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Inverse source problem for the pseudoparabolic equation associated with the Jacobi operator

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Abstract

In this paper, we investigate direct and inverse problems for time-fractional pseudoparabolic equations associated with the Jacobi operator. The existence and uniqueness of the solutions are proven. Also, the stability result of the inverse source problem (ISP) is established.

Mathematics Subject Classification: Primary 35R30; secondary 35R11; 35C15

Keywords: Jacobi operator; Jacobi transform; Time-fractional pseudoparabolic equation; Inverse source problem

1 Introduction

The main object of this paper is the following nonhomogeneous time-fractional pseudoparabolic equation:

$$\mathbb{D}^{\gamma}_{0^+,t}\big(u(t,x) - a\Delta_{\alpha,\beta}u(t,x)\big) - \Delta_{\alpha,\beta}u(t,x) + mu(t,x) = f(x)$$

on the domain $D = \{(t,x) : 0 < t < T < \infty, x \in \mathbb{R}^+ = (0,\infty)\}$, where $0 < \gamma \le 1$, with nonnegative constants m and a, and with the initial condition

$$u(0,x) = \phi(x), \quad x \in \mathbb{R}^+,$$

where $\mathbb{D}^{\gamma}_{0^+,t}$ is given by

$$\mathbb{D}^{\gamma}_{0^+,t} = \begin{cases} \mathcal{D}^{\gamma}_{0^+,t}, & 0 < \gamma < 1, \\ \frac{d}{dt}, & \gamma = 1, \end{cases}$$

where $\mathcal{D}^{\gamma}_{0^+,t}$ is the left-sided Caputo fractional derivative and $\Delta_{\alpha,\beta}$ is the Jacobi operator given by the expression

$$\Delta_{\alpha,\beta} = A^{-1}_{\alpha,\beta}(x)\frac{d}{dx}\left(A_{\alpha,\beta}(x)\frac{d}{dx}\right), \quad x \in (0,\infty). \tag{1.1}$$

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Here, we denote by $A_{\alpha,\beta}(x) = 2^{2\rho}(\sinh(x))^{2\alpha+1}(\cosh(x))^{2\beta+1}$, $\rho = \alpha + \beta + 1$, with $\alpha \geq \beta \geq -\frac{1}{2}$.

In our studies we question the well-posedness of the direct problem and the stability of the ISP with the additional information, an overdetermination condition

$$u(T,x) = \psi(x), \quad x \in \mathbb{R}^+.$$

For the ISP we will restore the pair (u,f) under some conditions on the function ψ.

One of the first mathematicians who studied the ISP was Rundell [1] in the 1980s. He considered the evolution-type equation

$$\frac{du}{dt} + Au = f \tag{1.2}$$

in a Banach space X, where A is a linear operator in X and f is a constant vector in X, with conditions

$$u(0) = u_0, \quad \text{and} \quad u(T) = u_1.$$

Using semigroups of operators, Rundell proved a general theorem about the existence of a unique solution pair $(u(t),f)$ of the problem, which then was applied to equations of parabolic and pseudoparabolic types. When the nonhomogeneous term is represented in the form $f(t) = \Phi(t)f$, where $\Phi(t)$ is a known operator and the element f is unknown, and A is a closed linear operator from $L_p(\Omega)$ into $L_p(\Omega)$, several ISPs for the equation (1.2) were studied by Prilepko and Tikhonov [2] in 1992. They applied the obtained results to the transport equation. In the general case, where the unknown source depends on time, under a sufficient condition, ISPs for equation (1.2) with the linear elliptic partial differential operator A of order $2m$ with the bounded measurable coefficients such that

$$(A\varphi, \varphi) \geq \|\varphi\|^2$$

for all $\varphi \in H^{2m}(\Omega) \cap H_0^m(\Omega)$, $\mu = constant > 0$ were investigated by Bushuyev [3] in 1995.

Nevertheless, there is no general closed theory for the abstract case of $F(x,t)$. Known results deal with separated source terms. In 2002, Tikhonov and Eidelman [4] considered ISPs for the generalization of the equation (1.2) of the form

$$\frac{d^N u(t)}{dt^N} = Au(t) + p, \quad 0 < t < T,$$

for some positive integer $N \geq 1$ and some real number $T > 0$ with an unknown parameter p and a closed linear operator A in the Banach space under the Cauchy conditions and "overdetermination condition" $u(T) = u_N$ (also in the Banach space).

For the Laplace operator $(-\Delta)$, which is one of the most interesting examples in physics, Choulli and Yamamoto in [5] established the uniqueness and conditional stability in determining a heat-source term from boundary measurements with $f = \sigma(t)\varphi(x)$, where $\sigma(t)$ is known.

Asymptotic behavior of the solution of the ISP for the pseudoparabolic equation

$$\big(u(x,t) - \Delta u(x,t)\big)_t - \Delta u(x,t) + \alpha u(x,t) = f(t)g(x,t), \qquad Q_\infty = \Omega \times (0,\infty)$$

with a integral overdetermination condition was studied by Yaman and Gözükızıl in [6] in 2004.

Fractional derivatives and fractional partial differential equations have received much attention both in analysis and application, which are used in modeling several phenomena in different areas of science such as biology, physics, and chemistry, so the fractional computation is increasingly attracted to mathematicians in the last several decades. The ISP for the time fractional parabolic equation

$$ {}^{c}D_t^{\alpha}u(x,t) = r^{\alpha}(Lu)(x,t) + f(x)h(x,t), \quad x \in \Omega, t \in (0,T), 0 < \alpha < 1, $$

where ${}^{c}D_t^{\alpha}$ is the Caputo derivative defined by

$$ {}^{c}D_t^{\alpha}g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} g(\tau)\,d\tau $$

and L is a symmetric uniformly elliptic operator was considered by Sakamoto and Yamamoto in [7] in 2011. The authors proved that the inverse problem is well-posed in the Hadamard sense except for a discrete set of values of diffusion constants using final overdetermining data. The blow-up solution and stability to ISP for the pseudoparabolic equation

$$ u_t - a\Delta u_t - \Delta u + \sum_{i=1}^{n} b_i u_{x_i} - |u|^p u = f(t)g(t), \quad x \in \Omega, t > 0 $$

with the integral overdetermination condition was studied by Metin Yaman in [8] in 2012. The ISP for equation (1.2) was considered by Slodička in [9] in 2013, when A is a linear differential operator of second order, strongly elliptic, and the right-hand side f is assumed to be separable in both variables x and t, i.e., $f(x,t) = g(x)h(t)$ (in this case $h(t)$ is unknown). The ISP for a semilinear time-fractional diffusion equation of second order in a bounded domain in $\mathbb{R}^d$

$$ \big(g_{1-\beta} * \partial_t u(x)\big)(t) + L(x,t)u(x,t) = h(t)f(x) + \int_0^t F\big(x,s,u(x,s)\big)\,ds $$

with a linear second-order differential operator $L(x,t)$, in the divergence form with space- and time-dependent coefficients, was studied by Slodička and Šiškova in [10] in 2016. The authors showed the existence, uniqueness, and regularity of a weak solution (u,h) ([10, Theorem 2.1, p. 1658]). One of the recent papers for ISPs for pseudoparabolic equations with fractional derivatives is [11] (in 2021). In [11], the authors considered the solvability of the ISP for the pseudoparabolic equation with the Caputo fractional derivative $\mathcal{D}_t^{\alpha}$, of order $0 < \alpha \leq 1$,

$$ \begin{aligned} &\mathcal{D}_t^{\alpha}\big(u(t) + \mathcal{L}u(t)\big) + \mathcal{M}u(t) = f(t) \quad \text{in } \mathcal{H}, \\ &u(0) = \phi \in \mathcal{H}, \qquad u(T) = \psi \in \mathcal{H}, \end{aligned} $$

where $\mathcal{H}$ is a separable Hilbert space and $\mathcal{L}$, $\mathcal{M}$ are operators with the corresponding discrete spectra on $\mathcal{H}$. The authors obtained well-posedness results.

A number of articles address the solvability of the inverse problems for the diffusion and subdiffusion equations ([12–18]) and fractional diffusion equations ([19–21]).

The semigroups $(H_t^{(\alpha,\beta)})_{t\geq 0}$ (the solution of the heat equation associated with the Jacobi–Dunkl operator $\Lambda^2_{\alpha,\beta}$) generate a new family of Markov processes on the real line. On some Riemannian symmetric spaces, this process is the radial part of the Brownian motion for particular values of (α,β) [22].

However, the ISP for the pseudoparabolic equations generated by the Jacobi operator $\Delta_{\alpha,\beta}$ (1.1) have not been considered yet. Hence, our goal is to consider the ISP for the pseudoparabolic equation with this special operator, the Jacobi operator $\Delta_{\alpha,\beta}$. Harmonic analysis associated with the Jacobi operator $\Delta_{\alpha,\beta}$ has been studied by Flensted-Jensen and Koornwinder, in a series of papers [23–26]. The spectral decomposition of the Jacobi operator was considered by Flensted-Jensen in 1972 [23]. There were obtained a generalization of the classical Paley–Wiener Theorem and a generalized Fourier transform $\mathcal{F}_{\alpha,\beta}$, is called the Jacobi–Fourier transform. Eigenfunctions $\varphi_\lambda^{\alpha,\beta}(x)$ of the Jacobi operator are called Jacobi functions, which are hypergeometric functions. The pseudodifferential operators (see [27]) and Sobolev-type spaces $G^{s,p}_{\alpha,\beta}$ (see [28]) associated with the Jacobi operator were studied by Ben Salem and Dachraoui. In [27], the authors proved that the pseudodifferential operator associated with the symbol in S_0^m is a continuous linear mapping from some subspace of the Schwartz space into itself.

Our main result reads as follows.

Theorem 1.1 *Let $0<\gamma\leq 1$. Assume that $\psi,\phi\in\mathcal{H}$. Then, the pair (u,f) is a unique solution of the ISP, which are functions $u\in C^\gamma([0,T],L^2(\mu))\cap C([0,T],\mathcal{H})$, $f\in L^2(\mu)$ can be represented by the formulas*

$$
\begin{aligned}
u(t,x) &= \int_0^\infty\int_0^\infty \frac{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\psi(y)\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda) \\
&\quad -\int_0^\infty\int_0^\infty \frac{\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)} \\
&\quad \times\phi(y)\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
f(x) &= \int_0^\infty\int_0^\infty\left(\lambda^2+\rho^2+m\right)\frac{\psi(y)-\phi(y)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)} \\
&\quad \times\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda).
\end{aligned}
$$

The contents of this paper are as follows. In Sect. 2, we collect some results about harmonic analysis associated with the Jacobi operator on $\mathbb{R}^+$ and here we introduce the Sobolev-type space $\mathcal{H}$, also given is some necessary information about fractional derivatives. In Sect. 3, we prove Theorem 3.1 for the direct problem. In Sect. 4, we prove our main Theorem 3.2 about the solvability of the ISP associated with the Jacobi operator on $\mathbb{R}^+$, also shown are the stability analysis and an example for the ISP.

2 Preliminaries

2.1 Jacobi analysis

The singular second-order differential equation ([23])

$$\Delta_{\alpha,\beta}\varphi_\lambda^{\alpha,\beta}(x) + (\lambda^2+\rho^2)\varphi_\lambda^{\alpha,\beta}(x) = 0 \quad \text{on } (0,\infty)$$

with initial conditions

$$\varphi_\lambda^{\alpha,\beta}(0) = 1, \qquad \frac{d}{dt}\varphi_\lambda^{\alpha,\beta}(0) = 0$$

has a unique solution, given by the expression

$$\varphi_\lambda^{\alpha,\beta}(x) = {}_2F_1\left(\frac{1}{2}(\rho+i\lambda), \frac{1}{2}(\rho-i\lambda); \alpha+1; -\sinh^2 x\right), \tag{2.1}$$

where ${}_2F_1$ is the Gauss hypergeometric function. The function $\varphi_\lambda^{\alpha,\beta}$ (2.1) is called the Jacobi function, is analytic for $x \in [0,\infty)$, and has the following properties

$$\varphi_\lambda^{\alpha,\beta}(x) = \varphi_{-\lambda}^{\alpha,\beta}(x) \quad \text{and} \quad \overline{\varphi_\lambda^{\alpha,\beta}(x)} = \varphi_{\bar{\lambda}}^{\alpha,\beta}(x).$$

In particular, we have

$$\varphi_\lambda^{-\frac{1}{2},-\frac{1}{2}}(x) = \cos(\lambda x).$$

Remark 2.1 ([23, Proposition 1, p. 144]) For each fixed $x \in (0,\infty)$, as a function of λ, $\varphi_\lambda^{\alpha,\beta}(x)$ is an entire function.

Properties of the Jacobi function:

1. For all $\lambda \in \mathbb{C}$ and $x \in [0,\infty)$, we have ([23, Lemma 11, p. 153])

i) $|\varphi_\lambda^{\alpha,\beta}(x)| \le \varphi_{i\,\mathrm{Im}\,\lambda}^{\alpha,\beta}(x)$;

ii) If $|\mathrm{Im}\,\lambda| \ge \rho$ then $|\varphi_\lambda^{\alpha,\beta}(x)| \le e^{(|\mathrm{Im}\,\lambda|-\rho)x}$;

iii) If $|\mathrm{Im}\,\lambda| \le \rho$ then $|\varphi_\lambda^{\alpha,\beta}(x)| \le 1$.

2. For all $n \in \mathbb{Z}^+$ there exists $K_n > 0$ such that ([23, Theorem 2, p. 145])

$$\left|\frac{d^n}{dx^n}\varphi_\lambda^{\alpha,\beta}(x)\right| \le K_n(1+x)\big(1+|\lambda|\big)^n e^{(|\mathrm{Im}\,\lambda|-\rho)x}$$

and

$$\left|\frac{d^n}{d\lambda^n}\varphi_\lambda^{\alpha,\beta}(x)\right| \le K_n(1+x)^{n+1} e^{(|\mathrm{Im}\,\lambda|-\rho)x}$$

for all $\lambda \in \mathbb{C}$, $x \in [0,\infty)$.

Let us introduce the following functions spaces ([23, p. 146–147], [27, p. 368]).

Let $\mathcal{S}_e(\mathbb{R})$ be the space of even, rapidly decreasing, and C^∞-functions on $\mathbb{R}$, equipped with usual Schwartz topology, and $\mathcal{S}_e^r(\mathbb{R}) = \{(\cosh x)^{\frac{-2\rho}{r}}\mathcal{S}_e(\mathbb{R})\}$, $0 < r \le 2$ be the space with the topology defined by the seminorms

$$N_{n,k}(f) = \sup_{x\ge 0}(\cosh x)^{\frac{2\rho}{r}}(1+x)^n\left|\frac{d^k}{dx^k}f(x)\right|.$$

Clearly, $\mathcal{S}_e^r(\mathbb{R})$ is invariant under $\Delta_{\alpha,\beta}$ and the seminorms defined by

$$N_{n,k}(f) = \sup_{x\geq 0}(\cosh x)^{\frac{2\rho}{r}}(1+x)^n\left|\Delta_{\alpha,\beta}^k f(x)\right|$$

are continuous on $\mathcal{S}_e^r(\mathbb{R})$.

Let $L^p(\mathbb{R}^+,\mu_{\alpha,\beta})$, $1\leq p<\infty$, be the space of measurable functions f on $\mathbb{R}^+$ such that

$$\|f\|_{p,\mu}^p = \int_0^\infty \left|f(x)\right|^p d\mu_{\alpha,\beta}(x)<\infty,$$

where $d\mu_{\alpha,\beta}(x) = (2\pi)^{-\frac{1}{2}}2^{2\rho}(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}\,dx$ or $d\mu_{\alpha,\beta}(x) = (2\pi)^{-\frac{1}{2}}A_{\alpha,\beta}(x)\,dx$.

Remark 2.2 [23, p. 146] Note that $\mathcal{S}_e^r(\mathbb{R})\subset L^r(\mathbb{R}^+,\mu_{\alpha,\beta})$ for all $0<r\leq 2$ and if $r\leq s$, then $\mathcal{S}_e^r(\mathbb{R})\subseteq\mathcal{S}_e^s(\mathbb{R})\subset L^2(\mathbb{R}^+,\mu_{\alpha,\beta})$.

Let $L^p(\mathbb{R}^+,\nu_{\alpha,\beta})$, $1\leq p<\infty$, be the space of measurable functions g on $\mathbb{R}^+$ such that

$$\|f\|_{p,\nu}^p = \int_0^\infty \left|g(\lambda)\right|^p d\nu_{\alpha,\beta}(\lambda)<\infty,$$

where $d\nu_{\alpha,\beta}(\lambda) = (2\pi)^{-\frac{1}{2}}|c_{\alpha,\beta}(\lambda)|^{-2}\,d\lambda$. Here, $c_{\alpha,\beta}(\lambda)$ is the Harish–Chandra function, given by

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{\alpha-\beta+1+i\lambda}{2})}.$$

For brevity, we use notations $L^p(\mu)$ and $L^p(\nu)$ instead $L^p(\mathbb{R}^+,\mu_{\alpha,\beta})$ and $L^p(\mathbb{R}^+,\nu_{\alpha,\beta})$, respectively.

For $f\in L^1(\mu)$, the Fourier–Jacobi transform $\mathcal{F}_{\alpha,\beta}$ of f is defined by ([23, Proposition 3, p. 146], [27, Definition 1.1, p. 369])

$$\widehat{f}(\lambda) = (\mathcal{F}_{\alpha,\beta}f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(x) \tag{2.2}$$

and for $g\in L^1(\nu)$ the inverse Fourier–Jacobi transform $\mathcal{F}_{\alpha,\beta}^{-1}$ is given by

$$\left(\mathcal{F}_{\alpha,\beta}^{-1}g\right)(x) = \int_0^\infty g(\lambda)\varphi_\lambda^{\alpha,\beta}(x)\,d\nu_{\alpha,\beta}(\lambda), \tag{2.3}$$

where $\varphi_\lambda^{\alpha,\beta}$ is the Jacobi function (2.1).

Proposition 2.3 ([23, *p.* 145–146]) *The operator, in* $L^2(\mu)$, *defined by* $\Delta_{\alpha,\beta}$ *with the domain*

$$D_{\alpha,\beta}^0 = \left\{u\in L^2(\mu): u \text{ and } u' \text{ are absolutely continuous and } \Delta_{\alpha,\beta}u\in L^2(\mu)\right\}$$

can be restricted to a domain $D_{\alpha,\beta}$, *such that the operator* $\Delta_{\alpha,\beta}$ *becomes self-adjoint. The operator* $\Delta_{\alpha,\beta}$ *contains at least functions in* $D_{\alpha,\beta}^0$, *which are differentiable at zero. The operator* $\Delta_{\alpha,\beta}$ *has a limit point at* ∞; *and at zero there is a limit point if* $2\alpha+1\geq 3$, *and a* *limit circle if* $2\alpha + 1 < 3$. *In this last case,* $D_{\alpha,\beta} \neq D^0_{\alpha,\beta}$ *and choosing* $\lambda_1 \in \mathbb{C}$ *with* $\operatorname{Im} \lambda_1^2 > 0$ *we can define*

$$D_{\alpha,\beta} = \left\{ u \in D^0_{\alpha,\beta} : \lim_{x\to 0} \left(A_{\alpha,\beta}(x) \cdot \left(\varphi^{\alpha,\beta}_{\lambda_1}(x)\overline{u'(x)} - \left(\frac{d}{dx}\varphi^{\alpha,\beta}_{\lambda_1}(x) \right) \overline{u(x)} \right) \right) = 0 \right\}.$$

Proposition 2.4 ([23, *Proposition* 3, *p.* 146]) *For* $f \in L^2(\mu)$ *and* $\lambda \in \mathbb{R}^+$ *define* $\widehat{f}$ *the integral converging in* $L^2(\nu)$. $f \to \widehat{f}$ *is a linear, normpreserving map of* $L^2(\mu)$ *onto* $L^2(\nu)$, *the inverse given by*

$$f(x) = \int_0^\infty g(\lambda)\varphi^{\alpha,\beta}_\lambda(x)\, d\nu_{\alpha,\beta}(\lambda)$$

the integral converging in $L^2(\mu)$. *A function* $f \in L^2(\mu)$ *belongs to* $D_{\alpha,\beta}$ *if and only if* $(\lambda^2 + \rho^2)\widehat{f}(\lambda) \in L^2(\nu)$ *and in that case*

$$\widehat{\Delta_{\alpha,\beta} f}(\lambda) = -(\lambda^2 + \rho^2)\widehat{f}(\lambda).$$

In particular, we have for Plancherel's identity

$$\|\widehat{f}\|_{2,\nu} = \|f\|_{2,\mu}. \tag{2.4}$$

Remark 2.5 For $\alpha = \beta = -\frac{1}{2}$, we have the Fourier-cosine transform

$$\widehat{f_c}(\lambda) = (\mathcal{F}_c f)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \cos(\lambda x) f(x)\, dx$$

and the inverse Fourier-cosine transform is defined by

$$\left(\mathcal{F}_c^{-1} g\right)(x) = \frac{4}{\sqrt{2\pi}} \int_0^\infty \cos(\lambda x) g(\lambda)\, d\lambda.$$

Definition 2.6 We define the space

$$\mathcal{H} := \left\{ u \in L^2(\mu) : \left(\cdot^2 + \rho^2\right)\widehat{u} \in L^2(\nu) \right\}$$

with norm

$$\|u\|^2_{\mathcal{H}} := \int_0^\infty \left| \left(\lambda^2 + \rho^2\right)\widehat{u}(\lambda) \right|^2 d\nu_{\alpha,\beta}(\lambda).$$

2.2 Fractional differentiation operators

In this subsection, we introduce fractional differentiation operators and other concepts.

Definition 2.7 [29, p. 69] Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The left and right Riemann–Liouville fractional integrals $I^\gamma_{a^+}$ and $I^\gamma_{b^-}$ of order $\gamma \in \mathbb{R}$ $(\gamma > 0)$ are defined by

$$I^\gamma_{a^+}[f](t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)\, ds, \quad t \in (a,b]$$

and

$$I_{b^-}^{\gamma}[f](t) := \frac{1}{\Gamma(\gamma)} \int_t^b (t-s)^{\gamma-1} f(s)\, ds, \quad t \in [a,b),$$

respectively. Here, Γ denotes the Euler gamma function.

Definition 2.8 [29, p. 70] The left and right Riemann–Liouville fractional derivatives $D_{a^+}^{\gamma}$ and $D_{b^-}^{\gamma}$ of order $\gamma \in \mathbb{R}$ $(0 < \gamma < 1)$ are given by

$$D_{a^+}^{\gamma}[f](t) := \frac{d}{dt} I_{a^+}^{1-\gamma}[f](t), \quad \forall t \in (a,b]$$

and

$$D_{b^-}^{\gamma}[f](t) := -\frac{d}{dt} I_{b^-}^{1-\gamma}[f](t), \quad \forall t \in [a,b),$$

respectively.

Definition 2.9 [29, p. 91] The left and right Caputo fractional derivatives $D_{a^+}^{\gamma}$ and $D_{b^-}^{\gamma}$ of order $\gamma \in \mathbb{R}$ $(0 < \gamma < 1)$ are defined by

$$\mathcal{D}_{a^+}^{\gamma}[f](t) := D_{a^+}^{\gamma}\big[f(t) - f(a)\big], \quad t \in (a,b]$$

and

$$\mathcal{D}_{b^-}^{\gamma}[f](t) := D_{b^-}^{\gamma}\big[f(t) - f(b)\big], \quad t \in [a,b),$$

respectively.

Definition 2.10 [30, p. 18, Definition 3] Let X be a Banach space. We say that $u \in C^{\gamma}([0,T],X)$ if $u \in C([0,T],X)$ and $\mathcal{D}_t^{\gamma} u \in C([0,T],X)$.

Computations involving fractional derivatives often require special functions, called Mittag–Leffler functions. Hence, let us give a brief introduction to these special functions. The classical Mittag–Leffler function $\mathbb{E}_{\gamma,1}(t)$ and the Mittag–Leffler-type function $\mathbb{E}_{\gamma,\gamma}(t)$ are given by the expressions

$$\mathbb{E}_{\gamma,1}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + 1)}, \quad \text{and} \quad \mathbb{E}_{\gamma,\gamma}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k + \gamma)}.$$

In the case $\gamma = 1$, we obtain $\mathbb{E}_{1,1}(t) = e^t$. For more information about the classical Mittag–Leffler function $\mathbb{E}_{\gamma,1}(t)$ and the Mittag–Leffler-type function $\mathbb{E}_{\gamma,\gamma}(t)$ see, e.g., [29, p. 40 and p. 42].

In [31, Theorem 4, p. 21] the following estimate for the Mittag–Leffler function is proved, when $0 < \gamma < 1$ (not true for $\gamma \geq 1$)

$$\frac{1}{1 + \Gamma(1-\gamma)t} \leq \mathbb{E}_{\gamma,1}(-t) \leq \frac{1}{1 + \Gamma(1+\gamma)^{-1}t}, \quad t > 0.$$

Then, it follows that

$$0 < \mathbb{E}_{\gamma,1}(-t) < 1, \quad t > 0. \tag{2.5}$$

Proposition 2.11 [32] *If* $0 < \gamma < 2$, β *is an arbitrary real number,* μ *is such that* $\pi\gamma/2 < \mu < \min\{\pi, \pi\gamma\}$, *then there exists a positive constant* C, *such that we have*

$$\left|\mathbb{E}_{\gamma,\beta}(z)\right| \leq \frac{C}{1+|z|}$$

for all $\mu \leq |\arg(z)| \leq \pi$.

Lemma 2.12 *If* $0 < \gamma \leq 1$ *and* $\gamma \leq \beta$, *then the generalized Mittag–Leffler function* $\mathbb{E}_{\gamma,\beta}(-z), z \geq 0$, *is completely monotonic, that is,*

$$(-1)^n \frac{d^n}{dz^n}\mathbb{E}_{\gamma,\beta}(-z) \geq 0, \quad \text{for } z \geq 0 \text{ and } n = 0, 1, 2, \dots.$$

Proof The proof can be found from [33]. □

Now, let us prove the following lemma, which is important for our calculations.

Lemma 2.13 *Assume that* $0 < t < T$, $0 < \gamma \leq 1$, *and* $\lambda \in \mathbb{R}^+$. *Then, the following inequalities*

$$0 < \frac{1-\mathbb{E}_{\gamma,1}(-\lambda t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\lambda T^\gamma)} \leq 1 \tag{2.6}$$

and

$$-1 < \frac{\mathbb{E}_{\gamma,1}(-\lambda T^\gamma)-\mathbb{E}_{\gamma,1}(-\lambda t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\lambda T^\gamma)} \leq 0 \tag{2.7}$$

hold.

Proof From Lemma 2.12, we obtain $0 < \mathbb{E}_{\gamma,1}(-\lambda T^\gamma) \leq \mathbb{E}_{\gamma,1}(-\lambda t^\gamma) < 1$ for $0 < t < T$. Then, this implies (2.6). Rewriting the expression

$$\frac{\mathbb{E}_{\gamma,1}(-\lambda T^\gamma)-\mathbb{E}_{\gamma,1}(-\lambda t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\lambda T^\gamma)} = \frac{1-\mathbb{E}_{\gamma,1}(-\lambda t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\lambda T^\gamma)} - 1$$

and using (2.6) we obtain (2.7). □

3 Main results

In this section we deal with the direct problem for the time-fractional pseudoparabolic equation associated with the Jacobi operator $\Delta_{\alpha,\beta}$ (1.1). Moreover, ISPs are subject to study. The existence, uniqueness, and stability results are established.

3.1 The direct problem for the time-fractional pseudoparabolic equation with the Jacobi operator

Let $0 < \gamma \le 1$. We consider the nonhomogeneous time-fractional pseudoparabolic equation

$$\mathbb{D}^{\gamma}_{0^+,t}\big(u(t,x) - a\Delta_{\alpha,\beta}u(t,x)\big) - \Delta_{\alpha,\beta}u(t,x) + mu(t,x) = f(t,x), \quad (t,x) \in D, \tag{3.1}$$

with initial condition

$$u(0,x) = \phi(x), \quad x \in \mathbb{R}^+, \tag{3.2}$$

where the functions f and ϕ are given functions. Our aim is to find the unique solution u of the problem (3.1) and (3.2).

Theorem 3.1 *Let $0 < \gamma \le 1$ and $\lambda \in \mathbb{R}^+$. Suppose that $f \in C^1([0,T], L^2(\mu))$ and $\phi \in \mathcal{H}$. Then, the problem (3.1) and (3.2) has a unique solution $u \in C^{\gamma}([0,T], L^2(\mu)) \cap C([0,T], \mathcal{H})$ and can be represented by the formula*

$$\begin{aligned} u(t,x) = {} & \int_0^\infty \int_0^\infty \int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^{\gamma}\right)\frac{f(\tau,y)}{1+a(\lambda^2+\rho^2)} \\ & \times \varphi^{\alpha,\beta}_{\lambda}(y)\varphi^{\alpha,\beta}_{\lambda}(x)\, d\tau\, d\mu_{\alpha,\beta}(y)\, d\nu_{\alpha,\beta}(\lambda) \\ & + \int_0^\infty \int_0^\infty \mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^{\gamma}\right)\phi(y)\varphi^{\alpha,\beta}_{\lambda}(y)\varphi^{\alpha,\beta}_{\lambda}(x)\, d\mu_{\alpha,\beta}(y)\, d\nu_{\alpha,\beta}(\lambda). \end{aligned}$$

Proof We assume that $0 < \gamma \le 1$, $\lambda \in \mathbb{R}^+$ and $u(t,\cdot) \in \mathcal{H}$. We first prove that the problem (3.1) and (3.2) has only one solution, if the latter exists. Suppose the proposition was false. Assume that there exist two different solutions $u_1(t,x)$ and $u_2(t,x)$. Denote $u_0(t,x) = u_1(t,x) - u_2(t,x)$. Then, $u_0(t,x)$ solves the following problem

$$\mathbb{D}^{\gamma}_{0^+,t}\big(u_0(t,x) - a\Delta_{\alpha,\beta}u_0(t,x)\big) - \Delta_{\alpha,\beta}u_0(t,x) + mu_0(t,x) = 0, \quad (t,x) \in D, \tag{3.3}$$

$$u_0(0,x) = 0, \quad x \in \mathbb{R}^+. \tag{3.4}$$

The problem (3.3) and (3.4) has only a trivial solution. This implies the uniqueness of the solution.

Now, we will prove the existence of the solutions. Using the Fourier–Jacobi transform $\mathcal{F}_{\alpha,\beta}$ (2.2) on both sides of (3.1) and (3.2), we have

$$\mathbb{D}^{\gamma}_{0^+,t}\widehat{u}(t,\lambda) + \frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}\widehat{u}(t,\lambda) = \frac{\widehat{f}(t,\lambda)}{1+a(\lambda^2+\rho^2)}, \tag{3.5}$$

$$\widehat{u}(0,\lambda) = \widehat{\phi}(\lambda) \tag{3.6}$$

for all $\lambda \in \mathbb{R}^+$ and $0 < t < T$. The solution (see [29, p. 231, ex. 4.9]) of the problem (3.5) and (3.6) is given by

$$\widehat{u}(t,\lambda) = \int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^{\gamma}\right)\frac{\widehat{f}(\tau,\lambda)}{1+a(\lambda^2+\rho^2)}\, d\tau$$

$$+\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right), \tag{3.7}$$

where $\mathbb{E}_{\gamma,1}(z)$ is the classical Mittag–Leffler function and $\mathbb{E}_{\gamma,\gamma}(z)$ is the Mittag–Leffler-type function. Now, by using the inverse Fourier–Jacobi transform $\mathcal{F}^{-1}_{\alpha,\beta}$ (2.3) to (3.7), we obtain the formula for the solution of the problem (3.1) and (3.2), given by

$$\begin{aligned} u(t,x) &= \int_0^\infty\int_0^\infty\int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^\gamma\right)\frac{f(\tau,y)}{1+a(\lambda^2+\rho^2)} \\ &\quad\times \varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\tau\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda) \\ &\quad+\int_0^\infty\int_0^\infty \mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right)\phi(y)\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda). \end{aligned}$$

By using the property

$$\frac{d}{d\tau}\left(\mathbb{E}_{\gamma,1}\left(c\tau^\gamma\right)\right) = c\tau^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(c\tau^\gamma\right), \quad c = \text{constant},$$

of the Mittag–Leffler function, we obtain

$$\begin{aligned} &\frac{d}{d\tau}\left(\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^\gamma\right)\right) \\ &\quad= \frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^\gamma\right) \end{aligned}$$

and we can write (3.7) in the form

$$\begin{aligned} \widehat{u}(t,\lambda) &= \int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^\gamma\right)\frac{\widehat{f}(\tau,\lambda)}{1+a(\lambda^2+\rho^2)}\,d\tau \\ &\quad+\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right) \\ &= \frac{1}{\lambda^2+\rho^2+m}\int_0^t \frac{d}{d\tau}\left(\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^\gamma\right)\right)\widehat{f}(\tau,\lambda)\,d\tau \\ &\quad+\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right) \\ &= \frac{1}{\lambda^2+\rho^2+m}\widehat{f}(t,\lambda) - \frac{1}{\lambda^2+\rho^2+m}\widehat{f}(0,\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right) \\ &\quad-\frac{1}{\lambda^2+\rho^2+m}\int_0^t \mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^\gamma\right)\frac{d}{d\tau}\widehat{f}(\tau,\lambda)\,d\tau \\ &\quad+\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right) \end{aligned}$$

by using the rule of integration by parts and $\mathbb{E}_{\gamma,1}(0) = 1$. Let $0 < \gamma < 1$ and $f \in C^1([0,T], L^2(\mu))$, $\phi \in \mathcal{H}$, then we can estimate u as follows:

$$\|u(t,\cdot)\|^2_{\mathcal{H}} = \int_0^\infty \left|\left(\lambda^2+\rho^2\right)\widehat{u}(t,\lambda)\right|^2\,d\nu_{\alpha,\beta}(\lambda)$$

$$
\begin{aligned}
&\lesssim \int_0^\infty \left|(\lambda^2+\rho^2)\frac{\widehat{f}(t,\lambda)}{\lambda^2+\rho^2+m}\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\quad + \int_0^\infty \left|(\lambda^2+\rho^2)\frac{\widehat{f}(0,\lambda)}{\lambda^2+\rho^2+m}\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\quad + \int_0^\infty \left|\frac{\lambda^2+\rho^2}{\lambda^2+\rho^2+m}\int_0^t \mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)^\gamma\right)\frac{d}{d\tau}\widehat{f}(\tau,\lambda)\,d\tau\right|^2 \\
&\quad \times d\nu_{\alpha,\beta}(\lambda) + \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \int_0^\infty \left|\widehat{f}(t,\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) + \int_0^\infty \left|\widehat{f}(0,\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\quad + \int_0^\infty \left(\int_0^t \left|\frac{d}{d\tau}\widehat{f}(\tau,\lambda)\right| d\tau\right)^2 d\nu_{\alpha,\beta}(\lambda) + \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{\phi}(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \left\|f(t,\cdot)\right\|_{2,\mu}^2 + \left\|f(0,\cdot)\right\|_{2,\mu}^2 + \int_0^T \left\|\frac{d}{dt}f(t,\cdot)\right\|_{2,\mu}^2 dt + \|\phi\|_{\mathcal{H}}^2,
\end{aligned}
$$

where we have used Definition 2.6, the Cauchy–Schwarz inequality, Fubibi's theorem, and $a \lesssim b$ denotes $a \le cb$ for some positive constant c independent of a and b. Thus,

$$
\left\|u(t,\cdot)\right\|_{\mathcal{H}}^2 \lesssim \left\|f(t,\cdot)\right\|_{2,\mu}^2 + \left\|f(0,\cdot)\right\|_{2,\mu}^2 + \int_0^T \left\|\frac{d}{dt}f(t,\cdot)\right\|_{2,\mu}^2 dt + \|\phi\|_{\mathcal{H}}^2.
$$

Then, we obtain

$$
\|u\|_{C([0,T],\mathcal{H})}^2 \lesssim \|f\|_{C^1([0,T],L^2(\mu))}^2 + \|\phi\|_{\mathcal{H}}^2 < \infty.
$$

In the case $\gamma = 1$, we have

$$
\begin{aligned}
\left\|u(t,\cdot)\right\|_{\mathcal{H}}^2 &= \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{u}(t,\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&= \int_0^\infty \left|(\lambda^2+\rho^2)\int_0^t \frac{\widehat{f}(\tau,\lambda)}{1+a(\lambda^2+\rho^2)}e^{-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)}\,d\tau \right. \\
&\quad \left. + (\lambda^2+\rho^2)\widehat{\phi}(\lambda)e^{-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t}\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \int_0^\infty \left|\int_0^t \widehat{f}(\tau,\lambda)e^{-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}(t-\tau)}\,d\tau\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\quad + \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{\phi}(\lambda)e^{-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t}\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \int_0^\infty \int_0^T \left|\widehat{f}(t,\lambda)\right|^2 dt\,d\nu_{\alpha,\beta}(\lambda) + \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{\phi}(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&= \int_0^T \left\|f(t,\cdot)\right\|_{2,\mu}^2 dt + \|\phi\|_{\mathcal{H}}^2
\end{aligned}
$$

by using Definition 2.6, the Cauchy–Schwarz inequality, and Fubini's theorem. Thus,

$$
\left\|u(t,\cdot)\right\|_{\mathcal{H}}^2 \lesssim \int_0^T \left\|f(t,\cdot)\right\|_{2,\mu}^2 dt + \|\phi\|_{\mathcal{H}}^2.
$$

Then, we have

$$\|u\|^2_{C([0,T],\mathcal{H})} \lesssim \|f\|^2_{C([0,T],L^2(\mu))} + \|\phi\|^2_{\mathcal{H}} < \infty.$$

Let us estimate the function $\mathbb{D}^{\gamma}_{0^+,t}u$

$$\begin{aligned}\left\|\mathbb{D}^{\gamma}_{0^+,t}u(t,\cdot)\right\|^2_{2,\mu} &= \left\|\mathbb{D}^{\gamma}_{0^+,t}\widehat{u}(t,\cdot)\right\|^2_{2,\nu} = \int_0^\infty \left|\mathbb{D}^{\gamma}_{0^+,t}\widehat{u}(t,\cdot)\right|^2 d\nu_{\alpha,\beta}(\lambda)\\ &= \int_0^\infty \left|\frac{\widehat{f}(t,\lambda)}{1+a(\lambda^2+\rho^2)} - \frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}\widehat{u}(t,\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda)\\ &\lesssim \left\|\widehat{f}(t,\cdot)\right\|^2_{2,\nu} + \left\|\widehat{u}(t,\cdot)\right\|^2_{2,\nu}.\end{aligned}$$

Thus, we have

$$\left\|\mathbb{D}^{\gamma}_{0^+,t}u(t,\cdot)\right\|^2_{2,\mu} \lesssim \left\|f(t,\cdot)\right\|^2_{2,\mu} + \left\|u(t,\cdot)\right\|^2_{2,\mu}$$

and

$$\left\|\mathbb{D}^{\gamma}_{0^+,t}u\right\|^2_{C([0,T],L^2(\mu))} \lesssim \|f\|^2_{C([0,T],L^2(\mu))} + \|u\|^2_{C([0,T],L^2(\mu))} < \infty.$$

Consequently, using Definition 2.10 we obtain $u \in C^{\gamma}([0,T],L^2(\mu))$. Our proof is completed. □

3.2 The ISP for the time-fractional pseudoparabolic equation

This subsection deals with the ISP for the time-fractional pseudoparabolic equation associated with the Jacobi operator $\Delta_{\alpha,\beta}$ (1.1).

3.2.1 Statement of the problem

Let $0 < \gamma \le 1$. We aim to find a couple of functions (u,f) satisfying equation

$$\mathbb{D}^{\gamma}_{0^+,t}\big(u(t,x) - a\Delta_{\alpha,\beta}u(t,x)\big) - \Delta_{\alpha,\beta}u(t,x) + mu(t,x) = f(x), \quad (t,x) \in D, \tag{3.8}$$

under conditions

$$u(0,x) = \phi(x), \quad x \in \mathbb{R}^+ \tag{3.9}$$

and

$$u(T,x) = \psi(x), \quad x \in \mathbb{R}^+. \tag{3.10}$$

Theorem 3.2 *Let $0 < \gamma \le 1$. Assume that $\psi,\phi \in \mathcal{H}$. Then, the pair (u,f) is a unique solution of ISP (3.8)–(3.10), which are functions $u \in C^{\gamma}([0,T],L^2(\mu)) \cap C([0,T],\mathcal{H}), f \in L^2(\mu)$ that can be represented by the formulas*

$$u(t,x) = \int_0^\infty \int_0^\infty \frac{1 - \mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^{\gamma})}{1 - \mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^{\gamma})}\psi(y)\varphi^{\alpha,\beta}_{\lambda}(y)\varphi^{\alpha,\beta}_{\lambda}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda)$$

$$-\int_0^\infty \int_0^\infty \frac{\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma) - \mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}$$
$$\times \phi(y)\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda)$$

and

$$f(x) = \int_0^\infty \int_0^\infty (\lambda^2+\rho^2+m)\frac{\psi(y)-\phi(y)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}$$
$$\times \varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda).$$

Proof We assume that $0<\gamma\leq 1$, and $u(t,\cdot), f \in \mathcal{H}$. Let us first prove the existence result. By using the Fourier–Jacobi transform $\mathcal{F}_{\alpha,\beta}$ (2.2) on both sides of (3.8)–(3.10), we obtain

$$\mathbb{D}_{0^+,t}^\gamma \widehat{u}(t,\lambda) + \frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}\widehat{u}(t,\lambda) = \frac{\widehat{f}(\lambda)}{1+a(\lambda^2+\rho^2)}, \quad (t,\lambda)\in D, \tag{3.11}$$

$$\widehat{u}(0,\lambda) = \widehat{\phi}(\lambda), \quad \lambda \in \mathbb{R}^+, \tag{3.12}$$

$$\widehat{u}(T,\lambda) = \widehat{\psi}(\lambda), \quad \lambda \in \mathbb{R}^+. \tag{3.13}$$

Solution of the equation (3.11) is given by

$$\widehat{u}(t,\lambda) = \frac{\widehat{f}(\lambda)}{\lambda^2+\rho^2+m} + C(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma\right), \tag{3.14}$$

for all $0<\gamma\leq 1$ and functions $\widehat{f}(\lambda)$ and $C(\lambda)$ are unknown functions. To determine these functions we use conditions (3.12) and (3.13). Then, we have

$$\widehat{u}(0,\lambda) = \frac{\widehat{f}(\lambda)}{\lambda^2+\rho^2+m} + C(\lambda) = \widehat{\phi}(\lambda)$$

and

$$\widehat{u}(T,\lambda) = \frac{\widehat{f}(\lambda)}{\lambda^2+\rho^2+m} + C(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma\right) = \widehat{\psi}(\lambda).$$

Thus, we have

$$C(\lambda) = \frac{\widehat{\phi}(\lambda)-\widehat{\psi}(\lambda)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}$$

and

$$\widehat{f}(\lambda) = (\lambda^2+\rho^2+m)\frac{\widehat{\psi}(\lambda)-\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}. \tag{3.15}$$

Substituting the resulting functions $C(\lambda)$ and $\widehat{f}(\lambda)$ into (3.14), we obtain

$$\widehat{u}(t,\lambda) = \frac{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\widehat{\psi}(\lambda) - \frac{\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\widehat{\phi}(\lambda).$$

Therefore, the solution of the problem (3.11)–(3.13) is the pair $(\widehat{u},\widehat{f})$. We obtain the solution of the problem (3.8)–(3.10) by applying the inverse Fourier–Jacobi transform $\mathcal{F}^{-1}_{\alpha,\beta}$ (2.3) to the functions $\widehat{u}$ and $\widehat{f}$, i.e.,

$$\begin{aligned} u(t,x) &= \int_0^\infty\int_0^\infty \frac{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\psi(y)\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda) \\ &\quad - \int_0^\infty\int_0^\infty \frac{\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)} \\ &\quad \times \phi(y)\varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} f(x) &= \int_0^\infty\int_0^\infty (\lambda^2+\rho^2+m)\frac{\psi(y)-\phi(y)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)} \\ &\quad \times \varphi_\lambda^{\alpha,\beta}(y)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(y)\,d\nu_{\alpha,\beta}(\lambda) \end{aligned} \tag{3.17}$$

for all $0<\gamma\le 1$.

Let $\psi,\phi\in\mathcal{H}$. Then, using Lemma 2.13 we can estimate the function u as follows:

$$\begin{aligned} \|u(t,\cdot)\|^2_{\mathcal{H}} &= \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{u}(t,\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\ &\lesssim \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{\psi}(\lambda)\frac{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\right|^2 d\nu_{\alpha,\beta}(\lambda) \\ &\quad + \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{\phi}(\lambda)\frac{\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\right|^2 d\nu_{\alpha,\beta}(\lambda) \\ &\lesssim \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{\psi}(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) + \int_0^\infty \left|(\lambda^2+\rho^2)\widehat{\phi}(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda), \end{aligned}$$

where we have used Definition 2.6. Thus,

$$\|u(t,\cdot)\|^2_{\mathcal{H}} \lesssim \|\psi\|^2_{\mathcal{H}} + \|\phi\|^2_{\mathcal{H}} < \infty.$$

Then, we have

$$\|u\|_{C([0,T],\mathcal{H})} \lesssim \|\psi\|_{\mathcal{H}} + \|\phi\|_{\mathcal{H}} < \infty.$$

Let us estimate the function f

$$\begin{aligned}
\|f\|^2_{2,\mu} &= \|\widehat{f}\|^2_{2,\nu} = \int_0^\infty \left|\widehat{f}(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&= \int_0^\infty \left|(\lambda^2+\rho^2+m)\frac{\widehat{\psi}(\lambda)-\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \int_0^\infty \left|(\lambda^2+\rho^2+m)\frac{\widehat{\psi}(\lambda)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\quad + \int_0^\infty \left|(\lambda^2+\rho^2+m)\frac{\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \|\psi\|^2_{\mathcal{H}} + \|\phi\|^2_{\mathcal{H}}.
\end{aligned}$$

Hence, we obtain

$$\|f\|^2_{2,\mu} \lesssim \|\psi\|^2_{\mathcal{H}} + \|\phi\|^2_{\mathcal{H}} < \infty.$$

Next, we estimate the function $\mathbb{D}^\gamma_{0^+,t}u$

$$\begin{aligned}
\left\|\mathbb{D}^\gamma_{0^+,t}u(t,\cdot)\right\|^2_{2,\mu} &= \left\|\mathbb{D}^\gamma_{0^+,t}\widehat{u}(t,\cdot)\right\|^2_{2,\nu} = \int_0^\infty \left|\mathbb{D}^\gamma_{0^+,t}\widehat{u}(t,\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&= \int_0^\infty \left|\frac{\widehat{f}(\lambda)}{1+a(\lambda^2+\rho^2)} - \frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}\widehat{u}(t,\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \|f\|^2_{2,\nu} + \left\|u(t,\cdot)\right\|^2_{2,\nu}.
\end{aligned}$$

Finally, we have

$$\left\|\mathbb{D}^\gamma_{0^+,t}u\right\|^2_{C([0,T],L^2(\mu))} \lesssim \|f\|^2_{2,\mu} + \|u\|^2_{C([0,T],L^2(\mu))} < \infty.$$

It is obvious that $\|u\|^2_{C([0,T],L^2(\mu))} < \infty$. The existence is proved.

Now, let us prove the uniqueness of the solution. Taking into account the property of the Fourier–Jacobi transform, Proposition 2.4, one observes that a pair of functions (u,f) is uniquely determined by the formulas (3.16) and (3.17). The uniqueness is proved. □

3.2.2 Stability theorem

Finally, we study a stability property of the solution (u,f) of the problem (3.8)–(3.10) given by the formulas (3.16) and (3.17).

Theorem 3.3 *Let (u,f) and (u_d,f_d) be solutions of the problem (3.8)–(3.10) corresponding to the data (ϕ,ψ) and its small perturbation (ϕ_d,ψ_d), respectively. Then, the solution of the problem (3.8)–(3.10) depends continuously on these data, namely, we have*

$$\|u-u_d\|^2_{C([0,T],\mathcal{H})} \lesssim \|\psi-\psi_d\|^2_{\mathcal{H}} + \|\phi-\phi_d\|^2_{\mathcal{H}}$$

and

$$\|f - f_d\|^2_{2,\mu} \lesssim \|\psi - \psi_d\|^2_{\mathcal{H}} + \|\phi - \phi_d\|^2_{\mathcal{H}}.$$

Proof From the definition of the Fourier–Jacobi transform (2.2)

$$\mathcal{F}_{\alpha,\beta}\big(u(t,\cdot)\big)(\lambda) = \widehat{u}(t,\lambda) = \int_0^\infty u(t,x)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(x),$$

we conclude that

$$\begin{aligned}\mathcal{F}_{\alpha,\beta}\big(u(t,\cdot) - u_d(t,\cdot)\big)(\lambda) &= \int_0^\infty \big(u(t,x) - u_d(t,x)\big)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(x)\\ &= \int_0^\infty u(t,x)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(x)\\ &\quad - \int_0^\infty u_d(t,x)\varphi_\lambda^{\alpha,\beta}(x)\,d\mu_{\alpha,\beta}(x)\\ &= \mathcal{F}_{\alpha,\beta}\big(u(t,\cdot)\big)(\lambda) - \mathcal{F}_{\alpha,\beta}\big(u_d(t,\cdot)\big)(\lambda)\\ &= \widehat{u}(t,\lambda) - \widehat{u}_d(t,\lambda),\end{aligned}$$

where we have used the property of the integral. According to the above statement and using Lemma 2.13, we have

$$\begin{aligned}&\big\|u(t,\cdot) - u_d(t,\cdot)\big\|^2_{\mathcal{H}}\\ &\quad= \int_0^\infty \big(\lambda^2+\rho^2\big)^2\big|\widehat{u}(t,\lambda) - \widehat{u}_d(t,\lambda)\big|^2\,d\nu_{\alpha,\beta}(\lambda)\\ &\quad= \int_0^\infty \big(\lambda^2+\rho^2\big)^2\bigg|\frac{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\big(\widehat{\psi}(\lambda) - \widehat{\psi}_d(\lambda)\big)\\ &\qquad - \frac{\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma) - \mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}t^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\big(\widehat{\phi}(\lambda) - \widehat{\phi}_d(\lambda)\big)\bigg|^2\,d\nu_{\alpha,\beta}(\lambda)\\ &\quad\lesssim \int_0^\infty \big(\lambda^2+\rho^2\big)^2\big|\widehat{\psi}(\lambda) - \widehat{\psi}_d(\lambda)\big|^2\,d\nu_{\alpha,\beta}(\lambda) + \int_0^\infty \big(\lambda^2+\rho^2\big)^2\big|\widehat{\phi}(\lambda) - \widehat{\phi}_d(\lambda)\big|^2\,d\nu_{\alpha,\beta}(\lambda)\\ &\quad= \|\psi - \psi_d\|^2_{\mathcal{H}} + \|\phi - \phi_d\|^2_{\mathcal{H}},\end{aligned}$$

where we have used Definition 2.6. Thus, one obtains

$$\big\|u(t,\cdot) - u_d(t,\cdot)\big\|^2_{\mathcal{H}} \lesssim \|\psi - \psi_d\|^2_{\mathcal{H}} + \|\phi - \phi_d\|^2_{\mathcal{H}}$$

and

$$\|u - u_d\|^2_{C([0,T],\mathcal{H})} \lesssim \|\psi - \psi_d\|^2_{\mathcal{H}} + \|\phi - \phi_d\|^2_{\mathcal{H}}.$$

By writing (3.15) in the form

$$\widehat{f}(\lambda) = \frac{\lambda^2+\rho^2+m}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\widehat{\psi}(\lambda) - \frac{(\lambda^2+\rho^2+m)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\widehat{\phi}(\lambda)$$

and applying similar estimates again we can observe that

$$\begin{aligned}
\|f-f_d\|^2_{2,\mu} &= \|\widehat{f}-\widehat{f}_d\|^2_{2,\nu} = \int_0^\infty \left|\widehat{f}(\lambda)-\widehat{f}_d(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&= \int_0^\infty \Bigg| \frac{\lambda^2+\rho^2+m}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\big(\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\big) \\
&\quad - \frac{(\lambda^2+\rho^2+m)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\big(\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\big)\Bigg|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \int_0^\infty \Bigg| \frac{\lambda^2+\rho^2+m}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\big(\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\big)\Bigg|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\quad + \int_0^\infty \Bigg| \frac{(\lambda^2+\rho^2+m)\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}{1-\mathbb{E}_{\gamma,1}(-\frac{\lambda^2+\rho^2+m}{1+a(\lambda^2+\rho^2)}T^\gamma)}\big(\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\big)\Bigg|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\lesssim \int_0^\infty (\lambda^2+\rho^2)^2\left|\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&\quad + \int_0^\infty (\lambda^2+\rho^2)^2\left|\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\right|^2 d\nu_{\alpha,\beta}(\lambda) \\
&= \|\psi-\psi_d\|^2_{\mathcal{H}} + \|\phi-\phi_d\|^2_{\mathcal{H}}.
\end{aligned}$$

It follows easily that

$$\|f-f_d\|^2_{2,\mu} \lesssim \|\psi-\psi_d\|^2_{\mathcal{H}} + \|\phi-\phi_d\|^2_{\mathcal{H}},$$

ending the proof. □

3.2.3 Stability test

Here, to check Theorem 3.3 we consider a ISP for the heat equation with a one-dimensional Sturm–Liouville operator

$$u_t(t,x) - u_{xx}(t,x) = f(x), \quad 0 < t < 1, x > 0, \tag{3.18}$$

with conditions

$$u(0,x) = u(1,x) = 0, \tag{3.19}$$

where we put $T = \gamma = 1$, $\alpha = \beta = -\frac{1}{2}$, $a = m = 0$, and $\phi(x) = \psi(x) = 0$ for all $x > 0$.

Also, consider a perturbed problem with some noise

$$u^\epsilon_t(t,x) - u^\epsilon_{xx}(t,x) = f^\epsilon(x), \quad 0 < t < 1, x > 0,$$

Table 1 Stability Test

ϵ	1	0.2	0.02
$\|\psi - \psi^\epsilon\|^2_{\mathcal{H}}$	1.5	0.06	0.0006
$\|u - u^\epsilon\|^2_{C([0,1],\mathcal{H})}$	0.75	0.03	0.0003
$\|f - f^\epsilon\|^2_{2,\mu}$	1.0474	0.041897	0.0004

with conditions

$$u^\epsilon(0,x) = 0, \quad \text{and} \quad u^\epsilon(1,x) = \epsilon \cdot e^{-x^2}, \quad x > 0,$$

and with additional information $\phi^\epsilon(x) = 0$ and $\psi^\epsilon(x) = \epsilon \cdot e^{-x^2}$, where ϵ is a positive constant. Then, by Theorem 3.2, we have

$$u^\epsilon(t,x) = \frac{\epsilon}{\sqrt{\pi}} \int_0^\infty \frac{1 - e^{-\lambda^2 t}}{1 - e^{-\lambda^2}} e^{-\frac{\lambda^2}{4}} \cos(\lambda x)\, d\lambda$$

and

$$f^\epsilon(x) = \frac{\epsilon}{\sqrt{\pi}} \int_0^\infty \frac{\lambda^2 e^{-\frac{\lambda^2}{4}}}{(1 - e^{-\lambda^2})} \cos(\lambda x)\, d\lambda.$$

Illustrations of our calculations above are given in Table 1.

4 Conclusions

In our paper, we considered one ISP for a pseudoparabolic equation generated by the Jacobi and Caputo fractional operators. We showed that the ISP has a unique solution, and the solution of the ISP continuously depends on the given data (Theorem 3.2 and Theorem 3.3). To examine our results we gave one simple example and in this example we did some calculations using the Maple 2021 program; the results are given in Table 1. We used a classical method based on the Fourier–Jacobi transform, the advantage of this method is that we can obtain an explicit solution. However, this method is only applicable for linear problems with constant coefficients, thus if we consider this problem with variable coefficients we are lost. Hence, continuation of this work might be a ISP with variable coefficients.

Appendix

Calculations in Table 1 were made by using the Maple 2021 program with the following codes:

$$psi := \exp\left(-x^2\right),$$

$$hat(psi) := int\left(\frac{1}{\sqrt{2\pi}} \cdot psi \cdot \cos(x \cdot \lambda), x = 0 \ldots \infty\right),$$

$$norm(psi) := 40 \cdot int\left(\frac{4}{\sqrt{2\pi}} \cdot \lambda^4 \cdot \left|hat(psi)\right|^2, \lambda = 0 \ldots \infty\right),$$

$$hat(u) := \frac{1 - \exp(-\lambda^2 \cdot t)}{1 - \exp(-\lambda^2)} \cdot hat(psi),$$

$$norm(u) := int\left(\frac{4}{\sqrt{2\pi}} \cdot \lambda^4 \cdot \left|hat(u)\right|^2, \lambda = 0 \ldots \infty\right),$$

$$\lim_{t\to 1}\bigl(norm(u)\bigr),$$

$$hat(f) := \frac{\lambda^2 \cdot hat(psi)}{1 - \exp(-\lambda^2)}$$

and

$$norm(f) := int\left(\frac{4}{\sqrt{2\pi}} \cdot \left|hat(f)\right|^2, \lambda = 0 \ldots \infty\right).$$

Author contributions
B. B. and N. T. wrote the manuscript text and solved all problems together. All authors reviewed the manuscript.

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References
1. Rundell, W.: Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data. Appl. Anal. **10**, 231–242 (1980)
2. Prilepko, A.I., Tikhonov, I.V.: Uniqueness of a solution of the inverse problem for the evolution equation and application to the transport equation. Math. Notes **51**, 158–165 (1992)
3. Bushuyev, I.: Global uniqueness for inverse parabolic problems with final observation. Inverse Probl. **11**, L11–L16 (1995)
4. Tikhonov, I.V., Eidelman, Y.S.: An inverse problem for a differential equation in a Banach space and distribution of zeros of an entire Mittag-Leffler function. Differ. Equ. **38**(5), 669–677 (2002)
5. Choulli, M., Yamamoto, M.: Conditional stability in determining a heat source. J. Inverse Ill-Posed Probl. **12**(3), 233–243 (2004)
6. Yaman, M., Gözükızıl, O.F.: Asymptotic behaviour of the solutions of inverse problems for pseudo-parabolic equations. Appl. Math. Comput. **154**, 69–74 (2004)
7. Sakamoto, K., Yamamoto, M.: Inverse source problem with a final overdetermination for a fractional diffusion equation. Math. Control Relat. Fields **1**(4), 509–518 (2011)
8. Yaman, M.: Blow-up solution and stability to an inverse problem for a pseudo-parabolic equation. J. Inequal. Appl. **2012**, 274 (2012)
9. Slodička, M.: A source identification problem in linear parabolic problems: a semigroup approach. J. Inverse Ill-Posed Probl. **21**, 579–600 (2013)
10. Slodička, M., Šiškova, M.: An inverse source problem in a semilinear time-fractional diffusion equation. Comput. Math. Appl. **72**, 1655–1669 (2016)
11. Ruzhansky, M., Serikbaev, D., Torebek, B.T., Tokmagambetov, N.: Direct and inverse problems for time-fractional pseudo-parabolic equations. Quaest. Math. (2021). https://doi.org/10.2989/16073606.2021.1928321
12. Cheng, J., Nakagawa, J., Yamamoto, M., Yamazaki, T.: Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. Inverse Probl. **25**, 115002 (2009)
13. Jin, B., Rundell, W.: A tutorial on inverse problems for anomalous diffusion processes. Inverse Probl. **31**, 035003 (2015)
14. Kaliev, I.A., Sabitova, M.M.: Problems of determining the temperature and density of heat sources from the initial and final temperatures. J. Appl. Ind. Math. **4**(3), 332–339 (2010)
15. Kirane, M., Samet, B., Torebek, B.T.: Determination of an unknown source term temperature distribution for the sub-diffusion equation at the initial and final data. Electron. J. Differ. Equ. **2017**, 257 (2017)
16. Orazov, I., Sadybekov, M.A.: One nonlocal problem of determination of the temperature and density of heat sources. Russ. Math. **56**(2), 60–64 (2012)

17. Orazov, I., Sadybekov, M.A.: On a class of problems of determining the temperature and density of heat sources given initial and final temperature. Sib. Math. J. **53**(1), 146–151 (2012)
18. Ruzhansky, M., Tokmagambetov, N., Torebek, B.T.: Inverse source problems for positive operators. I: hypoelliptic diffusion and subdiffusion equations. J. Inverse Ill-Posed Probl. (2019)
19. Slodička, M.M., Šiškova, M., Bockstal, K.V.: Uniqueness for an inverse source problem of determining a space dependent source in a time-fractional diffusion equation. Appl. Math. Lett. **91**, 15–21 (2019)
20. Torebek, B.T., Tapdigoglu, R.: Some inverse problems for the nonlocal heat equation with Caputo fractional derivative. Math. Methods Appl. Sci. **40**(18), 6468–6479 (2017)
21. Wang, W., Yamamoto, M., Han, B.: Numerical method in reproducing kernel space for an inverse source problem for the fractional diffusion equation. Inverse Probl. **29**, 095009 (2013)
22. Chouchene, F., Gallardo, L., Mili, M.: The heat semigroup for the Jacobi-Dunkl operator and the related Markov processes. Potential Anal. **25**(2), 103–119 (2006)
23. Flensted-Jensen, M.: Paley-Wiener type theorems for a differential operator connected with symmetric spaces. Ark. Mat. **10**, 143–162 (1972)
24. Flensted-Jensen, M., Koorwinder, T.H.: The convolution structure for Jacobi function expansions. Ark. Mat. **11**, 245–262 (1973)
25. Flensted-Jensen, M., Koorwinder, T.H.: Jacobi functions: the addition formula and the positivity of dual convolution structure. Ark. Mat. **17**, 139–151 (1979)
26. Koorwinder, T.H.: A new proof of a Paley-Wiener type theorem for the Jacobi transform. Ark. Mat. **13**, 145–159 (1975)
27. Salem, N.B., Dachraoui, A.: Pseudo-differential operators associated with the Jacobi differential operator. J. Math. Anal. Appl. **220**, 365–381 (1998)
28. Salem, N.B., Dachraoui, A.: Sobolev type spaces associated with Jacobi differential operators. Integral Transforms Spec. Funct. **9**, 163–184 (2000)
29. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies. Elsevier, Amsterdam (2006)
30. de Carvalho-Neto, P.M., Fehlberg Júnior, R.: Conditions for the absence of blowing up solutions to fractional differential equations. Acta Appl. Math. **154**, 15–29 (2018)
31. Simon, T.: Comparing Frechet and positive stable laws. Electron. J. Probab. **91**, 1–25 (2014)
32. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
33. Miller, K.S., Samko, S.G.: A note on the complete monotonicity of the generalized Mittag-Lefler function. Real Anal. Exch. **23**(2), 753–755 (1997–1998)

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