[\int_{u(t)}^{h(t)} f(t,s)\Delta s]^{\nabla} = \int_{u(t)}^{h(t)} f^{\nabla}(t,s)\Delta s + h^{\nabla}(t)f(\rho(t),h(t)) - u^{\nabla}(t)f(\rho(t),u(t))$;
(iii) $[\int_{u(t)}^{h(t)} f(t,s)\nabla s]^{\Delta} = \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\nabla s + h^{\Delta}(t)f(\sigma(t),h(t)) - u^{\Delta}(t)f(\sigma(t),u(t))$;
(iv) $[\int_{u(t)}^{h(t)} f(t,s)\nabla s]^{\nabla} = \int_{u(t)}^{h(t)} f^{\nabla}(t,s)\nabla s + h^{\nabla}(t)f(\rho(t),h(t)) - u^{\nabla}(t)f(\rho(t),u(t))$.

In this manuscript, by applying Theorem 1.5, we discuss the retarded time scale case of the inequalities obtained in [15]. Furthermore, these inequalities that are proved here extend some known results in [23, 31, 38] and also unify the continuous, the discrete, and the quantum cases.

2 Main results

Lemma 2.1 *Suppose that $\mathbb{T}_1$, $\mathbb{T}_2$ are two times scales and $a \in C(\Omega = \mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R}_+)$ is nondecreasing with respect to $(x,y) \in \Omega$. Assume that ϕ, $u, f \in C(\Omega, \mathbb{R}_+)$, $\theta \in C^1(\mathbb{T}_1, \mathbb{T}_1)$, and $\vartheta \in C^1(\mathbb{T}_2, \mathbb{T}_2)$ are nondecreasing functions with $\theta(x) \le x$ on $\mathbb{T}_1$, $\vartheta(y) \le y$ on $\mathbb{T}_2$. Furthermore, suppose that ψ, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\psi, \omega\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty} \psi(u) = +\infty$. If $u(x,y)$ satisfies*

$$\psi\big(u(x,y)\big) \le a(x,y) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi(s,t)f(s,t)\omega\big(u(s,t)\big)\nabla t\Delta s \tag{2.1}$$

for $(x,y) \in \Omega$, then

$$u(x,y) \le \psi^{-1}\left\{G^{-1}G\big(a(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi(s,t)f(s,t)\nabla t\Delta s\right\} \tag{2.2}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, where

$$G(v) = \int_{v_0}^{v} \frac{\Delta s}{\omega(\psi^{-1}(s))}, \quad v \ge v_0 > 0, \qquad G(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta s}{\omega(\psi^{-1}(s))} = +\infty \tag{2.3}$$

and $(x_1, y_1) \in \Omega$ *is chosen so that*

$$\left(G\big(a(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t) f(s,t) \nabla t \Delta s\right) \in \operatorname{Dom}\big(G^{-1}\big).$$

Proof First we assume that $a(x,y) > 0$. Fixing an arbitrary $(x_0, y_0) \in \Omega$, we define a positive and nondecreasing function $z(x,y)$ by

$$z(x,y) = a(x_0,y_0) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi(s,t) f(s,t) \omega\big(u(s,t)\big) \nabla t \Delta s \tag{2.4}$$

for $0 \le x \le x_0 \le x_1$, $0 \le y \le y_0 \le y_1$, then $z(x_0, y) = z(x, y_0) = a(x_0, y_0)$ and

$$u(x,y) \le \psi^{-1}\big(z(x,y)\big). \tag{2.5}$$

Taking Δ-derivative for (2.4) with employing Theorem 1.5(i), we have

$$\begin{aligned} z^{\Delta_x}(x,y) &= \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi\big(\theta(x),t\big) f\big(\theta(x),t\big) \omega\big(u\big(\theta(x),t\big)\big) \nabla t \\ &\le \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi\big(\theta(x),t\big) f\big(\theta(x),t\big) \omega\big(\psi^{-1}\big(z\big(\theta(x),t\big)\big)\big) \nabla t \\ &\le \omega\big(\psi^{-1}\big(z\big(\theta(x),\vartheta(y)\big)\big)\big) \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi\big(\theta(x),t\big) f\big(\theta(x),t\big) \nabla t. \end{aligned} \tag{2.6}$$

Inequality (2.6) can be written in the form

$$\frac{z^{\Delta_x}(x,y)}{\omega(\psi^{-1}(z(x,y)))} \le \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi\big(\theta(x),t\big) f\big(\theta(x),t\big) \nabla t. \tag{2.7}$$

Taking Δ-integral for inequality (2.7) leads to

$$\begin{aligned} G\big(z(x,y)\big) &\le G\big(z(x_0,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi(s,t) f(s,t) \nabla t \Delta s \\ &\le G\big(a(x_0,y_0)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi(s,t) f(s,t) \nabla t \Delta s. \end{aligned}$$

Since $(x_0, y_0) \in \Omega$ is chosen arbitrarily,

$$z(x,y) \le G^{-1}\left[G\big(a(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi(s,t) f(s,t) \nabla t \Delta s\right]. \tag{2.8}$$

From (2.8) and (2.5) we obtain the desired result (2.2). We carry out the above procedure with $\epsilon > 0$ instead of $a(x,y)$ when $a(x,y) = 0$ and subsequently let $\epsilon \to 0$. □

Now, as special cases of our results, we will give the continuous, discrete, and quantum inequalities. Namely, in the cases of time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

Remark 2.2 If we take $\mathbb{T} = \mathbb{R}$, $x_0 = 0$, and $y_0 = 0$ in Lemma 2.1, then, by relation (1.1), inequality (2.1) becomes the inequality obtained in [15, Lemma 2.1].

Corollary 2.3 *If we take $\mathbb{T} = h\mathbb{Z}$ in Lemma 2.1 by relation (1.3), then the following inequality*

$$\psi\big(u(x,y)\big) \le a(x,y) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi(sh,th) f(sh,th)\omega\big(u(sh,th)\big)$$

for $(x,y) \in \Omega$ implies

$$u(x,y) \le \psi^{-1}\left\{G^{-1}G\big(a(x,y)\big) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi(sh,th) f(sh,th)\right\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, where

$$G(v) = \sum_{s=\frac{v_0}{h}}^{\frac{v}{h}-1} \frac{h}{\omega(\psi^{-1}(sh))}, \quad v \ge v_0 > 0, \qquad G(+\infty) = \sum_{s=\frac{v_0}{h}}^{+\infty} \frac{h}{\omega(\psi^{-1}(sh))} = +\infty$$

and $(x_1,y_1) \in \Omega$ is chosen so that

$$\left(G\big(a(x,y)\big) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th) f(sh,th)\right) \in \mathrm{Dom}\big(G^{-1}\big).$$

Remark 2.4 In Corollary 2.3, if we take $h = 1$, then the following inequality

$$\psi\big(u(x,y)\big) \le a(x,y) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi(s,t) f(s,t)\omega\big(u(s,t)\big)$$

for $(x,y) \in \Omega$ implies

$$u(x,y) \le \psi^{-1}\left\{G^{-1}G\big(a(x,y)\big) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi(s,t) f(s,t)\right\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, where

$$G(v) = \sum_{s=v_0}^{v-1} \frac{1}{\omega(\psi^{-1}(s))}, \quad v \ge v_0 > 0, \qquad G(+\infty) = \sum_{s=v_0}^{+\infty} \frac{1}{\omega(\psi^{-1}(s))} = +\infty$$

and $(x_1,y_1) \in \Omega$ is chosen so that

$$\left(G\big(a(x,y)\big) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t) f(s,t)\right) \in \mathrm{Dom}\big(G^{-1}\big).$$

Corollary 2.5 *If we take* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Lemma* 2.1 *by relation* (1.4), *then the following inequality*

$$\psi\big(u(x,y)\big) \le a(x,y) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi\big(q^s,q^t\big)f\big(q^s,q^t\big)\omega\big(u\big(q^s,q^t\big)\big)$$

for $(x,y) \in \Omega$ *implies*

$$u(x,y) \le \psi^{-1}\left\{G^{-1}G\big(a(x,y)\big) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi\big(q^s,q^t\big)f\big(q^s,q^t\big)\right\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where*

$$G(v) = \sum_{s=(\log_q v_0)}^{(\log_q v)-1} \frac{(q-1)q^s}{\omega(\psi^{-1}(q^s))}, \quad v \ge v_0 > 0, \qquad G(+\infty) = \sum_{s=(\log_q v_0)}^{+\infty} \frac{(q-1)q^s}{\omega(\psi^{-1}(q^s))} = +\infty$$

and $(x_1,y_1) \in \Omega$ *is chosen so that*

$$\left(G\big(a(x,y)\big) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1\big(q^s,q^t\big)f\big(q^s,q^t\big)\right) \in \operatorname{Dom}\big(G^{-1}\big).$$

Theorem 2.6 *Let* u, a, f, θ, *and* ϑ *be as in Lemma* 2.1. *Let* $\phi_1, \phi_2 \in C(\Omega, \mathbb{R}_+)$. *If* $u(x,y)$ *satisfies*

$$\begin{aligned}\psi\big(u(x,y)\big) \le a(x,y) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\bigg[f(s,t)\omega\big(u(s,t)\big) \\ + \int_{x_0}^{s} \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\Delta\tau\bigg]\nabla t\Delta s \qquad (2.9)\end{aligned}$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \psi^{-1}\left\{G^{-1}\left(p(x,y) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\right)\right\} \qquad (2.10)$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where* G *is defined by* (2.3) *and*

$$p(x,y) = G\big(a(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\left(\int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\right)\nabla t\Delta s \qquad (2.11)$$

and $(x_1,y_1) \in \Omega$ *is chosen so that*

$$\left(p(x,y) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\right) \in \operatorname{Dom}\big(G^{-1}\big).$$

Proof By the same steps of the proof of Lemma 2.1, we can obtain (2.10) with suitable changes. □

Remark 2.7 If we take $\phi_2(x,y) = 0$, then Theorem 2.6 reduces to Lemma 2.1.

Corollary 2.8 *Let the functions u, f, ϕ_1, ϕ_2, a, θ, and ϑ be as in Theorem 2.6. Further, suppose that $q > p > 0$ are constants. If $u(x,y)$ satisfies*

$$u^q(x,y) \le a(x,y) + \frac{q}{q-p}\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\biggl[f(s,t)u^p(s,t) + \int_{x_0}^{s} \phi_2(\tau,t)u^p(\tau,t)\Delta\tau\biggr]\nabla t\Delta s \tag{2.12}$$

for $(x,y) \in \Omega$, then

$$u(x,y) \le \biggl\{p(x,y) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\biggr\}^{\frac{1}{q-p}}, \tag{2.13}$$

where

$$p(x,y) = \bigl(a(x,y)\bigr)^{\frac{q-p}{q}} + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\biggl(\int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\biggr)\nabla t\Delta s.$$

Proof In Theorem 2.6, by letting $\psi(u) = u^q$, $\omega(u) = u^p$, we have

$$G(v) = \int_{v_0}^{v} \frac{\Delta s}{\omega(\psi^{-1}(s))} = \int_{v_0}^{v} \frac{\Delta s}{s^{\frac{p}{q}}} \ge \frac{q}{q-p}\bigl(v^{\frac{q-p}{q}} - v_0^{\frac{q-p}{q}}\bigr), \quad v \ge v_0 > 0$$

and

$$G^{-1}(v) \ge \biggl\{v_0^{\frac{q-p}{q}} + \frac{q-p}{q}v\biggr\}^{\frac{1}{q-p}}.$$

We obtain inequality (2.13). □

Theorem 2.9 *Under the hypotheses of Theorem 2.6, further, let ψ, ω, $\eta \in C(\mathbb{R}_+,\mathbb{R}_+)$ be nondecreasing functions with $\{\psi,\omega,\eta\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty}\psi(u) = +\infty$. If $u(x,y)$ satisfies*

$$\psi\bigl(u(x,y)\bigr) \le a(x,y) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\biggl[f(s,t)\omega\bigl(u(s,t)\bigr)\eta\bigl(u(s,t)\bigr) + \int_{x_0}^{s} \phi_2(\tau,t)\omega\bigl(u(\tau,t)\bigr)\Delta\tau\biggr]\nabla t\Delta s \tag{2.14}$$

for $(x,y) \in \Omega$, then

$$u(x,y) \le \psi^{-1}\biggl\{G^{-1}\biggl(F^{-1}\biggl[F\bigl(p(x,y)\bigr) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\biggr]\biggr)\biggr\} \tag{2.15}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, where G and p are as in (2.3), (2.11) respectively and

$$F(v) = \int_{v_0}^{v} \frac{\Delta s}{\eta(\psi^{-1}(G^{-1}(s)))}, \quad v \ge v_0 > 0, \qquad F(+\infty) = +\infty, \tag{2.16}$$

and $(x_1, y_1) \in \Omega$ *is chosen so that*

$$\left[F\big(p(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t) f(s,t) \nabla t \Delta s\right] \in \mathrm{Dom}\big(F^{-1}\big).$$

Proof Assume that $a(x,y) > 0$. Fixing arbitrary $(x_0, y_0) \in \Omega$, we define a positive and non-decreasing function $z(x,y)$ by

$$z(x,y) = a(x_0,y_0) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\Big[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big) \tag{2.17}$$

$$+ \int_{x_0}^{s} \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\Delta\tau\Big]\nabla t \Delta s \tag{2.18}$$

for $0 \le x \le x_0 \le x_1$, $0 \le y \le y_0 \le y_1$, then $z(x_0,y) = z(x,y_0) = a(x_0,y_0)$ and

$$u(x,y) \le \psi^{-1}\big(z(x,y)\big). \tag{2.19}$$

Taking Δ-derivative for (2.17) with employing Theorem 1.5(i) gives

$$z^{\Delta_x}(x,y) = \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi_1\big(\theta(x),t\big)\Big[f\big(\theta(x),t\big)\omega\big(u(\theta(x),t)\big)\eta\big(u(\theta(x),t)\big) \tag{2.20}$$

$$+ \int_{x_0}^{\theta(x)} \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\Delta\tau\Big]\nabla t \tag{2.21}$$

$$\le \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi_1\big(\theta(x),t\big)\Big[f\big(\theta(x),t\big)\omega\big(\psi^{-1}\big(z(\theta(x),t)\big)\big) \tag{2.22}$$

$$\times \eta\big(\psi^{-1}\big(z(\theta(x),t)\big)\big) + \int_{x_0}^{\theta(x)} \phi_2(\tau,t)\omega\big(\psi^{-1}\big(z(\tau,t)\big)\big)\Delta\tau\Big]\nabla t \tag{2.23}$$

$$\le \theta^{\Delta}(x).\omega\big(\psi^{-1}\big(z(\theta(x),\vartheta(y))\big)\big) \tag{2.24}$$

$$\times \int_{y_0}^{\vartheta(y)} \phi_1\big(\theta(x),t\big)\Big[f\big(\theta(x),t\big)\eta\big(\psi^{-1}\big(z(\theta(x),t)\big)\big) \tag{2.25}$$

$$+ \int_{x_0}^{\theta(x)} \phi_2(\tau,t)\Delta\tau\Big]\nabla t. \tag{2.26}$$

From (2.20) we have

$$\frac{z^{\Delta_x}(x,y)}{\omega(\psi^{-1}(z(x,y)))} \le \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi_1\big(\theta(x),t\big)\Big[f\big(\theta(x),t\big)\eta\big(\psi^{-1}\big(z(\theta(x),t)\big)\big) \tag{2.27}$$

$$+ \int_{x_0}^{\theta(x)} \phi_2(\tau,t)\Delta\tau\Big]\nabla t. \tag{2.28}$$

Taking Δ-integral for (2.27) gives

$$G\big(z(x,y)\big) \le G\big(z(x_0,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\Big[f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big)$$

$$+ \int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\Big]\nabla t \Delta s$$

$$\le G\big(a(x_0,y_0)\big) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\bigg[f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big)$$
$$+ \int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\bigg]\nabla t\Delta s.$$

Since $(x_0,y_0)\in\Omega$ is chosen arbitrarily, the last inequality can be rewritten as

$$G\big(z(x,y)\big) \le p(x,y) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big)\nabla t\Delta s. \tag{2.29}$$

Since $p(x,y)$ is a nondecreasing function, an application of Lemma 2.1 to (2.29) gives us

$$z(x,y) \le G^{-1}\bigg(F^{-1}\bigg[F\big(p(x,y)\big) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\bigg]\bigg). \tag{2.30}$$

From (2.19) and (2.30) we obtain the desired inequality (2.15).

Now, we take the case $a(x,y)=0$ for some $(x,y)\in\Omega$. Let $a_\epsilon(x,y)=a(x,y)+\epsilon$ for all $(x,y)\in\Omega$, where $\epsilon>0$ is arbitrary, then $a_\epsilon(x,y)>0$ and $a_\epsilon(x,y)\in C(\Omega,\mathbb{R}_+)$ are nondecreasing with respect to $(x,y)\in\Omega$. We carry out the above procedure with $a_\epsilon(x,y)>0$ instead of $a(x,y)$, and we get

$$u(x,y) \le \psi^{-1}\bigg\{G^{-1}\bigg(F^{-1}\bigg[F\big(p_\epsilon(x,y)\big) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\bigg]\bigg)\bigg\},$$

where

$$p_\epsilon(x,y) = G\big(a_\epsilon(x,y)\big) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\bigg(\int_{x_0}^{s}\phi_2(\tau,t)\Delta\tau\bigg)\nabla t\Delta s.$$

Letting $\epsilon\to 0^+$, we obtain (2.15). The proof is complete. □

Now, as special cases of our results, we will give the continuous, discrete, and quantum inequalities. Namely, in the cases of time scales $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=h\mathbb{Z}$, $\mathbb{T}=\mathbb{Z}$, and $\mathbb{T}=\overline{q^{\mathbb{Z}}}$.

Remark 2.10 If we take $\mathbb{T}=\mathbb{R}$, $x_0=0$, and $y_0=0$ in Theorem 2.9, then, by relation (1.1), inequality (2.14) becomes the inequality obtained in [15, Theorem 2.2(A_2)].

Corollary 2.11 *If we take $\mathbb{T}=h\mathbb{Z}$ in Theorem 2.9 by relation (1.3), then the following inequality*

$$\psi\big(u(x,y)\big) \le a(x,y) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1}\sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)\bigg[f(sh,th)\omega\big(u(sh,th)\big)\eta\big(u(sh,th)\big)$$
$$+ h\sum_{t=\frac{x_0}{h}}^{\frac{s}{h}-1}\phi_2(\tau,th)\omega\big(u(\tau,th)\big)\bigg]$$

for $(x,y)\in\Omega$ *implies*

$$u(x,y)\le\psi^{-1}\left\{G^{-1}\left(F^{-1}\left[F\big(p(x,y)\big)+h^2\sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1}\sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1}\phi_1(sh,th)f(sh,th)\right]\right)\right\}$$

for $0\le x\le x_1$, $0\le y\le y_1$, *where* G *and* p *are as in* (2.3) *and* (2.11), *respectively, and*

$$F(v)=\sum_{s=\frac{v_0}{h}}^{\frac{v}{h}}\frac{h}{\eta(\psi^{-1}(G^{-1}(sh)))},\qquad v\ge v_0>0,\qquad F(+\infty)=+\infty$$

and $(x_1,y_1)\in\Omega$ *is chosen so that*

$$\left[F\big(p(x,y)\big)+h^2\sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1}\sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1}\phi_1(sh,th)f(sh,th)\right]\in\operatorname{Dom}\big(F^{-1}\big).$$

Remark 2.12 In Corollary 2.11, if we take $h=1$, then the following inequality

$$\psi\big(u(x,y)\big)\le a(x,y)+\sum_{s=x_0}^{\theta(x)-1}\sum_{t=y_0}^{\vartheta(y)+1}\phi_1(s,t)\left[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big)\right.$$
$$\left.+\sum_{t=x_0}^{s-1}\phi_2(\tau,t)\omega\big(u(\tau,t)\big)\right]$$

for $(x,y)\in\Omega$ implies

$$u(x,y)\le\psi^{-1}\left\{G^{-1}\left(F^{-1}\left[F\big(p(x,y)\big)+\sum_{s=x_0}^{\theta(x)-1}\sum_{t=y_0}^{\vartheta(y)+1}\phi_1(s,t)f(s,t)s\right]\right)\right\}$$

for $0\le x\le x_1$, $0\le y\le y_1$, where G and p are as in (2.3), and

$$F(v)=\sum_{s=v_0}^{v-1}\frac{1}{\eta(\psi^{-1}(G^{-1}(s)))},\qquad v\ge v_0>0,\qquad F(+\infty)=+\infty,$$

and $(x_1,y_1)\in\Omega$ is chosen so that

$$\left[F\big(p(x,y)\big)+\sum_{s=x_0}^{\theta(x)-1}\sum_{t=y_0}^{\vartheta(y)+1}\phi_1(s,t)f(s,t)\right]\in\operatorname{Dom}\big(F^{-1}\big).$$

Corollary 2.13 *If we take* $\mathbb{T}=\overline{q^{\mathbb{Z}}}$ *in Theorem* 2.9 *by relation* (1.4), *then the following inequality*

$$\psi\big(u(x,y)\big)\le a(x,y)+(q-1)^2\sum_{s=(\log_q x_0)}^{(\log_q\theta(x))-1}\sum_{t=(\log_q y_0)}^{(\log_q\vartheta(y))+1}q^{(s+t)}\phi_1\big(q^s,q^t\big)$$
$$\times\left[f\big(q^s,q^t\big)\omega\big(u\big(q^s,q^t\big)\big)\eta\big(u\big(q^s,q^t\big)\big)\right.$$

$$+ (q-1)\sum_{t=(\log_q x_0)}^{(\log_q s)-1} q^t \phi_2(\tau, q^t)\omega(u(\tau,t))\Bigg]$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \psi^{-1}\Bigg\{G^{-1}\Bigg(F^{-1}\Bigg[F\big(p(x,y)\big) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1(q^s,q^t)f(q^s,q^t)s\Bigg]\Bigg)\Bigg\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where* G *and* p *are as in* (2.3), *and*

$$F(\nu) = \sum_{s=(\log_q \nu_0)}^{(\log_q \nu)-1} \frac{(q-1)q^s}{\eta(\psi^{-1}(G^{-1}(q^s)))}, \qquad \nu \ge \nu_0 > 0, \qquad F(+\infty) = +\infty,$$

and $(x_1,y_1) \in \Omega$ *is chosen so that*

$$\Bigg[F\big(p(x,y)\big) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1(q^s,q^t)f(q^s,q^t)\Bigg] \in \mathrm{Dom}(F^{-1}).$$

Corollary 2.14 *Let the functions* u, a, f, ϕ_1, ϕ_2, θ, *and* ϑ *be as in Theorem* 2.6. *Further, suppose that* q, p, *and* r *are constants with* $p > 0$, $r > 0$, *and* $q > p + r$. *If* $u(x,y)$ *satisfies*

$$u^q(x,y) \le a(x,y) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\Bigg[f(s,t)u^p(s,t)u^r(s,t) + \int_{x_0}^{s} \phi_2(\tau,t)u^p(\tau,t)\Delta\tau\Bigg]\nabla t\Delta s \tag{2.31}$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \Bigg\{\big[p(x,y)\big]^{\frac{q-p-r}{q-p}} + \frac{q-p-r}{q}\int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\Bigg\}^{\frac{1}{q-p-r}}, \tag{2.32}$$

where

$$p(x,y) = \big(a(x,y)\big)^{\frac{q-p}{q}} + \frac{q-p}{q}\int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\bigg(\int_{x_0}^{s}\phi_2(\tau,t)\Delta\tau\bigg)\nabla t\Delta s.$$

Proof An application of Theorem 2.9 with $\psi(u) = u^q$, $\omega(u) = u^p$, and $\eta(u) = u^r$ yields the desired inequality (2.32). □

Theorem 2.15 *Under the hypotheses of Theorem* 2.9. *If* $u(x,y)$ *satisfies*

$$\psi\big(u(x,y)\big) \le a(x,y) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\Bigg[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big) + \int_{x_0}^{s} \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\eta\big(u(\tau,t)\big)\Delta\tau\Bigg]\nabla t\Delta s \tag{2.33}$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \psi^{-1}\left\{G^{-1}\left(F^{-1}\left[p_0(x,y) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\right]\right)\right\} \tag{2.34}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where*

$$p_0(x,y) = F\big(G\big(a(x,y)\big)\big) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\left(\int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\right)\nabla t\Delta s,$$

and $(x_1,y_1) \in \Omega$ *is chosen so that*

$$\left[p_0(x,y) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\right] \in \operatorname{Dom}\big(F^{-1}\big).$$

Proof Assume that $a(x,y) > 0$. Fixing arbitrary $(x_0,y_0) \in \Omega$, we define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z(x,y) = a(x_0,y_0) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\bigg[&f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big) \\ &+ \int_{x_0}^{s} \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\eta\big(u(\tau,t)\big)\Delta\tau\bigg]\nabla t\Delta s \end{aligned}$$

for $0 \le x \le x_0 \le x_1$, $0 \le y \le y_0 \le y_1$, then $z(x_0,y) = z(x,y_0) = a(x_0,y_0)$, and

$$u(x,y) \le \psi^{-1}\big(z(x,y)\big). \tag{2.35}$$

By the same steps as the proof of Theorem 2.9, we obtain

$$\begin{aligned} z(x,y) \le G^{-1}\bigg\{G\big(a(x_0,y_0)\big) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\bigg[&f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big) \\ &+ \int_{x_0}^{s} \phi_2(\tau,t)\eta\big(\psi^{-1}\big(z(\tau,t)\big)\big)\Delta\tau\bigg]\nabla t\Delta s\bigg\}. \end{aligned}$$

We define a nonnegative and nondecreasing function $v(x,y)$ by

$$\begin{aligned} v(x,y) = G\big(a(x_0,y_0)\big) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\bigg[&\big[f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big)\big] \\ &+ \int_{x_0}^{s} \phi_2(\tau,t)\eta\big(\psi^{-1}\big(z(\tau,t)\big)\big)\Delta\tau\bigg]\nabla t\Delta s, \end{aligned}$$

then $v(x_0,y) = v(x,y_0) = G(a(x_0,y_0))$,

$$z(x,y) \le G^{-1}\big[v(x,y)\big], \tag{2.36}$$

and then

$$\begin{aligned}
v^{\Delta x}(x,y) &\leq \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi_1\big(\theta(x),t\big)\Big[f\big(\theta(x),t\big)\eta\big(\psi^{-1}\big(G^{-1}\big(v\big(\theta(x),y\big)\big)\big)\big) \\
&\quad + \int_{x_0}^{\theta(x)} \phi_2(\tau,t)\eta\big(\psi^{-1}\big(G^{-1}\big(v(\tau,y)\big)\big)\big)\Delta\tau\Big]\nabla t \\
&\leq \theta^{\Delta}(x)\eta\big(\psi^{-1}\big(G^{-1}\big(v\big(\theta(x),\vartheta(y)\big)\big)\big)\big) \int_{y_0}^{\vartheta(y)} \phi_1\big(\theta(x),t\big)\Big[f\big(\theta(x),t\big) \\
&\quad + \int_{x_0}^{\theta(x)} \phi_2(\tau,t)\Delta\tau\Big]\nabla t
\end{aligned}$$

or

$$\frac{v^{\Delta x}(x,y)}{\eta(\psi^{-1}(G^{-1}(v(x,y))))} \leq \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi_1\big(\theta(x),t\big)\Big[f\big(\theta(x),t\big) + \int_{x_0}^{\theta(x)} \phi_2(\tau,t)\Delta\tau\Big]\nabla t.$$

Taking Δ-integral for the above inequality gives

$$F\big(v(x,y)\big) \leq F\big(v(x_0,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\Big[f(s,t) + \int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\Big]\nabla t\Delta s$$

or

$$\begin{aligned}
v(x,y) &\leq F^{-1}\Big\{F\big(G\big(a(x_0,y_0)\big)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\Big[f(s,t) \\
&\quad + \int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\Big]\nabla t\Delta s\Big\}. \qquad (2.37)
\end{aligned}$$

From (2.35)–(2.37), and since $(x_0,y_0) \in \Omega$ is chosen arbitrarily, we obtain the desired inequality (2.34). If $a(x,y) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(x,y)$ and subsequently let $\epsilon \to 0$. The proof is complete. □

Now, as special cases of our results, we will give the continuous, discrete, and quantum inequalities. Namely, in the cases of time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

Remark 2.16 If we take $\mathbb{T} = \mathbb{R}$ and $x_0 = 0$ and $y_0 = 0$ in Theorem 2.15, then, by relation (1.1), inequality (2.33) becomes the inequality obtained in [15, Theorem 2.2(A_3)].

Corollary 2.17 *If we take $\mathbb{T} = h\mathbb{Z}$ in Theorem 2.15 by relation (1.3), then the following inequality*

$$\begin{aligned}
\psi\big(u(x,y)\big) &\leq a(x,y) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)\Big[f(sh,th)\omega\big(u(sh,th)\big)\eta\big(u(sh,th)\big) \\
&\quad + h\sum_{t=x_0}^{\frac{s}{h}-1} \phi_2(\tau,th)\omega\big(u(\tau,th)\big)\eta\big(u(\tau,th)\big)\Big]
\end{aligned}$$

for $(x,y) \in \Omega$ *implies*

$$u(x,y) \le \psi^{-1}\left\{G^{-1}\left(F^{-1}\left[p_0(x,y) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)f(sh,th)\right]\right)\right\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where*

$$p_0(x,y) = F\big(G\big(a(x,y)\big)\big) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)\left(\sum_{t=\frac{x_0}{h}}^{\frac{s}{h}} \phi_2(\tau,th)\right),$$

and $(x_1,y_1) \in \Omega$ *is chosen so that*

$$\left[p_0(x,y) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)f(sh,th)\right] \in \mathrm{Dom}\big(F^{-1}\big).$$

Remark 2.18 In Corollary 2.17, if we take $h = 1$, then the following inequality

$$\begin{aligned}\psi\big(u(x,y)\big) \le a(x,y) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t)\Bigg[&f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big)\\ &+ \sum_{s=x_0}^{s-1} \phi_2(\tau,t)\omega\big(u(\tau,t)\big)\eta\big(u(\tau,t)\big)\Bigg]\end{aligned}$$

for $(x,y) \in \Omega$ implies

$$u(x,y) \le \psi^{-1}\left\{G^{-1}\left(F^{-1}\left[p_0(x,y) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t)f(s,t)\right]\right)\right\}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, where

$$p_0(x,y) = F\big(G\big(a(x,y)\big)\big) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t)\left(\sum_{t=x_0}^{s-1} \phi_2(\tau,t)\right),$$

and $(x_1,y_1) \in \Omega$ is chosen so that

$$\left[p_0(x,y) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t)f(s,t)\right] \in \mathrm{Dom}\big(F^{-1}\big).$$

Corollary 2.19 *If we take* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem* 2.15 *by relation* (1.4), *then the following inequality*

$$\begin{aligned}\psi\big(u(x,y)\big) \le a(x,y) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)} \\ \times \phi_1\big(q^s,q^t\big)\Bigg[f\big(q^s,q^t\big)\omega\big(u\big(q^s,q^t\big)\big)\eta\big(u\big(q^s,q^t\big)\big) \\ + (q-1)\sum_{s=(\log_q x_0)}^{(\log_q s)-1} q^t\phi_2\big(\tau,q^t\big)\omega\big(u\big(\tau,q^t\big)\big)\eta\big(u\big(\tau,q^t\big)\big)\Bigg]\end{aligned}$$

for $(x,y) \in \Omega$ *implies*

$$\begin{aligned}u(x,y) \le \psi^{-1}\Bigg\{G^{-1}\Bigg(F^{-1}\Bigg[p_0(x,y) \\ + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1\big(q^s,q^t\big)f\big(q^s,q^t\big)\Bigg]\Bigg)\Bigg\}\end{aligned}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where*

$$p_0(x,y) = F\big(G\big(a(x,y)\big)\big) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1\big(q^s,q^t\big)\Bigg(\sum_{t=(\log_q x_0)}^{(\log_q s)-1}\phi_2\big(\tau,q^t\big)\Bigg)$$

and $(x_1,y_1) \in \Omega$ *is chosen so that*

$$\Bigg[p_0(x,y) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1\big(q^s,q^t\big)f\big(q^s,q^t\big)\Bigg] \in \mathrm{Dom}\big(F^{-1}\big).$$

Corollary 2.20 *Under the hypotheses of Corollary* 2.14. *If* $u(x,y)$ *satisfies*

$$\begin{aligned}u^q(x,y) \le a(x,y) + \int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \phi_1(s,t)\Bigg[f(s,t)u^p(s,t)u^r(s,t) \\ + \int_{x_0}^{s}\phi_2(\tau,t)u^p(\tau,t)u^r(\tau,t)\Delta\tau\Bigg]\nabla t\Delta s\end{aligned} \tag{2.38}$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \Bigg\{p_0(x,y) + \frac{q-p-r}{q}\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)}\phi_1(s,t)f(s,t)\nabla t\Delta s\Bigg\}^{\frac{1}{q-p-r}}, \tag{2.39}$$

where

$$p_0(x,y) = \big(a(x,y)\big)^{\frac{q-p-r}{q}} + \frac{q-p-r}{q}\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)}\phi_1(s,t)\bigg(\int_{x_0}^{s}\phi_2(\tau,t)\Delta\tau\bigg)\nabla t\Delta s.$$

Proof An application of Theorem 2.15 with $\psi(u) = u^q$, $\omega(u) = u^p$, and $\eta(u) = u^r$ yields the desired inequality (2.39). □

Theorem 2.21 *Under the hypotheses of Theorem 2.9. If $u(x,y)$ satisfies*

$$\psi\big(u(x,y)\big) \le a(x,y) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\eta\big(u(s,t)\big) \times \left[f(s,t)\omega\big(u(s,t)\big) + \int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\right]\nabla t\Delta s \tag{2.40}$$

for $(x,y) \in \Omega$, then

$$u(x,y) \le \psi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1\big(p_1(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\right]\right)\right\} \tag{2.41}$$

for $0 \le x \le x_2$, $0 \le y \le y_2$, where

$$G_1(\nu) = \int_{\nu_0}^{\nu} \frac{\Delta s}{\eta(\psi^{-1}(s))}, \quad \nu \ge \nu_0 > 0, \qquad G_1(+\infty) = \int_{\nu_0}^{+\infty} \frac{\Delta s}{\eta(\psi^{-1}(s))} = +\infty$$

$$F_1(\nu) = \int_{\nu_0}^{\nu} \frac{\Delta s}{\omega[\psi^{-1}(G_1^{-1}(s))]}, \quad \nu \ge \nu_0 > 0, \qquad F_1(+\infty) = +\infty$$

$$p_1(x,y) = G_1\big(a(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\left(\int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\right)\nabla t\Delta s,$$

and $(x_2,y_2) \in \Omega$ is chosen so that

$$\left[F_1\big(p_1(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)f(s,t)\nabla t\Delta s\right] \in \operatorname{Dom}\big(F_1^{-1}\big).$$

Proof Suppose that $a(x,y) > 0$. Fixing an arbitrary $(x_0,y_0) \in \Omega$, we define a positive and nondecreasing function $z(x,y)$ by

$$z(x,y) = a(x_0,y_0) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\eta\big(u(s,t)\big)\left[f(s,t)\omega\big(u(s,t)\big) + \int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\right]\nabla t\Delta s$$

for $0 \le x \le x_0 \le x_2$, $0 \le y \le y_0 \le y_2$, then $z(x_0,y) = z(x,y_0) = a(x_0,y_0)$,

$$u(x,y) \le \psi^{-1}\big(z(x,y)\big) \tag{2.42}$$

and

$$z^{\Delta_x}(x,y) \le \theta^{\Delta}(x) \int_{y_0}^{\vartheta(y)} \phi_1\big(\theta(x),t\big)\eta\big[\psi^{-1}\big(z\big(\theta(x),t\big)\big)\big]\left[f\big(\theta(x),t\big)\omega\big(\psi^{-1}\big(z\big(\theta(x),t\big)\big)\big) + \int_{x_0}^{\theta(x)} \phi_2(\tau,t)\Delta\tau\right]\nabla t$$

$$\leq \theta^{\Delta}(x)\eta\big[\psi^{-1}\big(z\big(\theta(x),\vartheta(y)\big)\big)\big]\int_{y_0}^{\vartheta(y)}\phi_1\big(\theta(x),t\big)\bigg[f\big(\theta(x),t\big)\omega\big(\psi^{-1}\big(z\big(\theta(x),t\big)\big)\big)$$
$$+\int_{x_0}^{\theta(x)}\phi_2(\tau,t)\Delta\tau\bigg]\nabla t,$$

then

$$\frac{z^{\Delta_x}(x,y)}{\eta[\psi^{-1}(z(x,y))]}\leq \theta^{\Delta}(x)\int_{y_0}^{\vartheta(y)}\phi_1\big(\theta(x),t\big)\bigg[f\big(\theta(x),t\big)\omega\big(\psi^{-1}\big(z\big(\theta(x),t\big)\big)\big)$$
$$+\int_{x_0}^{\theta(x)}\phi_2(\tau,t)\Delta\tau\bigg]\nabla t.$$

Taking Δ-integral for the above inequality gives

$$G_1\big(z(x,y)\big)\leq G_1\big(z(0,y)\big)+\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)}\phi_1(s,t)\bigg[f(s,t)\omega\big(\psi^{-1}\big(z(s,t)\big)\big)$$
$$+\int_{x_0}^{s}\phi_2(\tau,t)\Delta\tau\bigg]\nabla t\Delta s,$$

then

$$G_1\big(z(x,y)\big)\leq G_1\big(a(x_0,y_0)\big)+\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)}\phi_1(s,t)\bigg[f(s,t)\omega\big(\psi^{-1}\big(z(s,t)\big)\big)$$
$$+\int_{x_0}^{s}\phi_2(\tau,t)\Delta\tau\bigg]\nabla t\Delta s.$$

Since $(x_0,y_0)\in\Omega$ is chosen arbitrarily, the last inequality can be restated as

$$G_1\big(z(x,y)\big)\leq p_1(x,y)+\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)}\phi_1(s,t)f(s,t)\omega\big(\psi^{-1}\big(z(s,t)\big)\big)\nabla t\Delta s. \tag{2.43}$$

It is easy to observe that $p_1(x,y)$ is a positive and nondecreasing function for all $(x,y)\in\Omega$, then an application of Lemma 2.1 to (2.43) yields the inequality

$$z(x,y)\leq G_1^{-1}\bigg(F_1^{-1}\bigg[F_1\big(p_1(x,y)\big)+\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)}\phi_1(s,t)f(s,t)\nabla t\Delta s\bigg]\bigg). \tag{2.44}$$

From (2.44) and (2.42) we get the desired inequality (2.41).

If $a(x,y)=0$, we carry out the above procedure with $\epsilon>0$ instead of $a(x,y)$ and subsequently let $\epsilon\to 0$. The proof is complete. □

Now, as special cases of our results, we will give the continuous, discrete, and quantum inequalities. Namely, in the cases of time scales $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=h\mathbb{Z}$, $\mathbb{T}=\mathbb{Z}$, and $\mathbb{T}=\overline{q^{\mathbb{Z}}}$.

Remark 2.22 If we take $\mathbb{T}=\mathbb{R}$ and $x_0=0$ and $y_0=0$ in Theorem 2.21, then, by relation (1.1), inequality (2.41) becomes the inequality obtained in [15, Theorem 2.7].

Corollary 2.23 *If we take $\mathbb{T} = h\mathbb{Z}$ in Theorem 2.15 by relation (1.3), then the following inequality*

$$\psi\big(u(x,y)\big) \le a(x,y) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)\eta\big(u(sh,th)\big) \times \Bigg[f(sh,th)\omega\big(u(sh,th)\big) + \sum_{t=\frac{x_0}{h}}^{\frac{s}{h}-1} \phi_2(\tau,th)\Bigg]$$

for $(x,y) \in \Omega$, then

$$u(x,y) \le \psi^{-1}\Bigg\{G_1^{-1}\Bigg(F_1^{-1}\Bigg[F_1\big(p_1(x,y)\big) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)f(sh,th)\Bigg]\Bigg)\Bigg\}$$

for $0 \le x \le x_2$, $0 \le y \le y_2$, where

$$G_1(v) = \sum_{s=\frac{v_0}{h}}^{\frac{v}{h}-1} \frac{h}{\eta(\psi^{-1}(sh))}, \quad v \ge v_0 > 0, \qquad G_1(+\infty) = \sum_{s=\frac{v_0}{h}}^{+\infty} \frac{h}{\eta(\psi^{-1}(sh))} = +\infty$$

$$F_1(v) = \sum_{s=\frac{v_0}{h}}^{\frac{v}{h}-1} \frac{h}{\omega[\psi^{-1}(G_1^{-1}(sh))]}, \quad v \ge v_0 > 0, \qquad F_1(+\infty) = +\infty$$

$$p_1(x,y) = G_1\big(a(x,y)\big) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)\Bigg(h\sum_{t=\frac{x_0}{h}}^{\frac{s}{h}-1} \phi_2(\tau,th)\Bigg),$$

and $(x_2,y_2) \in \Omega$ is chosen so that

$$\Bigg[F_1\big(p_1(x,y)\big) + h^2 \sum_{s=\frac{x_0}{h}}^{\frac{\theta(x)}{h}-1} \sum_{t=\frac{y_0}{h}}^{\frac{\vartheta(y)}{h}+1} \phi_1(sh,th)f(sh,th)\Bigg] \in \mathrm{Dom}\big(F_1^{-1}\big).$$

Corollary 2.24 *In Corollary 2.23, if we take $h = 1$, then the following inequality*

$$\psi\big(u(x,y)\big) \le a(x,y) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t)\eta\big(u(s,t)\big) \times \Bigg[f(s,t)\omega\big(u(s,t)\big) + \sum_{t=x_0}^{s-1} \phi_2(\tau,t)\Bigg]$$

for $(x,y) \in \Omega$, then

$$u(x,y) \le \psi^{-1}\Bigg\{G_1^{-1}\Bigg(F_1^{-1}\Bigg[F_1\big(p_1(x,y)\big) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t)f(s,t)\Bigg]\Bigg)\Bigg\}$$

for $0 \le x \le x_2$, $0 \le y \le y_2$, *where*

$$G_1(v) = \sum_{s=v_0}^{v-1} \frac{1}{\eta(\psi^{-1}(s))}, \quad v \ge v_0 > 0, \qquad G_1(+\infty) = \sum_{s=v_0}^{+\infty} \frac{1}{\eta(\psi^{-1}(s))} = +\infty,$$

$$F_1(v) = \sum_{s=v_0}^{v-1} \frac{1}{\omega[\psi^{-1}(G_1^{-1}(s))]}, \quad v \ge v_0 > 0, \qquad F_1(+\infty) = +\infty,$$

$$p_1(x,y) = G_1\bigl(a(x,y)\bigr) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t)\left(\sum_{t=x_0}^{s-1} \phi_2(\tau,t)\right)$$

and $(x_2,y_2) \in \Omega$ *is chosen so that*

$$\left[F_1\bigl(p_1(x,y)\bigr) + \sum_{s=x_0}^{\theta(x)-1} \sum_{t=y_0}^{\vartheta(y)+1} \phi_1(s,t)f(s,t)\right] \in \operatorname{Dom}\bigl(F_1^{-1}\bigr).$$

Corollary 2.25 *If we take* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *in Theorem* 2.21 *by relation* (1.4), *then the following inequality*

$$\begin{aligned}\psi\bigl(u(x,y)\bigr) \le{}& a(x,y) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1\bigl(q^s,q^t\bigr)\eta\bigl(u\bigl(q^s,q^t\bigr)\bigr)\\ &\times\left[f\bigl(q^s,q^t\bigr)\omega\bigl(u\bigl(q^s,q^t\bigr)\bigr) + (q-1)\sum_{t=(\log_q x_0)}^{(\log_q s)-1} q^t\phi_2\bigl(\tau,q^t\bigr)\right]\end{aligned}$$

for $(x,y) \in \Omega$, *then*

$$\begin{aligned}u(x,y) \le \psi^{-1}\Biggl\{G_1^{-1}\Biggl(F_1^{-1}\Biggl[&F_1\bigl(p_1(x,y)\bigr)\\ &+ (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1\bigl(q^s,q^t\bigr)f\bigl(q^s,q^t\bigr)\Biggr]\Biggr)\Biggr\}\end{aligned}$$

for $0 \le x \le x_2$, $0 \le y \le y_2$, *where*

$$G_1(v) = \sum_{s=(\log_q v_0)}^{(\log_q v)-1} \frac{(q-1)q^s}{\eta(\psi^{-1}(q^s))}, \quad v \ge v_0 > 0,$$

$$G_{(q-1)q^s}(+\infty) = \sum_{s=(\log_q v_0)}^{+\infty} \frac{(q-1)q^s}{\eta(\psi^{-1}(q^s))} = +\infty,$$

$$F_1(v) = \sum_{s=(\log_q v_0)}^{(\log_q v)-1} \frac{(q-1)q^s}{\omega[\psi^{-1}(G_1^{-1}(q^s))]}, \quad v \ge v_0 > 0, \qquad F_1(+\infty) = +\infty,$$

$$p_1(x,y) = G_1\bigl(a(x,y)\bigr) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1\bigl(q^s,q^t\bigr)\left(\sum_{t=x_0}^{s-1} \phi_2\bigl(\tau,q^t\bigr)\right)$$

and $(x_2, y_2) \in \Omega$ *is chosen so that*

$$\left[F_1\big(p_1(x,y)\big) + (q-1)^2 \sum_{s=(\log_q x_0)}^{(\log_q \theta(x))-1} \sum_{t=(\log_q y_0)}^{(\log_q \vartheta(y))+1} q^{(s+t)}\phi_1\big(q^s, q^t\big) f\big(q^s, q^t\big)\right] \in \operatorname{Dom}\big(F_1^{-1}\big).$$

Theorem 2.26 *Under the hypotheses of Theorem* 2.9, *and let* p *be a nonnegative constant. If* $u(x,y)$ *satisfies*

$$\begin{aligned} \psi\big(u(x,y)\big) \le a(x,y) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t) u^p(s,t) \\ \times \left[f(s,t)\omega\big(u(s,t)\big) + \int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\right]\nabla t \Delta s \end{aligned} \tag{2.45}$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \psi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1\big(p_1(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t) f(s,t)\nabla t \Delta s\right]\right)\right\} \tag{2.46}$$

for $0 \le x \le x_2$, $0 \le y \le y_2$, *where*

$$G_1(\nu) = \int_{\nu_0}^{\nu} \frac{\Delta s}{[\psi^{-1}(s)]^p}, \quad \nu \ge \nu_0 > 0, \qquad G_1(+\infty) = \int_{\nu_0}^{+\infty} \frac{\Delta s}{[\psi^{-1}(s)]^p} = +\infty, \tag{2.47}$$

and F_1, p_1 *are as in Theorem* 2.21 *and* $(x_2, y_2) \in \Omega$ *is chosen so that*

$$\left[F_1\big(p_1(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t) f(s,t)\nabla t \Delta s\right] \in \operatorname{Dom}\big(F_1^{-1}\big).$$

Proof An application of Theorem 2.21 with $\eta(u) = u^p$ yields the desired inequality (2.46). □

Remark 2.27 Taking $\mathbb{T} = \mathbb{R}$. The inequality established in Theorem 2.26 generalizes [38, Theorem 1] (with $p = 1$, $a(x,y) = b(x) + c(y)$, $x_0 = 0$, $y_0 = 0$, $\phi_1(s,t) f(s,t) = h(s,t)$, and $\phi_1(s,t)(\int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau) = g(s,t)$).

Corollary 2.28 *Under the hypotheses of Theorem* 2.26, *and let* $q > p > 0$ *be constants. If* $u(x,y)$ *satisfies*

$$\begin{aligned} u^q(x,y) \le a(x,y) + \frac{p}{p-q} \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t) u^p(s,t) \\ \times \left[f(s,t)\omega\big(u(s,t)\big) + \int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\right]\nabla t \Delta s \end{aligned} \tag{2.48}$$

for $(x,y) \in \Omega$, *then*

$$u(x,y) \le \left\{F_1^{-1}\left[F_1\big(p_1(x,y)\big) + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t) f(s,t)\nabla t \Delta s\right]\right\}^{\frac{1}{q-p}} \tag{2.49}$$

for $0 \le x \le x_2$, $0 \le y \le y_2$, *where*

$$p_1(x,y) = \left[a(x,y)\right]^{\frac{q-p}{q}} + \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \phi_1(s,t)\left(\int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau\right)\nabla t \Delta s$$

and F_1 *is defined in Theorem* 2.21.

Proof An application of Theorem 2.26 with $\psi(u(x,y)) = u^p$ to (2.48) yields inequality (2.49); to save space, we omit the details. □

Remark 2.29 Taking $\mathbb{T} = \mathbb{R}$, $x_0 = 0$, $y_0 = 0$, $a(x,y) = b(x) + c(y)$, $\phi_1(s,t)f(s,t) = h(s,t)$, and $\phi_1(s,t)(\int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau) = g(s,t)$ in Corollary 2.28, we obtain [39, Theorem 1].

Remark 2.30 Taking $\mathbb{T} = \mathbb{R}$, $x_0 = 0$, $y_0 = 0$, $a(x,y) = c^{\frac{p}{p-q}}$, $\phi_1(s,t)f(s,t) = h(t)$, and $\phi_1(s,t)(\int_{x_0}^{s} \phi_2(\tau,t)\Delta\tau) = g(t)$ and keeping y fixed in Corollary 2.28, we obtain [25, Theorem 2.1].

3 Application

In what follows, we discus the boundedness of the solutions of the initial boundary value problem for partial delay dynamic equation of the form

$$\begin{aligned} &\left(z^q\right)^{\Delta_x \nabla_y}(x,y) = A\left(x,y,z\big(x-h_1(x),y-h_2(y)\big),\int_{x_0}^{x} B\big(s,y,z\big(s-h_1(s),y\big)\big)\Delta s\right), \\ &z(x,y_0) = a_1(x), \qquad z(x_0,y) = a_2(y), \qquad a_1(x_0) = a_{y_0}(0) = 0 \end{aligned} \tag{3.1}$$

for $(x,y) \in \Omega$, where $z,b \in C(\Omega,\mathbb{R}_+)$, $A \in C(\Omega \times R^2,R)$, $B \in C(\Omega \times R,R)$, and $h_1 \in C^1(\mathbb{T}_1,\mathbb{R}_+)$, $h_2 \in C^1(\mathbb{T}_2,\mathbb{R}_+)$ are nondecreasing functions such that $h_1(x) \le x$ on $\mathbb{T}_1$, $h_2(y) \le y$ on $\mathbb{T}_2$, and $h_1^{\Delta}(x) < 1$, $h_2^{\Delta}(y) < 1$.

Theorem 3.1 *Assume that the functions* a_1, a_2, A, B *in* (3.1) *satisfy the conditions*

$$\left|a_1(x) + a_2(y)\right| \le a(x,y) \tag{3.2}$$

$$\left|A(s,t,z,u)\right| \le \frac{q}{q-p}\phi_1(s,t)\left[f(s,t)|z|^p + |u|\right] \tag{3.3}$$

$$\left|B(\tau,t,z)\right| \le \phi_2(\tau,t)|z|^p, \tag{3.4}$$

where $a(x,y)$, $\phi_1(s,t)$, $f(s,t)$, *and* $\phi_2(\tau,t)$ *are as in Theorem* 2.6, $q > p > 0$ *are constants. If* $z(x,y)$ *satisfies* (3.1), *then*

$$\left|z(x,y)\right| \le \left\{p(x,y) + M_1 M_2 \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \bar{\phi}_1(s,t)\bar{f}(s,t)\nabla t \Delta s\right\}^{\frac{1}{q-p}}, \tag{3.5}$$

where

$$\begin{aligned} p(x,y) = {} &\left(a(x,y)\right)^{\frac{q-p}{q}} \\ &+ M_1 M_2 \int_{x_0}^{\theta(x)} \int_{y_0}^{\vartheta(y)} \bar{\phi}_1(s,t)\left(M_1 \int_{x_0}^{s} \bar{\phi}_2(\tau,t)\Delta\tau\right)\nabla t \Delta s \end{aligned}$$

and

$$M_1 = \operatorname*{Max}_{x\in I_1} \frac{1}{1-h_1^{\Delta}(x)}, \qquad M_2 = \operatorname*{Max}_{y\in I_2} \frac{1}{1-h_2^{\Delta}(y)}$$

and $\bar{\phi}_1(\gamma,\xi) = \phi_1(\gamma + h_1(s), \xi + h_2(t))$, $\bar{\phi}_2(\mu,\xi) = \phi_2(\mu, \xi + h_2(t))$, $\bar{f}(\gamma,\xi) = f(\gamma + h_1(s), \xi + h_2(t))$.

Proof If $z(x,y)$ is any solution of (3.1), then

$$\begin{aligned} z^q(x,y) &= a_1(x) + a_2(y) \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} A\Big(s,t,z\big(s-h_1(s),t-h_2(t)\big), \\ &\quad \int_{x_0}^{s} B\big(\tau,t,z\big(\tau-h_1(\tau),t\big)\big)\Delta\tau\Big)\nabla t\Delta s. \end{aligned} \tag{3.6}$$

Using conditions (3.2)–(3.4) in (3.6), we obtain

$$\begin{aligned} \big|z(x,y)\big|^q &\le a(x,y) + \frac{q-p}{q}\int_{x_0}^{x}\int_{y_0}^{y} \phi_1(s,t)\Big[f(s,t)\big|z\big(s-h_1(s),t-h_2(t)\big)\big|^p \\ &\quad + \int_{x_0}^{s} \phi_2(\tau,t)\big|z(\tau,t)\big|^p\Delta\tau\Big]\nabla t\Delta s. \end{aligned} \tag{3.7}$$

Now, making a change of variables on the right-hand side of (3.7), $s-h_1(s)=\gamma$, $t-h_2(t)=\xi$, $x-h_1(x)=\theta(x)$ for $x\in\mathbb{T}_1$, $y-h_2(y)=\vartheta(y)$ for $y\in\mathbb{T}_2$, we obtain the inequality

$$\begin{aligned} \big|z(x,y)\big|^q &\le a(x,y) + \frac{q-p}{q}M_1M_2\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \bar{\phi}_1(\gamma,\xi)\Big[\bar{f}(\gamma,\xi)\big|z(\gamma,\xi)\big|^p \\ &\quad + M_1\int_{x_0}^{\gamma} \bar{\phi}_2(\mu,\xi)\big|z(\mu,t)\big|^p\Delta\mu\Big]\nabla\xi\Delta\gamma. \end{aligned} \tag{3.8}$$

We can rewrite inequality (3.8) as follows:

$$\begin{aligned} \big|z(x,y)\big|^q &\le a(x,y) + \frac{q-p}{q}M_1M_2\int_{x_0}^{\theta(x)}\int_{y_0}^{\vartheta(y)} \bar{\phi}_1(s,t)\Big[\bar{f}(s,t)\big|z(s,t)\big|^p \\ &\quad + M_1\int_{x_0}^{s} \bar{\phi}_2(\tau,t)\big|z(\tau,t)\big|^p\Delta\tau\Big]\nabla t\Delta s. \end{aligned} \tag{3.9}$$

As an application of Corollary 2.8 to (3.9) with $u(x,y) = |z(x,y)|$, we obtain the desired inequality (3.5). The proof is complete. □

4 Conclusion

In this article, we explored new generalizations of the integral retarded inequality given in [15] by the utilization of the integral rule on time scales. We generalized a number of those inequalities to a general time scale. Besides that, in order to obtain some new inequalities as special cases, we also extended our inequalities to discrete, quantum, and continuous calculus. Also, we studied the qualitative properties of solutions of some types of dynamic equations on time scales.

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