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Competing anisotropic and Finsler (p,q)-Laplacian problems

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Abstract

The aim of this paper is to prove the existence of generalized variational solutions for nonlinear Dirichlet problems driven by anisotropic and Finsler Laplacian competing operators. The main difficulty consists in the lack of ellipticity and monotonicity in the principal part of the equations. This difficulty is overcome by developing a Galerkin-type procedure.

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1 Introduction

Recently, Galewski and Motreanu [5] studied the coercive competitive equation

$$\begin{cases} -\operatorname{div}(g(\frac{1}{p}|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + \operatorname{div}(|\nabla u|^{q-2}\nabla u) = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

driven by a competing (p,q)-Laplacian-type operator with weight depending on the gradient. Problem (1.1) is variational, but the driving operator is competing which means that the ellipticity condition is not satisfied. For this reason, one cannot establish the existence of a weak solution, but one can find a so-called generalized variational solution. Moreover, an abstract setting related to a Galerkin-type scheme is built in [5] that we briefly recall here for treating new problems with competing operators.

Let E be a separable and reflexive Banach space. We recall that E is separable if there exists a countable dense subset $\{h_i\}_{i\geq 1}$ of E. It is said that a sequence of finite-dimensional subspaces $(E_n)_{n=1}^{\infty} \subset E$ has the approximation property if

$$\begin{cases} E_n \subset E_{n+1} \quad \text{for } n \geq 1 \quad \text{and} \\ \overline{\bigcup_{n=1}^{\infty} E_n} = E. \end{cases}$$

There always exists such a sequence $(E_n)_{n=1}^{\infty}$ by defining E_n for $n \in \mathbb{N}$ as the linear hull of $\{h_1, \dots, h_n\}$.

Let $A : E \to E^*$ be a potential operator, which means that $A = J'$ (the differential of J) for a Gâteaux differentiable function $J : E \to \mathbb{R}$ called the potential of A. Note that the critical points of J coincide with the solutions to the equation

$$A(u) = 0, \tag{1.2}$$

or equivalently (in the weak sense),

$$\langle A(u), v \rangle = 0 \quad \text{for all } v \in E. \tag{1.3}$$

Taking advantage of the variational structure of problem (1.2) involving the functional J, the following definition sets forth a new type of solution.

Definition 1.1 An element $u \in E$ is said to be a generalized variational solution to problem (1.2) if there exists a sequence of finite-dimensional subspaces $(E_n)_{n=1}^{\infty} \subset E$ with the approximation property and a sequence of elements $(u_n)_{n=1}^{\infty}$ with $u_n \in E_n$ such that
(a) $u_n \rightharpoonup u$ in E as $n \to \infty$;
(b) $\inf_{v \in E_n} J(v) = J(u_n)$;
(c) $A(u_n) \rightharpoonup 0$ in E^* and $\langle A(u_n), u_n - u \rangle \to 0$.

We quote the following abstract result from [5, Theorem 4] (stated here in the particular case $k = 0$).

Theorem 1.1 *Assume that the operator $A : E \to E^*$ is bounded* (*i.e., A maps bounded sets into bounded sets*) *and potential with a coercive potential $J : E \to \mathbb{R}$* (*i.e.,* $\lim_{\|u\| \to \infty} J(u) = +\infty$). *Then problem* (1.2) *has at least one generalized variational solution in the sense of Definition* 1.1.

Remark 1.1 Without additional assumptions, there is no relation between the notion of (weak) solution in (1.3) and the notion of generalized variational solution given in Definition 1.1. Indeed, the fact that $u \in E$ fulfills (1.3) amounts to saying that $J'(u) = 0$, but then one cannot generally expect that the minimization in Definition 1.1(b) is verified. Conversely, if $u \in E$ satisfies Definition 1.1, one cannot generally have the strong convergence $u_n \to u$ in E which would result in (1.3) unless the operator A fulfills the S-property (meaning that $u_n \rightharpoonup u$ and $\langle A(u_n), u_n - u \rangle \to 0$ imply $u_n \to u$).

The first aim of this paper is to investigate the anisotropic counterpart of problem (1.1), namely the nonlinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(g_i\left(\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) & \\ \quad + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\right) = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

on a bounded domain Ω in $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz boundary $\partial\Omega$. In the statement of problem (1.4), $g_i : \mathbb{R} \to \mathbb{R}$ are continuous functions for which there are constants $0 < a_{g_i} \leq g_i(t) \leq b_{g_i}$ for all $t \geq 0$ and $i = 1, 2, \ldots, N$, and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., $h(x,t)$ is measurable in x and continuous in t).

A prototype of the driving operator in the left-hand side of equation (1.4) is the competing anisotropic operator $-\Delta_{\overrightarrow{p}} + \Delta_{\overrightarrow{q}}$, where

$$\Delta_{\overrightarrow{p}} := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial(\cdot)}{\partial x_i}\right|^{p_i-2}\right)\frac{\partial(\cdot)}{\partial x_i},$$

is the anisotropic p-Laplacian with $\overrightarrow{p} = (p_1, \dots, p_n)$, and

$$\Delta_{\overrightarrow{q}} := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial(\cdot)}{\partial x_i}\right|^{q_i-2}\right)\frac{\partial(\cdot)}{\partial x_i}$$

is the anisotropic q-Laplacian with $\overrightarrow{q} = (q_1, \dots, q_n)$. In (1.4) we have an extension of $\Delta_{\overrightarrow{p}}$ constructed by means of the weights $(g_1, \dots, g_N)$, specifically,

$$u \mapsto \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(g_i\left(\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right).$$

We assume that $1 < p_1, \dots, p_N < \infty$, $1 < q_1, \dots, q_N < \infty$, $q_i < p_i$ for all $i = 1, \dots, N$, and

$$\sum_{i=1}^{N} \frac{1}{p_i} > 1.$$

Let us introduce

$$p^+ := \max\{p_1, \dots, p_N\}, \qquad p^- := \min\{p_1, \dots, p_N\}, \qquad p^* := \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i} - 1},$$

and assume that

$$p^+ < p^*.$$

The anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ is defined as the completion of the set of smooth functions with compact support $C_c^{\infty}(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} := \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}.$$

This space is separable and uniformly convex, thus a reflexive Banach space. The dual of the space $W_0^{1,\overrightarrow{p}}(\Omega)$ is denoted $W^{-1,\overrightarrow{p}'}(\Omega)$.

We also introduce

$$\mu := \inf_{u \in W_0^{1,\overrightarrow{p}}(\Omega), u \neq 0} \frac{\sum_{i=1}^{N} \frac{1}{p_i} \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}^{p^-}}{\|u\|_{L^{p^-}(\Omega)}^{p^-}}. \tag{1.5}$$

The quantity μ in (1.5) is finite due to the compact embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset L^{p^-}(\Omega)$ (see [4, Theorem 1]). For various results regarding anisotropic Sobolev spaces, we refer to [1, 3, 4, 6–12].

In order to simplify the notation, for any real number $r > 1$ we denote $r' := r/(r-1)$ (the Hölder conjugate of r).

The following condition is assumed to hold:

$(H1)$ There exist a nonnegative function $\sigma \in L^{(p^*)'}(\Omega)$ and a constant $b \geq 0$ such that

$$\left|h(x,t)\right| \leq \sigma(x) + b|t|^{p^*-1}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

In addition, we formulate the condition:

$(H2)_{\xi,\alpha}$ Given positive constants ξ and α, it holds

$$H(x,t) := \int_0^t h(x,s)\,ds \leq c_1\left(|t|^\alpha + 1\right)$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, with a positive constant $c_1 < \xi$.

The usual arguments fail to apply for obtaining a weak solution to problem (1.4), which means an element $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ satisfying

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega g_i\left(\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx\\&\quad-\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx = \int_\Omega h(x,u)v\,dx\end{aligned}$$

for all $v \in W_0^{1,\overrightarrow{p}}(\Omega)$. The reason is the lack of ellipticity condition for equation (1.4). Notice also that the driving operator in (1.4) is not monotone. The idea is to weaken the notion of solution, still keeping the main characteristics of problem (1.4) as, for instance, its variational structure. Hence the Euler functional $J : W_0^{1,\overrightarrow{p}}(\Omega) \to \mathbb{R}$ associated to problem (1.4) is well defined and given by

$$J(u) = \sum_{i=1}^N \int_\Omega G_i\left(\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)dx - \sum_{i=1}^N \frac{1}{q_i}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q_i}dx - \int_\Omega H\big(x,u(x)\big)\,dx \tag{1.6}$$

for all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, where

$$G_i(x,t) := \int_0^t g_i(x,s)\,ds$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and $1 \leq i \leq N$.

Now we state the existence result.

Theorem 1.2 *Assume that the conditions* $(H1)$ *and* $(H2)_{\xi,\alpha}$ *hold with* $\xi = \mu a^*$ *for* $a^* = \min\{a_{g_i} : 1 \leq i \leq N\}$ *and* $\alpha = p^-$. *Then there exists a generalized variational solution to problem* (1.4) *in the sense of Definition* 1.1.

The second aim of the paper is to study the Dirichlet problem

$$\begin{cases} -\mathcal{Q}_p^g u + \mathcal{Q}_q u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

on a bounded domain Ω in $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz boundary $\partial\Omega$. Problem (1.7) is driven by the competing Finsler (p,q)-Laplacian-type operator $-\mathcal{Q}_p^g + \mathcal{Q}_q$, with $1 < q < p < +\infty$, that we now describe.

Let $F : \mathbb{R}^N \to [0,+\infty)$ be a convex function of class $C^2(\mathbb{R}^N \setminus \{0\})$, which is even and satisfies $F(\xi) > 0$ for each $\xi \neq 0$, and $F(t\xi) = |t|F(\xi)$ for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$. Given $p \in (1,+\infty)$, we assume that there exists a constant $\gamma > 0$ such that

$$\sum_{i,j=1}^{N} \nabla^2\bigl(F^p\bigr)(\eta)\zeta_i\zeta_j \geq \gamma|\eta|^{p-2}|\zeta|^2$$

with some positive constant γ, for all $\eta \in \mathbb{R}^N \setminus \{0\}$ and $\zeta \in \mathbb{R}^N$.

The Finsler p-Laplacian operator $\mathcal{Q}_p : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ is defined as

$$\mathcal{Q}_p u := \operatorname{div}\bigl(F^{p-1}(\nabla u)(\nabla F)(\nabla u)\bigr), \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.8}$$

If $F(\xi) = |\xi|$ (the Euclidean norm) and $p = 2$, then it becomes the ordinary Laplacian.

We denote by λ_1 the first eigenvalue of $-\mathcal{Q}_p$, that is,

$$\lambda_1 = \min_{\varphi \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega F^p(\nabla\varphi)\,dx}{\int_\Omega |\varphi|^p\,dx}. \tag{1.9}$$

For more details on the operator $-\mathcal{Q}_p$, we refer to [2, 13]. As a real life application, we mention Wulff's work [14] on crystal shapes.

In (1.7) we have a weighted version of the Finsler p-Laplacian $\mathcal{Q}_p$ extending (1.8). Specifically, corresponding to a continuous function $g : \mathbb{R} \to \mathbb{R}$ for which there exist constants $a_g > 0$ and $b_g > 0$ such that $a_g \leq g(t) \leq b_g$ for all $t \geq 0$, one sets

$$\mathcal{Q}_p^g u = \operatorname{div}\left(g\left(\frac{1}{p}F^p(\nabla u)\right)F^{p-1}(\nabla u)(\nabla F)(\nabla u)\right), \quad \forall u \in W_0^{1,p}(\Omega).$$

The underlying space for problem (1.7) is $W_0^{1,p}(\Omega)$. Denote by p^* the critical Sobolev exponent, that is, $p^* = \frac{Np}{N-p}$ if $N > p$ and $p^* = +\infty$ if $N \leq p$.

We are in a position to state our result regarding problem (1.7).

Theorem 1.3 *Assume that $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function for which the conditions* $(H1)$ *and* $(H2)_{\xi,\alpha}$ *with $\xi = \frac{\lambda_1 a_g}{p}$ and $\alpha = p$ hold. Then there exists a generalized variational solution to problem* (1.7) *in the sense of Definition* 1.1.

Remark 1.2 Problems (1.4) and (1.7) cannot be reduced one to another due to the different structure of the leading operators. For example, the driving operator in (1.4) is orthotropic whose properties depend on directions, whereas the driving operator in (1.7) is homogeneous. Such features are reflected in the distinct choices for the constants ξ and α in hypothesis $(H2)_{\xi,\alpha}$, as well as in the different proofs of Theorems 1.2 and 1.3.

Remark 1.3 The limit cases when $q_i = p_i$ for all $i = 1,\dots,N$ in problem (1.4) and $q = p$ in problem (1.7) generally do not give rise to competing operators, which is the object of our work. For instance, taking $g_i \equiv 2$ for all $i = 1,\dots,N$, we obtain an equation driven by the (negative) pseudo-p-Laplacian which is an elliptic operator.

In the rest of the paper, we prove the existence of generalized variational solutions for problems (1.4) and (1.7). Sections 2 and 3 contain the proofs of Theorems 1.2 and 1.3, respectively.

2 Generalized variational solutions for competing anisotropic Laplacian

In this section, we prove the existence of generalized variational solutions for problem (1.4) by Theorem 1.2, i.e., we present the proof of Theorem 1.2.

We show that we can fit in the setting of Theorem 1.1. Problem (1.4) can be regarded as an operator equation (1.2) with $E = W_0^{1,\overrightarrow{p}}(\Omega)$ and

$$\begin{aligned} Au = &-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(g_i\left(\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) \\ &+\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\right) - h(\cdot,u), \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega). \end{aligned} \tag{2.1}$$

Consider the Nemytskij operator $N_h : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ induced by the Carathéodory function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$N_h(w) = h\bigl(\cdot, w(\cdot)\bigr) \quad \text{for all } w \in W_0^{1,\overrightarrow{p}}(\Omega).$$

Assumption $(H1)$, Hölder's inequality, and the continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset L^{p^*}(\Omega)$ (see [4, Theorem 1]) imply that there is a constant $C > 0$ such that

$$\begin{aligned} \int_\Omega \bigl|h\bigl(x,w(x)\bigr)v(x)\bigr|\,dx &\le \int_\Omega \bigl|\sigma(x)\bigr|\bigl|v(x)\bigr|\,dx + b\int_\Omega \bigl|w(x)\bigr|^{p^*-1}v(x)\,dx \\ &\le C\bigl(\|\sigma\|_{L^{(p^*)'}(\Omega)} + \|w\|_{L^{p^*}(\Omega)}^{p^*-1}\bigr)\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \end{aligned}$$

for all $v, w \in W^{1,\overrightarrow{p}}(\Omega)$. It turns out that

$$\bigl\|N_h(w)\bigr\|_{W^{-1,\overrightarrow{p}'}(\Omega)} \le C\bigl(\|\sigma\|_{L^{(p^*)'}(\Omega)} + \|w\|_{L^{(p^*)}(\Omega)}^{p^*-1}\bigr), \quad \forall w \in W_0^{1,\overrightarrow{p}}(\Omega).$$

By Hölder's inequality, we see that

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}^{q_i} \le |\Omega|^{\frac{p_i-q_i}{p_i}}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{q_i}, \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega), i = 1,\dots,N, \tag{2.2}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. This ensures the continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset W_0^{1,\overrightarrow{q}}(\Omega)$, guaranteeing that the sum in (2.1) is well defined on $W_0^{1,\overrightarrow{p}}(\Omega)$. It follows that the operator $A : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ expressed by (2.1) is well defined and bounded.

Standard arguments relying on assumption $(H1)$ and Lebesgue's dominated convergence theorem ensure that the functional $J : W_0^{1,\overrightarrow{p}}(\Omega) \to \mathbb{R}$ in (1.6) is Gâteaux differentiable and its differential J' satisfies $J' = A$. Therefore the operator $A : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega)$ introduced in (2.1) is a potential operator with the potential given by the functional $J : W_0^{1,\overrightarrow{p}}(\Omega) \to \mathbb{R}$ in (1.6).

We claim that the functional $J : W_0^{1,\overrightarrow{p}}(\Omega) \to \mathbb{R}$ in (1.6) is coercive. Towards this, we note that assumption $(H2)_{\xi,\alpha}$, with $\xi = \mu a^*$, $a^* = \min\{a_{g_i} : 1 \le i \le N\}$, $\alpha = p^-$, and (1.5) imply

$$\begin{aligned}\int_\Omega H\bigl(x,u(x)\bigr)\,dx &\le c_1\bigl(\|u\|_{L^{p^-}(\Omega)}^{p^-} + |\Omega|\bigr)\\ &\le c_1\left(\mu^{-1}\sum_{i=1}^N \frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p^-} + |\Omega|\right)\\ &\le c_1\left(\mu^{-1}\sum_{i=1}^N \frac{1}{p_i}\left(\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} + 1\right) + |\Omega|\right).\end{aligned}$$

Then, in view of (1.6), (2.2), and $G_i(t) \ge a_{g_i} t$ for all $t \ge 0$ and $i = 1,\dots,N$, we are led to

$$\begin{aligned}J(u) \ge{}& \sum_{i=1}^N \frac{a_{g_i}}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} - \sum_{i=1}^N \frac{1}{q_i}|\Omega|^{\frac{p_i-q_i}{p_i}}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{q_i}\\ &- c_1\left(\mu^{-1}\sum_{i=1}^N \frac{1}{p_i}\left(\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} + 1\right) + |\Omega|\right), \quad \forall u \in W_0^{1,\overrightarrow{p}}(\Omega).\end{aligned}$$

Since $q_i < p_i$ for all $i = 1,\dots,N$ and $c_1 < \mu a^*$, we obtain that J coercive.

We have checked all the hypotheses required to apply Theorem 1.1 to the functional J in (1.6). As a consequence, according to Definition 1.1, the existence of a generalized variational solution to problem (1.2) with A given in (2.1) is established. This is just the stated result for the original problem (1.4), thus completing the proof of Theorem 1.2.

3 Generalized variational solutions for competing Finsler operator

In this section, we prove the existence of generalized variational solutions for problem (1.7) by Theorem 1.3, i.e., we present the proof of Theorem 1.3.

We apply Theorem 1.1 taking $E = W_0^{1,p}(\Omega)$ and $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ given by

$$\begin{aligned}Au = {}&-\mathrm{div}\left(g\left(\frac{1}{p}F^p(\nabla u)\right)F^{p-1}(\nabla u)(\nabla F)(\nabla u)\right)\\ &+ \mathrm{div}\bigl(F^{q-1}(\nabla u)(\nabla F)(\nabla u)\bigr) - h(\cdot,u), \quad \forall u \in W_0^{1,p}(\Omega).\end{aligned} \tag{3.1}$$

Observe that problem (1.7) can be written as the operator equation (1.2) with A in (3.1).

By Hölder's inequality, we see that

$$\bigl\|F(\nabla u)\bigr\|_{L^q(\Omega)}^q \le |\Omega|^{\frac{p-q}{p}}\bigl\|F(\nabla u)\bigr\|_{L^p(\Omega)}^q, \quad \forall u \in W_0^{1,p}(\Omega).$$

Since for the function F there exist two constants $0 < a < b < +\infty$ such that $a|\xi| \le F(\xi) \le b|\xi|$ for all $\xi \in \mathbb{R}^N$, the operator $-\mathcal{Q}_p^g + \mathcal{Q}_q$ is well defined, continuous, and bounded on $W_0^{1,p}(\Omega)$.

The Carathéodory function $h(x,t)$ entering equation (1.7) determines the Nemytskij operator $N_h : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$N_h(u) = h\bigl(\cdot, u(\cdot)\bigr) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Assumption $(H1)$, Hölder's inequality, and Sobolev embedding theorem imply that there is a constant $C > 0$ such that

$$\begin{aligned}\int_\Omega \bigl|h\bigl(x, w(x)\bigr)v(x)\bigr| \, dx &\le \int_\Omega \bigl|\sigma(x)\bigr|v(x) \, dx + b \int_\Omega \bigl|w(x)\bigr|^{p^*-1} v(x) \, dx \\ &\le C\bigl(\|\sigma\|_{L^{(p^*)'}(\Omega)} + \|w\|_{L^{p^*}(\Omega)}^{p^*-1}\bigr)\|\nabla v\|_{L^p(\Omega)}\end{aligned}$$

for all $v, w \in W^{1,p}(\Omega)$. Hence we find the estimate

$$\bigl\|N_h(w)\bigr\|_{W^{-1,p'}(\Omega)} \le C\bigl(\|\sigma\|_{L^{(p^*)'}(\Omega)} + \|w\|_{L^{p^*}(\Omega)}^{p^*-1}\bigr), \quad \forall w \in W_0^{1,p}(\Omega). \tag{3.2}$$

We infer from (3.2) that the operator $N_h : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is well defined and bounded. Taking into account (3.1), it follows that the operator $A = -\mathcal{Q}_p^g + \mathcal{Q}_q - N_h$ is well defined and bounded from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$.

We are going to show that the operator A in (3.1) is potential. To this end, we define the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega G\left(\frac{1}{p}F^p(\nabla u)\right) dx - \frac{1}{q}\int_\Omega F^q(\nabla u) \, dx - \int_\Omega H\bigl(x, u(x)\bigr) \, dx \tag{3.3}$$

for all $u \in W_0^{1,p}(\Omega)$, where

$$G(t) = \int_0^t g(x,s) \, ds$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

The boundedness of g implies the existence of a constant $c > 0$ with $|G(t)| \le c(|t| + 1)$ for all $t \in \mathbb{R}$. Then, arguing on the basis of assumption $(H1)$, we can prove through Lebesgue's dominated convergence theorem that the functional J in (3.3) is Gâteaux differentiable with the differential

$$\begin{aligned}\bigl\langle J'(u), v\bigr\rangle = {} & \int_\Omega g\left(\frac{1}{p}F^p(\nabla u)\right)F^{p-1}(\nabla u)F'(\nabla u)\nabla v \, dx \\ & - \int_\Omega F^{q-1}(\nabla u)F'(\nabla u)\nabla v \, dx \\ & - \int_\Omega h\bigl(x, u(x)\bigr)v(x) \, dx, \quad \forall v \in W_0^{1,p}(\Omega).\end{aligned} \tag{3.4}$$

By (3.1) and (3.4), we note that $Au = J'(u)$ for all $u \in W_0^{1,p}(\Omega)$. As a consequence, we can infer that A in (3.1) is a potential operator with the potential J given by (3.3).

Now we focus on the coerciveness of the functional J in (3.3). Assumption $(H2)_{\xi,\alpha}$, with $\xi = \frac{\lambda_1 a_g}{p}$ and $\alpha = p$, and (1.9) imply

$$\int_\Omega H\big(x, u(x)\big) \le c_1\big(\lambda_1^{-1}\big\|F^p(\nabla u)\big\|^p_{L^p(\Omega)} + |\Omega|\big).$$

Then, in view of (3.3) and $G(t) \ge a_g t$ for all $t \ge 0$, we are led to

$$J(u) \ge \left(\frac{a_g}{p} - c_1\lambda_1^{-1}\right)\big\|F(\nabla u)\big\|^p_{L^p(\Omega)} - \frac{1}{q}|\Omega|^{\frac{p-q}{p}}\big\|F(\nabla u)\big\|^q_{L^p(\Omega)} - c_1|\Omega|$$

for all $u \in W_0^{1,p}(\Omega)$. Since $q < p$ and $c_1 < \frac{a_g\lambda_1}{p}$, we obtain that J is coercive.

All the hypotheses required to apply Theorem 1.1 to the functional J in (3.3) are fulfilled. Then the existence of a generalized variational solution to problem $Au = 0$ with A given in (3.1) is established. This completes the proof concerning the original problem (1.7).

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Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

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