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Global attractors of the ^{3}He-^{4}He system in H_α spaces

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Abstract

In this paper, the existence of a global attractor for the ^{3}He-^{4}He system is investigated. By using an iteration procedure, combining with the classical existence theorem of global attractors, we prove that this system possesses a global attractor, which attracts any bounded set of H_α in H_α-norm.

MSC: 35B40; 35Q56

Keywords: helium-3; helium-4; operator semigroup; global attractor

1 Introduction

Superfluidity is a phase of matter in which 'unusual' effects are observed when liquids, typically of helium-3 or helium-4, overcome friction by surface interaction when at a stage, known as the 'lambda point' for helium-4, at which viscosity of the liquid becomes zero. Experiments have indicated that helium atoms have two stable isotopes ^{3}He and ^{4}He, and liquid ^{3}He and ^{4}He can be dissolved into each other. The general features of phase separation and superfluidity in mixtures of liquid ^{3}He and ^{4}He have been known for some time [1–3].

Up to now, we find several mathematical results on model for liquid mixture of ^{3}He-^{4}He. In [4, 5], Ma and Wang introduced a dynamical Ginzburg-Landau phase transition/separation model for the mixture of liquid helium-3 and helium-4, and studied phase separations between liquid helium-3 and liquid helium-4 from both the modeling and analysis points of view. The analysis leads to three critical length scales $L_1 < L_2 < L_3$ and the corresponding λ-transition and phase separation diagrams. In [6], Ma studied existence of solutions to model for liquid mixture of ^{3}He-^{4}He by using spatial sequence techniques and linear operator theories. In [7], Luo and Pu obtained the existence and regularity of solutions to ^{3}He-^{4}He system by using the Galerkin method.

As we know, the dynamical properties of the ^{3}He-^{4}He system such as the global asymptotical behaviors of solutions and existence of global attractors, are important for the study of phase transition and separation for mixture of liquid ^{3}He and ^{4}He, which ensure the stability of phase transition and provide the mathematical foundation for the study of phase transition dynamics.

In this article, we concerned with the global asymptotical behaviors of solutions and existence of global attractors of the ^{3}He-^{4}He system, to wit the following initial-boundary problem:

$$\begin{cases} \frac{\partial u}{\partial t} = -\mu_2\triangle^2 u + \triangle(\lambda_2 u + b_1 u^2 + b_2 u^3 + \frac{1}{2}\alpha_3|\psi|^2), & x \in \Omega, \\ \frac{\partial \psi}{\partial t} = \mu_1\triangle\psi - \lambda_1\psi - \alpha_2|\psi|^2\psi - \alpha_3\psi u, & x \in \Omega, \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \qquad \frac{\partial \triangle u}{\partial n}|_{\partial\Omega} = 0, \qquad \frac{\partial \psi}{\partial n}|_{\partial\Omega} = 0, \qquad \int_\Omega u\,\mathrm{d}x = 0, \\ u(x,0) = u_0(x), \qquad \psi(x,0) = \psi_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where the unknown function u represents the mol fraction of liquid ^{3}He, and the complex-valued function $\psi = \psi_1 + i\psi_2$ describes the phase transition of liquid ^{4}He between the normal and superfluid states. $\triangle$ is the Laplace operator, $\Omega \subset \mathbb{R}^n$ $(1 \le n \le 3)$ is a C^∞ bounded domain. Due to the Landau average field theory (also see [4, 5]), we have

$$\mu_1, \mu_2, \alpha_2, \alpha_3, b_2 > 0. \tag{1.2}$$

We shall use the regularity estimates for the linear semigroups, combining with the classical existence theorem of global attractors, to prove that the ^{3}He-^{4}He system (1.1) possesses, in space H_α, a global attractor, which attracts any bounded set of H_α in H_α-norm. Attractor in H_α have studied by some authors, we refer the reader to [8–10]. In [8], Luo studied the extended Fisher-Kolmogorov equation, to wit the following initial-boundary problem:

$$\begin{cases} \frac{\partial u}{\partial t} = -\beta\triangle^2 u + \triangle u - u^3 + u, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \qquad \triangle u|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.3}$$

and obtained the existence of global attractors of (1.3) in H^k spaces by using an iteration procedure. Although [8] and this paper all use the iteration procedure, in this paper we not only consider the iteration procedures of the unknown functions u and ψ, but also the interaction between u and ψ. Because of that and because the unknown function ψ is a complex-valued function, the iteration procedure in this paper is more difficult than in [8].

The rest of the paper is arranged as follows. In Section 2, we will iterate some notations and theorems for the abstract nonlinear evolution equation. In Section 3, we will state and prove our main result.

2 Preliminaries and auxiliary results

In this section, we iterate some notations, abstract theorems, and auxiliary results, which are important for getting our main result.

Let H and H_1 be Hilbert space, $H_1 \subset H$ a compact and dense inclusion. Consider the abstract evolution equation defined on H, given by

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = Lu + G(u), \\ u(0) = \varphi, \end{cases} \tag{2.1}$$

where u is an unknown function, $L : H_1 \to H$ a linear operator, and $G : H_1 \to H$ a nonlinear operator.

A family of operators $S(t): H \to H$ $(t \geq 0)$ is called a semigroup generated by (2.1) provided $S(t)$ satisfies the properties:

(1) $S(t): H \to H$ is a continuous mapping for any $t \geq 0$,
(2) $S(0) = \mathrm{id}: H \to H$ the identity,
(3) $S(t+s) = S(t) \cdot S(s)$, $\forall t, s \geq 0$,

and the solution of (2.1) can be expressed as

$$u(t, \varphi) = S(t)\varphi.$$

Next, we introduce the concepts and definitions of invariant sets, global attractors, and ω-limit sets for the semigroup $S(t)$.

Definition 2.1 [11] Let $S(t)$ be a semigroup defined on H. A set Σ is called an invariant set of $S(t)$ if $S(t)\Sigma = \Sigma$, $\forall t \geq 0$. An invariant set Σ is an attractor of $S(t)$ if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that for any $\varphi \in U$,

$$\inf_{v \in \Sigma} \| S(t)\varphi - v \|_H \to 0, \quad \text{as } t \to \infty.$$

In this case, we say that Σ attracts U. Especially, if Σ attracts any bounded set of H, Σ is called a global attractor of $S(t)$.

For a set $D \subset H$, we define the ω-limit set of D as follows:

$$\omega(D) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)D},$$

where the closure is taken in the H-norm.

Lemma 2.1 is the classical existence theorem of global attractors by Temam [11].

Lemma 2.1 *Let $S(t): H \to H$ be the semigroup generated by* (2.1). *Assume the following conditions hold*:

(1) *$S(t)$ has a bounded absorbing set $B \subset H$, i.e., for any bounded set $A \subset H$ there exists a time $t_A > 0$ such that $S(t)\varphi \in B$, $\forall \varphi \in A$ and $t > t_A$,*
(2) *$S(t)$ is uniformly compact, i.e., for any bounded set $U \subset H$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t \geq T} S(t)U}$ is compact in H.*

Then the ω-limit set $\mathcal{A} = \omega(B)$ of B is a global attractor of (2.1), *and $\mathcal{A}$ is connected provided B is connected.*

We used to assume that the linear operator L in (2.1) is a sectorial operator which generates an analytic semigroup e^{tL}. It is known that there exists a constant $\lambda \geq 0$ such that $L - \lambda I$ generates the fractional power operator $\mathcal{L}^\alpha$ and fractional order space H_α for $\alpha \in R^1$, where $\mathcal{L} = -(L - \lambda I)$. Without loss of generality, we assume that L generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order space H_α as follows:

$$\mathcal{L}^\alpha = (-L)^\alpha : H_\alpha \to H, \quad \alpha \in R^1,$$

where $H_\alpha = D(\mathcal{L}^\alpha)$ is the domain of $\mathcal{L}^\alpha$.

For sectorial operators, we also have the following properties, which can be found in [12].

Lemma 2.2 *Let $L : H_1 \to H$ be a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy* $\operatorname{Re}\lambda < -\lambda_0$ *for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\alpha$ ($\mathcal{L} = -L$) we have*

(1) *$T(t) : H \to H_\alpha$ is bounded for all $\alpha \in R^1$ and $t > 0$,*
(2) *$T(t)\mathcal{L}^\alpha x = \mathcal{L}^\alpha T(t)x$, $\forall x \in H_\alpha$,*
(3) *for each $t > 0$, $\mathcal{L}^\alpha T(t) : H \to H$ is bounded, and*

$$\left\| \mathcal{L}^\alpha T(t) \right\| \le C_\alpha t^{-\alpha} e^{-\delta t},$$

where $\delta > 0$ and $C_\alpha > 0$ are constants only depending on α,
(4) *the H_α-norm can be defined by*

$$\|x\|_{H_\alpha} = \left\| \mathcal{L}^\alpha x \right\|_H,$$

(5) *if $\mathcal{L}$ is symmetric, for any $\alpha, \beta \in R^1$ we have*

$$\langle \mathcal{L}^\alpha u, v \rangle_H = \langle \mathcal{L}^{\alpha-\beta} u, \mathcal{L}^\beta v \rangle_H.$$

For convenience, we introduce the following result.

Lemma 2.3 [6] *Under the condition* (1.2), *for $(u_0, \psi_0) \in [H^1(\Omega) \times H^1(\Omega, \mathbb{C})] \cap H$ the system* (1.1) *has a unique solution*

$$(u, \psi) \in C^0\big((0, T), H^4(\Omega) \times H^1(\Omega, \mathbb{C})\big) \cap C^1\big((0, T), L^2(\Omega) \times L^2(\Omega, \mathbb{C})\big),$$

here the space H is defined as follows:

$$H = \left\{ u \in L^2(\Omega) \;\Big|\; \int_\Omega u \, \mathrm{d}x = 0 \right\} \times L^2(\Omega, \mathbb{C}).$$

Next we convert (1.1) into the abstract form (2.1). To do so, we need the following space:

$$H_1 = \left\{ (u, \psi) \in \left[H^4(\Omega) \times H^2(\Omega, \mathbb{C})\right] \cap H \;\Big|\; \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0, \frac{\partial \triangle u}{\partial n}\bigg|_{\partial\Omega} = 0, \frac{\partial \psi}{\partial n}\bigg|_{\partial\Omega} = 0 \right\}.$$

We define the operators $L, G : H_1 \to H$ by

$$\begin{aligned} L(U) &= \begin{pmatrix} -\mu_2 \triangle^2 u \\ \mu_1 \triangle \psi \end{pmatrix}, \\ G(U) &= \begin{pmatrix} \triangle(\lambda_2 u + b_1 u^2 + b_2 u^3 + \frac{1}{2}\alpha_3 |\psi|^2) \\ -\lambda_1 \psi - \alpha_2 |\psi|^2 \psi - \alpha_3 \psi u \end{pmatrix}, \end{aligned} \tag{2.2}$$

where $U = (u, \psi)$. Then the ^{3}He-^{4}He system (1.1) can be written in the abstract form (2.1).

It is easy to see that

$$L = \begin{pmatrix} -\mu_2 \triangle^2 & 0 \\ 0 & \mu_1 \triangle \end{pmatrix} : H_1 \to H$$

is a sectorial operator and L generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order space H_α as follows:

$$\mathcal{L}^\alpha = (-L)^\alpha : H_\alpha \to H, \quad \alpha \in R^1,$$

where the space H_α is given by

$$H_\alpha = H^{4\alpha}(\Omega) \times H^{2\alpha}(\Omega, \mathbb{C}).$$

Especially, the fractional power operator $(-L)^{\frac{1}{2}} : H_{\frac{1}{2}} \to H$ is given by

$$(-L)^{\frac{1}{2}} = \begin{pmatrix} -\sqrt{\mu_2}\triangle & 0 \\ 0 & \sqrt{\mu_1}(-\triangle)^{\frac{1}{2}} \end{pmatrix},$$

where

$$H_{\frac{1}{2}} = H^2(\Omega) \times H^1(\Omega, \mathbb{C}).$$

3 Main result

In this section we state and prove our main result; some of our important ideas come from Ma's recent books [4, 6].

The main result in this article is given by the following theorem, which provides the existence of global attractors of the ^{3}He-^{4}He system (1.1) in space H_α.

Theorem 3.1 *Assume* (1.2) *hold. Then for any* $\alpha \geq 0$ *system* (1.1) *has a global attractor* $\mathcal{A}_\alpha \subset H_\alpha$, *and* $\mathcal{A}_\alpha$ *attracts any bounded set of* H_α *in the* H_α*-norm.*

Remark 3.1 The attractors $\mathcal{A}_\alpha \subset H_\alpha$ in Theorem 3.1 are the same for all $\alpha \geq 0$, *i.e.*, $\mathcal{A}_\alpha = \mathcal{A}$, $\alpha \geq 0$. Hence, $\mathcal{A} \subset C^\infty$. Theorem 3.1 implies that for any $\varphi = (u_0, \psi_0) \in H$, the solution $U(t, \varphi)$ of (1.1) satisfies

$$\lim_{t\to\infty} \inf_{v\in\mathcal{A}} \left\| U(t,\varphi) - v \right\|_{C^k} = 0, \quad \forall k \geq 1.$$

Proof From Lemma 2.3, we know that the solution of system (1.1) is a strong solution for any $\varphi = (u_0, \psi_0) \in [H^1(\Omega) \times H^1(\Omega, \mathbb{C})] \cap H$. Hence, the solution $U(t, \varphi) = (u, \psi)$ of system (1.1) can be written as

$$U(t,\varphi) = e^{tL}\varphi + \int_0^t e^{(t-\tau)L} G(U)\, \mathrm{d}\tau. \tag{3.1}$$

Let

$$\tilde{L} = -\triangle^2, \qquad \bar{L} = -\triangle.$$

Then $\tilde{L}$, $\bar{L}$ generate the fractional power operators $\tilde{\mathcal{L}}^\alpha$, $\bar{\mathcal{L}}^\alpha$ and fractional order spaces $\tilde{H}_\alpha$, $\bar{H}_\alpha$ as follows:

$$\tilde{\mathcal{L}}^\alpha = (-\tilde{L})^\alpha : \tilde{H}_\alpha \to L^2(\Omega), \quad \alpha \in R^1,$$

$$\bar{\mathcal{L}}^{\alpha} = (-\bar{L})^{\alpha} : \bar{H}_{\alpha} \to L^2(\Omega, \mathbb{C}), \quad \alpha \in R^1,$$

where $\tilde{H}_{\alpha} = H^{4\alpha}(\Omega)$, $\bar{H}_{\alpha} = H^{2\alpha}(\Omega, \mathbb{C})$.

Next, according to Lemma 2.1, we prove Theorem 3.1 in the following five steps.

Step 1. We prove that for any bounded set $W \subset H_{\frac{1}{2}}$ there is a constant $C > 0$ such that the solution $U(t, \varphi)$ of system (1.1) is uniformly bounded by the constant C for any $\varphi \in W$ and $t \geq 0$, *i.e.*,

$$\|u(t,\varphi)\|_{\tilde{H}_{\alpha}} \leq C, \qquad \|\psi(t,\varphi)\|_{\bar{H}_{2\alpha}} \leq C, \quad \forall t > 0, \varphi \in W, \frac{1}{4} \leq \alpha < \frac{1}{2}.$$

To do that, we firstly check that system (1.1) has a global Lyapunov function as follows:

$$\begin{aligned} F(U) = \frac{1}{2} \int_{\Omega} \bigg[&\mu_1 |\nabla \psi|^2 + \mu_2 |\nabla u|^2 + \lambda_1 |\psi|^2 + \lambda_2 u^2 \\ &+ \frac{1}{2}\alpha_2 |\psi|^4 + \frac{1}{2} b_2 u^4 + \frac{2}{3} b_1 u^3 + \alpha_3 |\psi|^2 u \bigg] \,\mathrm{d}x. \end{aligned} \tag{3.2}$$

In fact, if $U(t,\varphi)$ is a solution of system (1.1), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} F\big(U(t,\varphi)\big) &= \left\langle DF(U), \frac{\mathrm{d}U}{\mathrm{d}t} \right\rangle_H \\ &= \big\langle DF(U), L(U) + G(U) \big\rangle_H \\ &= \bigg\langle -\mu_2 \triangle u + \lambda_2 u + b_1 u^2 + b_2 u^3 + \frac{1}{2}\alpha_3 |\psi|^2, \\ &\quad - \mu_2 \triangle^2 u + \triangle \bigg(\lambda_2 u + b_1 u^2 + b_2 u^3 + \frac{1}{2}\alpha_3 |\psi|^2 \bigg) \bigg\rangle_{L^2} \\ &\quad + \big\langle -\mu_1 \triangle \psi + \lambda_1 \psi + \alpha_2 |\psi|^2 \psi + \alpha_3 \psi u, \\ &\quad \mu_1 \triangle \psi - \lambda_1 \psi - \alpha_2 |\psi|^2 \psi - \alpha_3 \psi u \big\rangle_{L^2} \\ &= \int_{\Omega} \bigg[\mu_2 \triangle u - \lambda_2 u - b_1 u^2 - b_2 u^3 - \frac{1}{2}\alpha_3 |\psi|^2 \bigg] \\ &\quad \times \triangle \bigg(\mu_2 \triangle u - \lambda_2 u - b_1 u^2 - b_2 u^3 - \frac{1}{2}\alpha_3 |\psi|^2 \bigg) \,\mathrm{d}x \\ &\quad - \int_{\Omega} \big| \mu_1 \triangle \psi - \lambda_1 \psi - \alpha_2 |\psi|^2 \psi - \alpha_3 \psi u \big|^2 \,\mathrm{d}x \\ &= - \bigg\| \mu_2 \triangle u - \lambda_2 u - b_1 u^2 - b_2 u^3 - \frac{1}{2}\alpha_3 |\psi|^2 \bigg\|_{H^1}^2 \\ &\quad - \big\| \mu_1 \triangle \psi - \lambda_1 \psi - \alpha_2 |\psi|^2 \psi - \alpha_3 \psi u \big\|_{L^2}^2 \\ &=: - \big\| DF(U) \big\|_{\hat{H}_{\frac{1}{4}}}^2, \end{aligned} \tag{3.3}$$

where $\hat{H}_{\frac{1}{4}} = H^1(\Omega) \times L^2(\Omega, \mathbb{C})$. Hence (3.2) is a Lyapunov function.

Integrating (3.3) from 0 to t gives

$$F\big(U(t,\varphi)\big) = -\int_0^t \big\| DF(U) \big\|_{\hat{H}_{\frac{1}{4}}}^2 \,\mathrm{d}t + F(\varphi). \tag{3.4}$$

Using (1.2) and (3.2), we have

$$\begin{aligned}
F(U) &= \frac{1}{2}\int_\Omega\bigg[\mu_1|\nabla\psi|^2+\mu_2|\nabla u|^2+\lambda_1|\psi|^2+\lambda_2u^2\\
&\quad+\frac{1}{2}\alpha_2|\psi|^4+\frac{1}{2}b_2u^4+\frac{2}{3}b_1u^3+\alpha_3|\psi|^2u\bigg]\,\mathrm{d}x\\
&\geq\frac{1}{2}\int_\Omega\bigg[\mu_1|\nabla\psi|^2+\mu_2|\nabla u|^2-|\lambda_1||\psi|^2-|\lambda_2||u|^2\\
&\quad+\frac{1}{2}\alpha_2|\psi|^4+\frac{1}{2}b_2|u|^4-\frac{2}{3}|b_1||u|^3-\alpha_3|\psi|^2|u|\bigg]\,\mathrm{d}x\\
&\geq\frac{1}{2}\int_\Omega\bigg[\mu_1|\nabla\psi|^2+\mu_2|\nabla u|^2-\varepsilon_1|\psi|^4-\frac{|\lambda_1|^2}{\varepsilon_1}\\
&\quad-\varepsilon_1|u|^4-\frac{|\lambda_2|^2}{\varepsilon_1}+\frac{1}{2}\alpha_2|\psi|^4+\frac{1}{2}b_2|u|^4-\varepsilon_1|u|^4\\
&\quad-\varepsilon_1^{-3}\bigg(\frac{2}{3}|b_1|\bigg)^4-\varepsilon_1|u|^4-\varepsilon_1|\psi|^4-\varepsilon_1^{-3}\alpha_3^4\bigg]\,\mathrm{d}x\\
&=\frac{1}{2}\int_\Omega\bigg\{\mu_1|\nabla\psi|^2+\mu_2|\nabla u|^2+\bigg(\frac{1}{2}b_2-3\varepsilon_1\bigg)|u|^4+\bigg(\frac{1}{2}\alpha_2-2\varepsilon_1\bigg)|\psi|^4\\
&\quad-\bigg[\frac{|\lambda_1|^2}{\varepsilon_1}+\frac{|\lambda_2|^2}{\varepsilon_1}+\varepsilon_1^{-3}\bigg(\frac{2}{3}|b_1|\bigg)^4+\varepsilon_1^{-3}\alpha_3^4\bigg]\bigg\}\,\mathrm{d}x.
\end{aligned}$$

Choosing $\varepsilon_1>0$ such $\frac{1}{2}b_2-3\varepsilon_1>0$, $\frac{1}{2}\alpha_2-2\varepsilon_1>0$, we get

$$F(U)\geq C_1\int_\Omega\big[\big(|\nabla u|^2+|u|^4\big)+\big(|\nabla\psi|^2+|\psi|^4\big)\big]\,\mathrm{d}x-C_2.$$

Combining with (3.4) yields

$$\begin{aligned}
&C_1\int_\Omega\big[\big(|\nabla u|^2+|u|^4\big)+\big(|\nabla\psi|^2+|\psi|^4\big)\big]\,\mathrm{d}x-C_2\leq-\int_0^t\big\|DF(U)\big\|^2_{\hat{H}_{\frac{1}{4}}}\,\mathrm{d}t+F(\varphi),\\
&C_1\int_\Omega\big[\big(|\nabla u|^2+|u|^4\big)+\big(|\nabla\psi|^2+|\psi|^4\big)\big]\,\mathrm{d}x+\int_0^t\big\|DF(U)\big\|^2_{\hat{H}_{\frac{1}{4}}}\,\mathrm{d}t\leq F(\varphi)+C_2,\\
&\int_\Omega\big[\big(|\nabla u|^2+|u|^4\big)+\big(|\nabla\psi|^2+|\psi|^4\big)\big]\,\mathrm{d}x\leq C,
\end{aligned}$$

which implies

$$\|u\|_{H^1}+\|\psi\|_{H^1}\leq C,\quad\forall t\geq0,\varphi\in W\subset H_{\frac{1}{2}},\tag{3.5}$$

where C_1, C_2, and C are positive constants. C only depends on φ.

Set

$$\begin{aligned}
\tilde{G}(U)&=\triangle\bigg(\lambda_2u+b_1u^2+b_2u^3+\frac{1}{2}\alpha_3|\psi|^2\bigg),\\
\bar{G}(U)&=-\lambda_1\psi-\alpha_2|\psi|^2\psi-\alpha_3\psi u,\\
g(U)&=\lambda_2u+b_1u^2+b_2u^3+\frac{1}{2}\alpha_3|\psi|^2.
\end{aligned}$$

Next, we show that for any bounded set $W \subset H_{\frac{1}{2}}$ there exists $C > 0$ such that

$$\|u(t,\varphi)\|_{\tilde{H}_\alpha} \le C, \qquad \|\psi(t,\varphi)\|_{\bar{H}_{2\alpha}} \le C, \quad \forall t > 0, \varphi \in W, \frac{1}{4} \le \alpha < \frac{1}{2}. \tag{3.6}$$

We claim that $g : H^1(\Omega) \times H^1(\Omega,\mathbb{C}) \to L^2(\Omega)$ is bounded. By

$$H^1(\Omega) \hookrightarrow L^6(\Omega),$$

we have

$$\begin{aligned}\|g(U)\|_{L^2}^2 &= \int_\Omega \left|\lambda_2 u + b_1 u^2 + b_2 u^3 + \frac{1}{2}\alpha_3|\psi|^2\right|^2 \mathrm{d}x \\ &\le C\left(\int_\Omega |u|^6 \,\mathrm{d}x + \int_\Omega |\psi|^6 \,\mathrm{d}x + 1\right) \le C\left(\|u\|_{H^1}^6 + \|\psi\|_{H^1}^6 + 1\right),\end{aligned}$$

which implies that $g : H^1(\Omega) \times H^1(\Omega,\mathbb{C}) \to L^2(\Omega)$ is bounded. Hence, by (3.5) and Lemma 2.2 we deduce that

$$\begin{aligned}\|u(t,\varphi)\|_{\tilde{H}_\alpha} &= \left\| e^{t\tilde{L}}u_0 + \int_0^t e^{(t-\tau)\tilde{L}}\tilde{G}(U)\,\mathrm{d}\tau\right\|_{\tilde{H}_\alpha} \\ &\le \|u_0\|_{\tilde{H}_\alpha} + \int_0^t \left\|(-\tilde{L})^{\alpha+\frac{1}{2}}e^{(t-\tau)\tilde{L}}g(U)\right\|_{L^2}\mathrm{d}\tau \\ &\le \|u_0\|_{\tilde{H}_{\frac{1}{2}}} + \int_0^t \left\|(-\tilde{L})^{\alpha+\frac{1}{2}}e^{(t-\tau)\tilde{L}}\right\| \cdot \left\|g(U)\right\|_{L^2}\mathrm{d}\tau \\ &\le \|\varphi\|_{H_{\frac{1}{2}}} + C\int_0^t \tau^{-\beta}e^{-\delta\tau}\,\mathrm{d}\tau \\ &\le C, \quad \forall t \ge 0, \varphi \in W \subset H_{\frac{1}{2}}, \frac{1}{4} \le \alpha < \frac{1}{2},\end{aligned}$$

where $\beta = \alpha + \frac{1}{2}$ $(0 < \beta < 1)$.

We claim that $\bar{G} : H^1(\Omega) \times H^1(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ is bounded. By

$$H^1(\Omega,\mathbb{C}) \hookrightarrow L^6(\Omega,\mathbb{C}),$$

we have

$$\begin{aligned}\|\bar{G}(U)\|_{L^2}^2 &= \int_\Omega \left|-\lambda_1\psi - \alpha_2|\psi|^2\psi - \alpha_3\psi u\right|^2 \mathrm{d}x \\ &\le C\left(\int_\Omega |u|^6 \,\mathrm{d}x + \int_\Omega |\psi|^6 \,\mathrm{d}x + 1\right) \le C\left(\|u\|_{H^1}^6 + \|\psi\|_{H^1}^6 + 1\right),\end{aligned}$$

which implies that $\bar{G} : H^1(\Omega) \times H^1(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ is bounded. Hence, by (3.5) and Lemma 2.2 we deduce that

$$\begin{aligned}\|\psi(t,\varphi)\|_{\bar{H}_{2\alpha}} &\le \left\| e^{t\bar{L}}\psi_0 + \int_0^t e^{(t-\tau)\bar{L}}\bar{G}(U)\,\mathrm{d}\tau\right\|_{\bar{H}_{2\alpha}} \\ &\le \|\psi_0\|_{\bar{H}_{2\alpha}} + \int_0^t \left\|(-\bar{L})^{2\alpha}e^{(t-\tau)\bar{L}}\bar{G}(U)\right\|_{L^2}\mathrm{d}\tau\end{aligned}$$

$$\begin{aligned}
&\leq \|\psi_0\|_{\bar{H}_1} + \int_0^t \left\|(-\bar{L})^{2\alpha} e^{(t-\tau)\bar{L}}\right\| \cdot \left\|\bar{G}(U)\right\|_{L^2} \,\mathrm{d}\tau \\
&\leq \|\varphi\|_{H_{\frac{1}{2}}} + C\int_0^t \tau^{-2\alpha} e^{-\delta\tau} \,\mathrm{d}\tau \\
&\leq C, \quad \forall t \geq 0, \varphi \in W \subset H_{\frac{1}{2}}, \frac{1}{4} \leq \alpha < \frac{1}{2},
\end{aligned}$$

which implies that

$$\left\|\psi(t,\varphi)\right\|_{\bar{H}_{2\alpha}} \leq C, \quad \forall t > 0, \varphi \in W, \frac{1}{4} \leq \alpha < \frac{1}{2}.$$

Hence, (3.6) holds.

Step 2. We prove that for any bounded set $W \subset H_\alpha$ ($1 \leq \alpha < \frac{3}{2}$) there is a constant $C > 0$ such that

$$\left\|u(t,\varphi)\right\|_{\tilde{H}_\alpha} \leq C, \qquad \left\|\psi(t,\varphi)\right\|_{\bar{H}_{2\alpha}} \leq C, \quad \forall t > 0, \varphi \in W, \alpha < \frac{3}{4}. \tag{3.7}$$

In fact, by the embedding theorems of fractional order spaces (see Pazy [12]):

$$\tilde{H}_\alpha \hookrightarrow C^0(\Omega) \cap H^1(\Omega), \qquad \bar{H}_{2\alpha} \hookrightarrow C^0(\Omega,\mathbb{C}) \cap H^1(\Omega,\mathbb{C}), \quad \text{as } \alpha > \frac{3}{8},$$

we have

$$\begin{aligned}
\left\|g(U)\right\|^2_{\tilde{H}_{\frac{1}{4}}} &= \int_\Omega \left|\nabla\left(\lambda_2 u + b_1 u^2 + b_2 u^3 + \frac{1}{2}\alpha_3|\psi|^2\right)\right|^2 \mathrm{d}x \\
&= \int_\Omega \left|\lambda_2 \nabla u + 2b_1 u \nabla u + 3b_2 u^2 \nabla u + \alpha_3 \psi_1 \nabla \psi_1 + \alpha_3 \psi_2 \nabla \psi_2\right|^2 \mathrm{d}x \\
&\leq C\int_\Omega \left[\left(|u|^4 + 1\right)|\nabla u|^2 + |\psi_1|^2|\nabla\psi_1|^2 + |\psi_2|^2|\nabla\psi_2|^2\right] \mathrm{d}x \\
&\leq C\int_\Omega \left[\left(\sup_{x\in\Omega}|u|^4 + 1\right)|\nabla u|^2 + \left(\sup_{x\in\Omega}|\psi_1|^2\right)|\nabla\psi_1|^2 + \left(\sup_{x\in\Omega}|\psi_2|^2\right)|\nabla\psi_2|^2\right] \mathrm{d}x \\
&\leq C\left[\left(\|u\|^4_{\tilde{H}_\alpha} + 1\right)\|u\|^2_{\tilde{H}_\alpha} + \|\psi_1\|^4_{\bar{H}_\alpha} + \|\psi_2\|^4_{\bar{H}_\alpha}\right],
\end{aligned}$$

where $\psi = \psi_1 + i\psi_2$. Hence for $\alpha > \frac{3}{8}$ we see that

$$g : \tilde{H}_\alpha \times \bar{H}_{2\alpha} \to \tilde{H}_{\frac{1}{4}} \quad \text{is bounded.} \tag{3.8}$$

Therefore, it follows from (3.6) and (3.8) that

$$\left\|g\big(U(t,\varphi)\big)\right\|_{\tilde{H}_{\frac{1}{4}}} < C, \quad \forall t \geq 0, \varphi \in W, \frac{3}{8} < \alpha < \frac{1}{2}. \tag{3.9}$$

Then, by using the same method as that in Step 1, we get from (3.9)

$$\begin{aligned}
\left\|u(t,\varphi)\right\|_{\tilde{H}_\alpha} &= \left\|e^{t\tilde{L}} u_0 - \int_0^t (-\tilde{L})^{\frac{1}{2}} e^{(t-\tau)\tilde{L}} g(U) \,\mathrm{d}\tau\right\|_{\tilde{H}_\alpha} \\
&\leq \|u_0\|_{\tilde{H}_\alpha} + \int_0^t \left\|(-\tilde{L})^{\alpha+\frac{1}{2}} e^{(t-\tau)\tilde{L}} g(U)\right\|_{L^2} \mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq \|u_0\|_{\tilde{H}_\alpha} + \int_0^t \big\|(-\tilde{L})^{\alpha+\frac{1}{4}} e^{(t-\tau)\tilde{L}}\big\| \cdot \big\|g(U)\big\|_{\tilde{H}_{\frac{1}{4}}} \,\mathrm{d}\tau \\
&\leq \|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta\tau}\,\mathrm{d}\tau \\
&\leq C, \quad \forall t \geq 0, \varphi \in W, \frac{1}{2} \leq \alpha < \frac{3}{4},
\end{aligned}
$$

where $\beta = \alpha + \frac{1}{4}$ $(0 < \beta < 1)$.

We also have

$$
\begin{aligned}
\big\|\bar{G}(U)\big\|^2_{\bar{H}_{\frac{1}{2}}} &= \int_\Omega \big|\nabla\big(-\lambda_1\psi - \alpha_2|\psi|^2\psi - \alpha_3\psi u\big)\big|^2\,\mathrm{d}x \\
&= \int_\Omega \big|-\lambda_1\nabla\psi - \alpha_2\nabla\big(|\psi|^2\psi\big) - \alpha_3\nabla(\psi u)\big|^2\,\mathrm{d}x \\
&= \int_\Omega \big|-\lambda_1\nabla\psi - \alpha_2\big(\psi_1^2\nabla\psi + 2\psi_1\psi\nabla\psi_1 + \psi_2^2\nabla\psi \\
&\quad + 2\psi_2\psi\nabla\psi_2\big) - \alpha_3(\psi\nabla u + u\nabla\psi)\big|^2\,\mathrm{d}x \\
&\leq C\int_\Omega \big[|\nabla\psi|^2 + |\psi_1|^4|\nabla\psi|^2 + |\psi_1\psi|^2|\nabla\psi_1|^2 + |\psi_2|^4|\nabla\psi|^2 \\
&\quad + |\psi_2\psi|^2|\nabla\psi_2|^2 + |\psi|^2|\nabla u|^2 + |u|^2|\nabla\psi|^2\big]\,\mathrm{d}x \\
&\leq C\int_\Omega \Big[|\nabla\psi|^2 + \Big(\sup_\Omega|\psi_1|^4\Big)|\nabla\psi|^2 + \Big(\sup_\Omega|\psi_1\psi|^2\Big)|\nabla\psi_1|^2 \\
&\quad + \Big(\sup_\Omega|\psi_2|^4\Big)|\nabla\psi|^2 + \Big(\sup_\Omega|\psi_2\psi|^2\Big)|\nabla\psi_2|^2 \\
&\quad + \Big(\sup_\Omega|\psi|^2\Big)|\nabla u|^2 + \Big(\sup_\Omega|u|^2\Big)|\nabla\psi|^2\Big]\,\mathrm{d}x \\
&\leq C\big[\|\psi\|^2_{H^1} + \|\psi_1\|^4_{\tilde{H}_\alpha}\|\psi\|^2_{H^1} + \|\psi_1\|^2_{\tilde{H}_\alpha}\|\psi\|^2_{\bar{H}_{2\alpha}}\|\psi_1\|^2_{H^1} \\
&\quad + \|\psi_2\|^4_{\tilde{H}_\alpha}\|\psi\|^2_{H^1} + \|\psi_2\|^2_{\tilde{H}_\alpha}\|\psi\|^2_{\bar{H}_{2\alpha}}\|\psi_2\|^2_{H^1} \\
&\quad + \|\psi\|^2_{\bar{H}_{2\alpha}}\|u\|^2_{H^1} + \|u\|^2_{\tilde{H}_\alpha}\|\psi\|^2_{H^1}\big] \\
&\leq C\big[\|\psi\|^2_{\bar{H}_{2\alpha}} + \|\psi_1\|^4_{\tilde{H}_\alpha}\|\psi\|^2_{\bar{H}_{2\alpha}} + \|\psi_2\|^4_{\tilde{H}_\alpha}\|\psi\|^2_{\bar{H}_{2\alpha}} + \|u\|^2_{\tilde{H}_\alpha}\|\psi\|^2_{\bar{H}_{2\alpha}}\big],
\end{aligned}
$$

which implies that

$$
\bar{G} : \tilde{H}_\alpha \times \bar{H}_{2\alpha} \to \bar{H}_{\frac{1}{2}} \quad \text{is bounded for } \alpha > \frac{3}{8}. \tag{3.10}
$$

Therefore, it follows from (3.6) and (3.10) that

$$
\big\|\bar{G}\big(U(t,\varphi)\big)\big\|_{\bar{H}_{\frac{1}{2}}} < C, \quad \forall t \geq 0, \varphi \in W, \frac{3}{8} < \alpha < \frac{1}{2}. \tag{3.11}
$$

Then, by using the same method as that in Step 1, we get from (3.11)

$$
\begin{aligned}
\big\|\psi(t,\varphi)\big\|_{\bar{H}_{2\alpha}} &= \bigg\|e^{t\bar{L}}\psi_0 + \int_0^t e^{(t-\tau)\bar{L}}\bar{G}(U)\,\mathrm{d}\tau\bigg\|_{\bar{H}_{2\alpha}} \\
&\leq \|\psi_0\|_{\bar{H}_{2\alpha}} + \int_0^t \big\|(-\bar{L})^{2\alpha-\frac{1}{2}} e^{(t-\tau)\bar{L}}(-\bar{L})^{\frac{1}{2}}\bar{G}(U)\big\|_{L^2}\,\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\psi_0\|_{\bar{H}_{2\alpha}} + \int_0^t \big\|(-\bar{L})^{2\alpha-\frac{1}{2}} e^{(t-\tau)\bar{L}}\big\| \cdot \big\|\bar{G}(U)\big\|_{\bar{H}_{\frac{1}{2}}} \,\mathrm{d}\tau \\
&\leq \|\varphi\|_{H_{2\alpha}} + C\int_0^t \tau^{-2(\alpha-\frac{1}{4})} e^{-\delta\tau} \,\mathrm{d}\tau \\
&\leq C, \quad \forall t \geq 0, \varphi \in W, \frac{1}{2} \leq \alpha < \frac{3}{4},
\end{aligned}
$$

which implies

$$
\big\|\psi(t,\varphi)\big\|_{\bar{H}_{2\alpha}} \leq C, \quad \forall t \geq 0, \varphi \in W, \frac{1}{2} \leq \alpha < \frac{3}{4}.
$$

Hence, (3.7) holds.

Step 3. We prove that for any bounded set $W \subset H_\alpha$ ($\frac{3}{2} \leq \alpha < 2$) there is a constant $C > 0$ such that

$$
\big\|u(t,\varphi)\big\|_{\tilde{H}_\alpha} \leq C, \qquad \big\|\psi(t,\varphi)\big\|_{\bar{H}_{2\alpha}} \leq C, \quad \forall t > 0, \varphi \in W, \frac{3}{4} \leq \alpha < 1. \tag{3.12}
$$

In fact, by the embedding theorems of fractional order spaces (see Pazy [12]):

$$
\begin{aligned}
&H^2(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow W^{1,4}(\Omega), \\
&\tilde{H}_\alpha \hookrightarrow C^0(\Omega) \cap H^2(\Omega), \qquad \bar{H}_{2\alpha} \hookrightarrow C^0(\Omega,\mathbb{C}) \cap H^2(\Omega,\mathbb{C}), \quad \text{as } \alpha \geq \frac{1}{2},
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
\big\|g(U)\big\|^2_{\tilde{H}_{\frac{1}{2}}} &= \int_\Omega \left|\triangle\left(\lambda_2 u + b_1 u^2 + b_2 u^3 + \frac{1}{2}\alpha_3|\psi|^2\right)\right|^2 \mathrm{d}x \\
&= \int_\Omega \big|\lambda_2 \triangle u + b_1\big(2u\triangle u + 2|\nabla u|^2\big) + b_2\big(3u^2\triangle u + 6u|\nabla u|^2\big) \\
&\quad + \alpha_3\big(\psi_1\triangle\psi_1 + |\nabla\psi_1|^2 + \psi_2\triangle\psi_2 + |\nabla\psi_2|^2\big)\big|^2 \,\mathrm{d}x \\
&\leq \int_\Omega \big\{|\lambda_2||\triangle u| + |b_1|\big[\big(u^2+4\big)|\triangle u| + 2|\nabla u|^2\big] \\
&\quad + b_2\big(3u^2|\triangle u| + 6|u||\nabla u|^2\big) + \alpha_3\big(|\psi_1||\triangle\psi_1| \\
&\quad + |\nabla\psi_1|^2 + |\psi_2||\triangle\psi_2| + |\nabla\psi_2|^2\big)\big\}^2 \,\mathrm{d}x \\
&\leq C\int_\Omega \big(|\triangle u| + u^2|\triangle u| + |\nabla u|^2 + |u||\nabla u|^2 \\
&\quad + |\psi_1||\triangle\psi_1| + |\nabla\psi_1|^2 + |\psi_2||\triangle\psi_2| + |\nabla\psi_2|^2\big)^2 \,\mathrm{d}x \\
&\leq C\int_\Omega \big(|\triangle u|^2 + u^4|\triangle u|^2 + |\nabla u|^4 + |u|^2|\nabla u|^4 \\
&\quad + |\psi_1|^2|\triangle\psi_1|^2 + |\nabla\psi_1|^4 + |\psi_2|^2|\triangle\psi_2|^2 + |\nabla\psi_2|^4\big) \,\mathrm{d}x \\
&\leq C\big(\|u\|^2_{H^2} + \|u\|^4_{\tilde{H}_\alpha}\|u\|^2_{H^2} + \|u\|^4_{W^{1,4}} + \|u\|^2_{\tilde{H}_\alpha}\|u\|^4_{W^{1,4}} \\
&\quad + \|\psi_1\|^2_{\tilde{H}_\alpha}\|\psi_1\|^2_{H^2} + \|\psi_1\|^4_{W^{1,4}} + \|\psi_2\|^2_{\tilde{H}_\alpha}\|\psi_2\|^2_{H^2} + \|\psi_2\|^4_{W^{1,4}}\big) \\
&\leq C\big(\|u\|^6_{\tilde{H}_\alpha} + \|u\|^4_{\tilde{H}_\alpha} + \|u\|^2_{\tilde{H}_\alpha} + \|\psi_1\|^4_{\tilde{H}_\alpha} + \|\psi_2\|^4_{\tilde{H}_\alpha}\big),
\end{aligned}
$$

where $\psi = \psi_1 + i\psi_2$. Hence for $\alpha \geq \frac{1}{2}$ we see that

$$g : \tilde{H}_\alpha \times \bar{H}_{2\alpha} \to \tilde{H}_{\frac{1}{2}} \quad \text{is bounded.} \tag{3.13}$$

Therefore, it follows from (3.7) and (3.13) that

$$\|g(U(t,\varphi))\|_{\tilde{H}_{\frac{1}{2}}} < C, \quad \forall t \geq 0, \varphi \in W, \frac{1}{2} \leq \alpha < \frac{3}{4}. \tag{3.14}$$

Then, by using the same method as that in Step 2, we get from (3.14)

$$\begin{aligned}
\|u(t,\varphi)\|_{\tilde{H}_\alpha} &= \left\| e^{t\tilde{L}}u_0 - \int_0^t (-\tilde{L})^{\frac{1}{2}} e^{(t-\tau)\tilde{L}} g(U)\,\mathrm{d}\tau \right\|_{\tilde{H}_\alpha} \\
&\leq \|u_0\|_{\tilde{H}_\alpha} + \int_0^t \left\|(-\tilde{L})^{\alpha+\frac{1}{2}} e^{(t-\tau)\tilde{L}} g(U)\right\|_{L^2} \mathrm{d}\tau \\
&\leq \|u_0\|_{\tilde{H}_\alpha} + \int_0^t \left\|(-\tilde{L})^{\alpha} e^{(t-\tau)\tilde{L}}\right\| \cdot \|g(U)\|_{\tilde{H}_{\frac{1}{2}}} \mathrm{d}\tau \\
&\leq \|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-\alpha} e^{-\delta\tau}\,\mathrm{d}\tau \\
&\leq C, \quad \forall t \geq 0, \varphi \in W, \frac{3}{4} \leq \alpha < 1.
\end{aligned}$$

We also have

$$\begin{aligned}
\|\bar{G}(U)\|^2_{\bar{H}_1} &= \int_\Omega \left|\triangle\left(-\lambda_1\psi - \alpha_2|\psi|^2\psi - \alpha_3\psi u\right)\right|^2 \mathrm{d}x \\
&= \int_\Omega \Big|-\lambda_1\triangle\psi - \alpha_2\big(\psi_1^2\triangle\psi + 2\psi_1\psi\triangle\psi_1 + 2\psi|\nabla\psi_1|^2 \\
&\quad + 4\psi_1\nabla\psi\nabla\psi_1 + \psi_2^2\triangle\psi + 2\psi_2\psi\triangle\psi_2 + 2\psi|\nabla\psi_2|^2 \\
&\quad + 4\psi_2\nabla\psi\nabla\psi_2\big) - \alpha_3(\psi\triangle u + u\triangle\psi + 2\nabla\psi\nabla u)\Big|^2 \mathrm{d}x \\
&\leq C\int_\Omega \big(|\triangle\psi| + \psi_1^2|\triangle\psi| + |\psi_1||\psi||\triangle\psi_1| + |\psi||\nabla\psi_1|^2 \\
&\quad + |\psi_1||\nabla\psi||\nabla\psi_1| + \psi_2^2|\triangle\psi| + |\psi_2||\psi||\triangle\psi_2| + |\psi||\nabla\psi_2|^2 \\
&\quad + |\psi_2||\nabla\psi||\nabla\psi_2| + |\psi||\triangle u| + |u||\triangle\psi| + |\nabla\psi||\nabla u|\big)^2 \mathrm{d}x \\
&\leq C\int_\Omega \big(|\triangle\psi|^2 + \psi_1^4|\triangle\psi|^2 + \psi_1^2|\psi|^2|\triangle\psi_1|^2 + |\psi|^2|\nabla\psi_1|^4 \\
&\quad + \psi_1^2|\nabla\psi|^2|\nabla\psi_1|^2 + \psi_2^4|\triangle\psi|^2 + \psi_2^2|\psi|^2|\triangle\psi_2|^2 \\
&\quad + |\psi|^2|\nabla\psi_2|^4 + \psi_2^2|\nabla\psi|^2|\nabla\psi_2|^2 + |\psi|^2|\triangle u|^2 \\
&\quad + u^2|\triangle\psi|^2 + |\nabla\psi|^2|\nabla u|^2\big)\,\mathrm{d}x \\
&\leq C\big(\|\psi\|^2_{H^2} + \|\psi_1\|^4_{\tilde{H}_\alpha}\|\psi\|^2_{H^2} + \|\psi_1\|^2_{\tilde{H}_\alpha}\|\psi\|^2_{\bar{H}_{2\alpha}}\|\psi_1\|^2_{H^2} \\
&\quad + \|\psi\|^2_{\bar{H}_{2\alpha}}\|\psi_1\|^4_{W^{1,4}} + \|\psi_1\|^2_{\tilde{H}_\alpha}\|\psi\|^2_{H^1}\|\psi_1\|^2_{H^1} \\
&\quad + \|\psi_2\|^4_{\tilde{H}_\alpha}\|\psi\|^2_{H^2} + \|\psi_2\|^2_{\tilde{H}_\alpha}\|\psi\|^2_{\bar{H}_{2\alpha}}\|\psi_2\|^2_{H^2} \\
&\quad + \|\psi\|^2_{\bar{H}_{2\alpha}}\|\psi_2\|^4_{W^{1,4}} + \|\psi_2\|^2_{\tilde{H}_\alpha}\|\psi\|^2_{H^1}\|\psi_2\|^2_{H^1}
\end{aligned}$$

$$
\begin{aligned}
&+\|\psi\|_{\bar{H}_{2\alpha}}^2\|u\|_{H^2}^2+\|u\|_{\tilde{H}_\alpha}^2\|\psi\|_{H^2}^2+\|\psi\|_{H^1}^2\|u\|_{H^1}^2\big)\\
&\le C\big(\|\psi\|_{\bar{H}_{2\alpha}}^2+\|\psi_1\|_{\tilde{H}_\alpha}^4\|\psi\|_{\bar{H}_{2\alpha}}^2+\|\psi_2\|_{\tilde{H}_\alpha}^4\|\psi\|_{\bar{H}_{2\alpha}}^2+\|\psi\|_{\bar{H}_{2\alpha}}^2\|u\|_{\tilde{H}_\alpha}^2\big),
\end{aligned}
$$

which implies that

$$\bar{G}:\tilde{H}_\alpha\times\bar{H}_{2\alpha}\to\bar{H}_1\quad\text{is bounded for }\alpha\ge\frac{1}{2}.\tag{3.15}$$

Therefore, it follows from (3.7) and (3.15) that

$$\big\|\bar{G}\big(U(t,\varphi)\big)\big\|_{\bar{H}_1}<C,\quad\forall t\ge 0,\varphi\in W,\frac{1}{2}\le\alpha<\frac{3}{4}.\tag{3.16}$$

Then, by using the same method as that in Step 2, we get from (3.16)

$$
\begin{aligned}
\big\|\psi(t,\varphi)\big\|_{\bar{H}_{2\alpha}}&=\left\|e^{t\bar{L}}\psi_0+\int_0^t e^{(t-\tau)\bar{L}}\bar{G}(U)\,\mathrm{d}\tau\right\|_{\bar{H}_{2\alpha}}\\
&\le\|\psi_0\|_{\bar{H}_{2\alpha}}+\int_0^t\big\|(-\bar{L})^{2\alpha}e^{(t-\tau)\bar{L}}\bar{G}(U)\big\|_{L^2}\,\mathrm{d}\tau\\
&=\|\psi_0\|_{\bar{H}_{2\alpha}}+\int_0^t\big\|(-\bar{L})^{2\alpha-1}e^{(t-\tau)\bar{L}}(-\bar{L})^1\bar{G}(U)\big\|_{L^2}\,\mathrm{d}\tau\\
&\le\|\psi_0\|_{\bar{H}_{2\alpha}}+\int_0^t\big\|(-\bar{L})^{2\alpha-1}e^{(t-\tau)\bar{L}}\big\|\cdot\big\|\bar{G}(U)\big\|_{\bar{H}_1}\,\mathrm{d}\tau\\
&\le\|\varphi\|_{H_{2\alpha}}+C\int_0^t\tau^{-2(\alpha-\frac{1}{2})}e^{-\delta\tau}\,\mathrm{d}\tau\\
&\le C,\quad\forall t\ge 0,\varphi\in W\subset H_\alpha,\frac{3}{4}\le\alpha<1,
\end{aligned}
$$

which implies

$$\big\|\psi(t,\varphi)\big\|_{\bar{H}_{2\alpha}}\le C,\quad\forall t\ge 0,\varphi\in W,\frac{3}{4}\le\alpha<1.$$

Hence, (3.12) holds.

Step 4. We prove that for any bounded set $W\subset H_\alpha$ ($\alpha\ge 0$) there is a constant $C>0$ such that

$$\big\|u(t,\varphi)\big\|_{\tilde{H}_\alpha}\le C,\qquad\big\|\psi(t,\varphi)\big\|_{\bar{H}_{2\alpha}}\le C,\quad\forall t>0,\varphi\in W,\alpha\ge 0.\tag{3.17}$$

In fact, by the embedding theorems of fractional order spaces (see Pazy [12]), we obtain

$$
\begin{aligned}
&H^3(\Omega)\hookrightarrow W^{2,6}(\Omega)\hookrightarrow W^{1,6}(\Omega),\\
&\tilde{H}_\alpha\hookrightarrow C^0(\Omega)\cap H^3(\Omega),\qquad\bar{H}_{2\alpha}\hookrightarrow C^0(\Omega,\mathbb{C})\cap H^3(\Omega,\mathbb{C}),\quad\text{as }\alpha\ge\frac{3}{4}.
\end{aligned}\tag{3.18}
$$

Hence, it follows from (3.18) that

$$
\begin{aligned}
&g:\tilde{H}_\alpha\times\bar{H}_{2\alpha}\to\tilde{H}_{\frac{3}{4}}\quad\text{is bounded for }\alpha\ge\frac{3}{4},\\
&\bar{G}:\tilde{H}_\alpha\times\bar{H}_{2\alpha}\to\bar{H}_{\frac{3}{2}}\quad\text{is bounded for }\alpha\ge\frac{3}{4}.
\end{aligned}
$$

Therefore, by (3.12) for any bounded set $W \subset H_\alpha$ ($2 \le \alpha < \frac{5}{2}$) we derive that

$$\begin{aligned} &\big\| g\big(U(t,\varphi)\big)\big\|_{\tilde{H}_{\frac{3}{4}}} \le C, \\ &\big\| \bar{G}\big(U(t,\varphi)\big)\big\|_{\bar{H}_{\frac{3}{2}}} \le C, \quad \forall t \ge 0, \varphi \in W, \frac{3}{4} \le \alpha < 1. \end{aligned} \tag{3.19}$$

Then, it follows from (3.19)

$$\begin{aligned} \big\|u(t,\varphi)\big\|_{\tilde{H}_\alpha} &= \left\| e^{t\tilde{L}}u_0 - \int_0^t (-\tilde{L})^{\frac{1}{2}} e^{(t-\tau)\tilde{L}} g(U)\,\mathrm{d}\tau \right\|_{\tilde{H}_\alpha} \\ &\le \|u_0\|_{\tilde{H}_\alpha} + \int_0^t \big\|(-\tilde{L})^{\alpha+\frac{1}{2}} e^{(t-\tau)\tilde{L}} g(U)\big\|_{L^2}\,\mathrm{d}\tau \\ &\le \|u_0\|_{\tilde{H}_\alpha} + \int_0^t \big\|(-\tilde{L})^{\alpha-\frac{1}{4}} e^{(t-\tau)\tilde{L}} (-\tilde{L})^{\frac{3}{4}} g(U)\big\|_{L^2}\,\mathrm{d}\tau \\ &\le \|u_0\|_{\tilde{H}_\alpha} + \int_0^t \big\|(-\tilde{L})^{\alpha-\frac{1}{4}} e^{(t-\tau)\tilde{L}}\big\| \cdot \big\|g(U)\big\|_{\tilde{H}_{\frac{3}{4}}}\,\mathrm{d}\tau \\ &\le \|\varphi\|_{H_\alpha} + C\int_0^t \tau^{-(\alpha-\frac{1}{4})} e^{-\delta\tau}\,\mathrm{d}\tau \\ &\le C, \quad \forall t \ge 0, \varphi \in W, 1 \le \alpha < \frac{5}{4}, \\ \big\|\psi(t,\varphi)\big\|_{\bar{H}_{2\alpha}} &= \left\| e^{t\bar{L}}\psi_0 + \int_0^t e^{(t-\tau)\bar{L}} \bar{G}(U)\,\mathrm{d}\tau \right\|_{\bar{H}_{2\alpha}} \\ &\le \|\psi_0\|_{\bar{H}_{2\alpha}} + \int_0^t \big\|(-\bar{L})^{2\alpha} e^{(t-\tau)\bar{L}} \bar{G}(U)\big\|_{L^2}\,\mathrm{d}\tau \\ &= \|\psi_0\|_{\bar{H}_{2\alpha}} + \int_0^t \big\|(-\bar{L})^{2\alpha-\frac{3}{2}} e^{(t-\tau)\bar{L}} (-\bar{L})^{\frac{3}{2}} \bar{G}(U)\big\|_{L^2}\,\mathrm{d}\tau \\ &\le \|\psi_0\|_{\bar{H}_{2\alpha}} + \int_0^t \big\|(-\bar{L})^{2\alpha-\frac{3}{2}} e^{(t-\tau)\bar{L}}\big\| \cdot \big\|\bar{G}(U)\big\|_{\bar{H}_{\frac{3}{2}}}\,\mathrm{d}\tau \\ &\le \|\varphi\|_{H_{2\alpha}} + C\int_0^t \tau^{-2(\alpha-\frac{3}{4})} e^{-\delta\tau}\,\mathrm{d}\tau \\ &\le C, \quad \forall t \ge 0, \varphi \in W, 1 \le \alpha < \frac{5}{4}. \end{aligned}$$

Hence, (3.17) is valid for $1 \le \alpha < \frac{5}{4}$.

By doing the same procedures as steps 1-3, we can prove that (3.17) holds for all $\alpha \ge 0$.

Step 5. We show that for any $\alpha \ge 0$, system (1.1) has a bounded absorbing set in H_α.

We first consider the case of $\alpha = \frac{1}{2}$. It is well known that (1.1) possesses a global attractor in H space, and the global attractor of these equations consists of equilibria with their stable and unstable manifolds. Thus, each trajectory has to converge to a critical point. From (3.17) and (3.3), we deduce that for any $\varphi \in H_{\frac{1}{2}}$ the solution $U(t,\varphi)$ of system (1.1) converges to a critical point of F. Hence, we only need to prove the following two properties:

(1) $F(U) \to \infty \Leftrightarrow \|U\|_{H_{\frac{1}{2}}}$;

(2) the set $S = \{U \in H_{\frac{1}{2}} \mid DF(U) = 0\}$ is bounded.

Property (1) is obviously true, we now prove (2) in the following. It is easy to check if $DF(U) = 0$, U is a solution of the following equations:

$$\begin{cases} -\mu_2\triangle u + g(U) = 0, \\ -\mu_1\triangle\psi - \bar{G}(U) = 0, \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \qquad \frac{\partial\psi}{\partial n}|_{\partial\Omega} = 0, \qquad \int_\Omega u\,\mathrm{d}x = 0. \end{cases} \tag{3.20}$$

Taking the scalar product of (3.20) with $(-\triangle u, \psi)$, then we derive that

$$\begin{aligned} 0 &= \langle -\mu_2\triangle u + g(U), -\triangle u\rangle_{L^2} + \langle -\mu_1\triangle\psi - \bar{G}(U), \psi\rangle_{L^2} \\ &= \int_\Omega \biggl[\mu_2|\triangle u|^2 + \biggl(\lambda_2 u + b_1u^2 + b_2u^3 + \frac{1}{2}\alpha_3|\psi|^2\biggr)(-\triangle u) \\ &\quad + \mu_1|\nabla\psi|^2 + \lambda_1|\psi|^2 + \alpha_2|\psi|^4 + \alpha_3|\psi|^2u\biggr]\,\mathrm{d}x \\ &= \int_\Omega \bigl[\mu_2|\triangle u|^2 + \lambda_2|\nabla u|^2 + 2b_1u|\nabla u|^2 + 3b_2u^2|\nabla u|^2 + \alpha_3\psi_1\nabla\psi_1\nabla u \\ &\quad + \alpha_3\psi_2\nabla\psi_2\nabla u + \mu_1|\nabla\psi|^2 + \lambda_1\psi^2 + \alpha_2|\psi|^4 + \alpha_3|\psi|^2u\bigr]\,\mathrm{d}x \\ &\geq \int_\Omega \bigl[\mu_2|\triangle u|^2 - |\lambda_2||\nabla u|^2 - 2|b_1||u||\nabla u|^2 + 3b_2u^2|\nabla u|^2 - \alpha_3|\psi_1||\nabla\psi_1\nabla u| \\ &\quad - \alpha_3|\psi_2||\nabla\psi_2\nabla u| + \mu_1|\nabla\psi|^2 - |\lambda_1||\psi|^2 + \alpha_2|\psi|^4 - |\alpha_3||\psi|^2|u|\bigr]\,\mathrm{d}x. \end{aligned}$$

Using the Hölder inequality and the above inequality, we have

$$\int_\Omega \bigl(|\triangle u|^2 + |\nabla\psi|^2\bigr)\,\mathrm{d}x \leq C,$$

where $C > 0$ is a constant. Thus, property (2) is proved.

Now, we show that system (1.1) has a bounded absorbing set in H_α for any $\alpha \geq \frac{1}{2}$, *i.e.*, for any bounded set $W \subset H_\alpha$ there are $T > 0$ and a constant $C > 0$ independent of φ such that

$$\bigl\|U(t,\varphi)\bigr\|_{H_\alpha} \leq C, \quad \forall t \geq T, \varphi \in W. \tag{3.21}$$

From the above discussion, we know that (3.21) holds as $\alpha = \frac{1}{2}$. By (3.1) we have

$$\begin{aligned} u(t,\varphi) &= e^{(t-T)\tilde{L}}u(T,\varphi) - \int_T^t (-\tilde{L})^{\frac{1}{2}} e^{(t-\tau)\tilde{L}} g(U)\,\mathrm{d}\tau, \\ \psi(t,\varphi) &= e^{(t-T)\bar{L}}\psi(T,\varphi) + \int_T^t e^{(t-\tau)\bar{L}}\bar{G}(U)\,\mathrm{d}\tau. \end{aligned} \tag{3.22}$$

Let $B \subset H_{\frac{1}{2}}$ be the bounded absorbing set of system (1.1), and $T_0 > 0$ such that

$$U(t,\varphi) \in B, \quad \forall t > T_0, \varphi \in W \subset H_\alpha \left(\alpha \geq \frac{1}{2}\right).$$

It is well known that

$$\bigl\|e^{t\tilde{L}}\bigr\| \leq Ce^{-t\tilde{\lambda}_1^2}, \qquad \bigl\|e^{t\bar{L}}\bigr\| \leq Ce^{-t\bar{\lambda}_1^2}, \tag{3.23}$$

where $\tilde{\lambda}_1, \bar{\lambda}_1 > 0$, is the first eigenvalue of the equation

$$\begin{cases} \triangle^2 u = \lambda u, \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \qquad \frac{\partial \triangle u}{\partial n}|_{\partial\Omega} = 0, \\ \int_\Omega u \,\mathrm{d}x = 0, \end{cases}$$

$$\begin{cases} -\triangle \psi = \lambda \psi, \\ \frac{\partial \psi}{\partial n}|_{\partial\Omega} = 0, \end{cases}$$

respectively. Hence, for any given $T > 0$ and $\varphi \in W \subset H_\alpha$ ($\alpha \geq \frac{1}{2}$) we have

$$\begin{aligned} &\|e^{(t-T)\tilde{L}} u(T,\varphi)\|_{\tilde{H}_\alpha} = \|(-\tilde{L})^\alpha e^{(t-T)\tilde{L}} u(T,\varphi)\|_{L^2} \to 0, \quad \text{as } t \to \infty, \\ &\|e^{(t-T)\bar{L}} \psi(T,\varphi)\|_{\bar{H}_\alpha} = \|(-\bar{L})^\alpha e^{(t-T)\bar{L}} \psi(T,\varphi)\|_{L^2} \to 0, \quad \text{as } t \to \infty. \end{aligned} \tag{3.24}$$

From (3.22), (3.23), and Lemma 2.2, for any $\frac{1}{2} \leq \alpha < 1$ we get

$$\begin{aligned} \|u(t,\varphi)\|_{\tilde{H}_\alpha} &\leq \|e^{(t-T_0)\tilde{L}} u(T_0,\varphi)\|_{\tilde{H}_\alpha} + \int_{T_0}^t \|(-\tilde{L})^{\alpha+\frac{1}{2}} e^{(t-\tau)\tilde{L}} g(U)\|_{L^2} \,\mathrm{d}\tau \\ &= \|e^{(t-T_0)\tilde{L}} u(T_0,\varphi)\|_{\tilde{H}_\alpha} + \int_{T_0}^t \|(-\tilde{L})^{\alpha} e^{(t-\tau)\tilde{L}} (-\tilde{L})^{\frac{1}{2}} g(U)\|_{L^2} \,\mathrm{d}\tau \\ &\leq \|e^{(t-T_0)\tilde{L}} u(T_0,\varphi)\|_{\tilde{H}_\alpha} + \int_{T_0}^t \|(-\tilde{L})^{\alpha} e^{(t-\tau)\tilde{L}}\| \cdot \|g(U)\|_{\tilde{H}_{\frac{1}{2}}} \,\mathrm{d}\tau \\ &\leq \|e^{(t-T_0)\tilde{L}} u(T_0,\varphi)\|_{\tilde{H}_\alpha} + C\int_{T_0}^t \tau^{-\alpha} e^{-\tilde{\lambda}_1 \tau} \,\mathrm{d}\tau, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \|\psi(t,\varphi)\|_{\bar{H}_\alpha} &\leq \|e^{(t-T_0)\bar{L}} \psi(T_0,\varphi)\|_{\bar{H}_\alpha} + \int_{T_0}^t \|(-\bar{L})^{\alpha} e^{(t-\tau)\bar{L}} \bar{G}(U)\|_{L^2} \,\mathrm{d}\tau \\ &\leq \|e^{(t-T_0)\bar{L}} \psi(T_0,\varphi)\|_{\bar{H}_\alpha} + \int_{T_0}^t \|(-\bar{L})^{\alpha} e^{(t-\tau)\bar{L}}\| \cdot \|\bar{G}(U)\|_{L^2} \,\mathrm{d}\tau \\ &\leq \|e^{(t-T_0)\bar{L}} \psi(T_0,\varphi)\|_{\bar{H}_\alpha} + C\int_{T_0}^t \|(-\bar{L})^{\alpha} e^{(t-\tau)\bar{L}}\| \cdot \|\bar{G}(U)\|_{\bar{H}_1} \,\mathrm{d}\tau \\ &\leq \|e^{(t-T_0)\bar{L}} \psi(T_0,\varphi)\|_{\bar{H}_\alpha} + C\int_{T_0}^t \tau^{-\alpha} e^{-\bar{\lambda}_1 \tau} \,\mathrm{d}\tau, \end{aligned} \tag{3.26}$$

where $C > 0$ is a constant independent of φ.

Then, we infer from (3.24)-(3.26) that (3.21) holds for all $\frac{1}{2} \leq \alpha < 1$. By the iteration method, we find that (3.21) holds for all $\alpha \geq \frac{1}{2}$.

Finally, this theorem follows from (3.17), (3.21), and Lemma 2.2. The proof is completed. □

Competing interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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