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A family of singular ordinary differential equations of the third order with an integral boundary condition

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Abstract

We establish in this paper the equivalence between a Volterra integral equation of the second kind and a singular ordinary differential equation of the third order with two initial conditions and an integral boundary condition, with a real parameter. This equivalence allows us to obtain the solution to some problems for non-classical heat equation, the continuous dependence of the solution with respect to the parameter and the corresponding explicit solution to the considered problem.

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1 Introduction

We consider the family of singular ordinary differential equations of the third order with an integral boundary condition, indexed by a parameter $\lambda \in \mathbb{R}$ given by

$$\left.\begin{array}{l} y^{(3)}(t) - \lambda^2 y(t) = \frac{\lambda}{2\sqrt{\pi}} \frac{1}{t^{3/2}}, \quad 0 < t < 1, \\ y(0) = 1, \qquad y'(0) = 0, \qquad y^{(2)}(1) = -\frac{\lambda}{\sqrt{\pi}} + \lambda^2 \int_0^1 y(t)\, dt, \end{array}\right\} \tag{1.1}$$

where $y^{(n)}$ denotes the n-derivative of the function y.

Singular boundary value problems arise very frequently in fluid mechanics and in other branches of applied mathematics. There are results on the existence and asymptotic estimates of solutions for third order ordinary differential equations with singularly perturbed boundary value problems, which depend on a small positive parameter, see, for example [1–3], on third order ordinary differential equations with singularly perturbed boundary value problems and with nonlinear coefficients or boundary conditions, see for example [4–7], on third order ordinary differential equations with nonlinear boundary value problems, see for example [8, 9], on existence results for third order ordinary differential equations, see for example [10–12], and particularly third order ordinary differential equations with integral boundary conditions, see for example [13–21].

In the last years there have been published several papers which consider integral or nonlocal boundary conditions on different branches of applications, e.g., for the heat equations, see for example [22–32], for the wave equations [33], for the second order ordinary differential equations, see for example [34–40], for the fourth order ordinary differential equations, see for example [41, 42], for higher order ordinary differential equations, see for example [43], for fractional differential equations, see for example [44–46].

Our goal is to prove in Sect. 2 that the system (1.1) is equivalent to the following Volterra integral equation of the second kind:

$$y(t) = 1 - \frac{2\lambda}{\sqrt{\pi}} \int_0^t y(\tau)\sqrt{t-\tau}\,d\tau, \quad 0 < t < 1\ (\lambda \in \mathbb{R}), \tag{1.2}$$

which allows us to obtain the solution to some problems for non-classical heat equation for any real parameter λ (see [47–53]).

We remark that the Volterra integral equation (1.2) can also be considered and extended for $t > 0$, that is,

$$y(t) = 1 - \frac{2\lambda}{\sqrt{\pi}} \int_0^t y(\tau)\sqrt{t-\tau}\,d\tau, \quad t > 0\ (\lambda \in \mathbb{R}). \tag{1.3}$$

In Sect. 3, we establish the dependence of the family of boundary value problems for singular ordinary differential equations of third order (1.1) with respect to the parameter $\lambda \in \mathbb{R}$ by using the equivalence with the Volterra integral equation (1.2).

2 Equivalence and existence results

Preliminarily, we give some results useful in the next sections.

Lemma 2.1 *We have the following properties for all* $t > 0$:

$$\int_0^t \left(\int_0^\tau y(\xi)\,d\xi \right) d\tau = \int_0^t y(\tau)(t-\tau)\,d\tau, \tag{2.1}$$

$$\int_0^t \left(\int_0^\xi y(\tau)(t-\tau)\,d\tau \right) d\xi = \int_0^t y(\tau)(t-\tau)^2\,d\tau, \tag{2.2}$$

$$\int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau = 2\sqrt{t} + 2\int_0^t y'(\tau)\sqrt{t-\tau}\,d\tau, \tag{2.3}$$

$$\int_\sigma^t \frac{\sqrt{\tau-\sigma}}{\sqrt{t-\tau}}\,d\tau = \frac{\pi}{2}(t-\sigma), \tag{2.4}$$

$$\int_\sigma^t \frac{d\tau}{\sqrt{t-\tau}\sqrt{\tau-\sigma}} = \pi. \tag{2.5}$$

Proof The first three properties (2.1)–(2.3) follow from the simple integration process. To prove (2.4), we use the change of variable $\tau = \sigma + (t-\sigma)\xi$ then we obtain

$$\begin{aligned}\int_\sigma^t \frac{\sqrt{\tau-\sigma}}{\sqrt{t-\tau}}\,d\tau &= (t-\sigma)\int_0^1 \sqrt{\frac{\xi}{1-\xi}}\,d\xi = (t-\sigma)\int_0^1 \xi^{\frac{3}{2}-1}(1-\xi)^{\frac{1}{2}-1}\,d\xi \\ &= (t-\sigma)B\left(\frac{3}{2},\frac{1}{2}\right) = (t-\sigma)\frac{\Gamma(\frac{3}{2})\Gamma((\frac{1}{2})}{\Gamma(2)} = \frac{\pi}{2}(t-\sigma),\end{aligned}$$

where B and Γ are the known Beta and Gamma functions defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x>0, y>0,$$
$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt, \quad x>0,$$

with the well-known relations

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \Gamma(x+1) = x\Gamma(x) \quad \forall x>0,$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma(n+1) = n! \quad \forall n \in \mathbb{N}.$$

To prove (2.5) we use the same change of variable, so we obtain

$$\int_\sigma^t \frac{d\tau}{\sqrt{t-\tau}\sqrt{\tau-\sigma}} = \int_0^1 \frac{d\xi}{\sqrt{\xi(1-\xi)}} = B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \pi. \qquad \square$$

Theorem 2.2 *y is a solution to the singular boundary value problems* (1.1) *if and only if y is a solution to the Volterra integral equation* (1.2) *for any real parameter* $\lambda > 0$.

Proof Firstly, we consider that y is a solution to the singular boundary value problems (1.1). Then, by using an integration in variable t we obtain

$$y^{(2)}(t) = y^{(2)}(0) + \lambda^2 \int_0^t y(\tau)\,d\tau - \frac{\lambda}{\sqrt{\pi t}}, \quad 0<t<1, \tag{2.6}$$

thus

$$y^{(2)}(1) = y^{(2)}(0) + \lambda^2 \int_0^1 y(\tau)\,d\tau - \frac{\lambda}{\sqrt{\pi}}.$$

And using the integral boundary condition we get

$$y^{(2)}(1) = -\frac{\lambda}{\sqrt{\pi}} + \lambda^2 \int_0^1 y(t)\,dt,$$

so $y^{(2)}(0) = 0$. Thus taking this new condition into account, from (2.6) by using an integration in variable t, the condition $y'(0) = 0$ and (2.1) we get

$$\begin{aligned} y'(t) &= \lambda^2 \int_0^t \left(\int_0^\tau y(\sigma)\,d\sigma\right) d\tau - \frac{2\lambda\sqrt{t}}{\sqrt{\pi}} \\ &= \lambda^2 \int_0^t y(\tau)(t-\tau)\,d\tau - \frac{2\lambda\sqrt{t}}{\sqrt{\pi}}, \quad 0<t<1. \end{aligned} \tag{2.7}$$

Finally, from (2.7) by using another integration in the variable t, and the condition $y(0) = 1$, we obtain

$$\begin{aligned} y(t) &= 1 + \lambda^2 \int_0^t \left(\int_0^\tau y(\sigma)(\tau - \sigma)\,d\sigma \right) d\tau - \frac{4\lambda 3}{3\sqrt{\pi}} t^{3/2} \\ &= 1 + \lambda^2 \int_0^t y(\tau)(t-\tau)^2\,d\tau - \frac{4\lambda}{3\sqrt{\pi}} t^{3/2}, \quad 0 < t < 1. \end{aligned} \tag{2.8}$$

We cannot arrive directly at the Volterra equation (1.2), but we can define the auxiliary function

$$\varphi(t) = y(t) - 1 + \frac{2\lambda}{\sqrt{\pi}} \int_0^t y(\tau)\sqrt{t-\tau}\,d\tau, \tag{2.9}$$

and now our goal is to prove that $\varphi = 0$. We have $\varphi(0) = 0$, by using the boundary $y(0) = 1$.

Now, we compute the first derivative of φ using the property (2.3), we get

$$\begin{aligned} \varphi'(t) &= y'(t) + \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau \\ &= y'(t) + \frac{\lambda}{\sqrt{\pi}} \left(2\sqrt{t} + 2\int_0^t y'(\tau)\sqrt{t-\tau}\,d\tau \right), \quad 0 < t < 1. \end{aligned} \tag{2.10}$$

On the other hand, by using (2.9), (2.7), (2.10) and the property (2.4) we obtain

$$\begin{aligned} \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}}\,d\tau &= \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau - 2\sqrt{t} + \frac{2\lambda}{\sqrt{\pi}} \int_0^t \frac{\int_0^\tau y(\sigma)\sqrt{\tau-\sigma}\,d\sigma}{\sqrt{t-\tau}}\,d\tau \\ &= \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau - 2\sqrt{t} + \lambda\sqrt{\pi} \int_0^t y(\tau)(t-\tau)\,d\tau \\ &= \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau - 2\sqrt{t} + \lambda\sqrt{\pi} \left(\frac{y'(t)}{\lambda^2} + \frac{2}{\lambda\sqrt{\pi}}\sqrt{t} \right) \\ &= \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\,d\tau + \frac{\sqrt{\pi}}{\lambda} y'(t) = \frac{\sqrt{\pi}}{\lambda}\varphi'(t), \quad 0 < t < 1. \end{aligned}$$

That is,

$$\varphi'(t) = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}}\,d\tau, \quad 0 < t < 1, \tag{2.11}$$

thus $\varphi'(0) = 0$. Therefore, we have

$$\varphi'(t) = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\varphi(t-\tau)}{\sqrt{\tau}}\,d\tau, \quad 0 < t < 1, \tag{2.12}$$

and then we obtain

$$\varphi^{(2)}(t) = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\varphi'(t-\tau)}{\sqrt{\tau}}\,d\tau = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\varphi'(\tau)}{\sqrt{t-\tau}}\,d\tau, \quad 0 < t < 1, \tag{2.13}$$

thus $\varphi^{(2)}(0) = 0$, and so on we obtain $\varphi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, then this part holds.

Secondly, we consider that y is a solution of the Volterra integral equation (1.2), then we have the condition $y(0) = 1$, which is automatically satisfied.

Then, by derivation of (1.2) and by using the property (2.4) we have

$$\begin{aligned} y'(t) &= -\frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}}\, d\tau \\ &= -\frac{\lambda}{\sqrt{\pi}} \left(2\sqrt{t} - \frac{2\lambda}{\sqrt{\pi}} \left(\int_0^t \frac{1}{\sqrt{t-\tau}} \left(\int_0^\tau y(\sigma)\sqrt{\tau-\sigma}\, d\sigma \right) d\tau \right) \right) \\ &= -\frac{2\lambda}{\sqrt{\pi}}\sqrt{t} + \frac{2\lambda^2}{\pi} \int_0^t \left(\int_\sigma^t \frac{\sqrt{\tau-\sigma}}{\sqrt{t-\tau}}\, d\tau \right) y(\sigma)\, d\sigma \\ &= -\frac{2\lambda}{\sqrt{\pi}}\sqrt{t} + \lambda^2 \int_0^t y(\sigma)(t-\sigma)\, d\sigma, \quad 0 < t < 1, \end{aligned} \tag{2.14}$$

and the boundary condition $y'(0) = 0$ holds. Therefore, from (2.14) we have

$$y^{(2)}(t) = -\frac{\lambda}{\sqrt{\pi t}} + \lambda^2 \int_0^t y(\tau)\, d\tau, \quad 0 < t < 1, \tag{2.15}$$

thus for $t = 1$ we get the integral boundary condition.

Finally, from (2.15) we have

$$y^{(3)}(t) = \frac{\lambda}{\sqrt{\pi}} t^{-3/2} + \lambda^2 y(t), \quad 0 < t < 1, \tag{2.16}$$

so the singular boundary value problems (1.1) holds for any real parameter $\lambda > 0$, thus the proof of the theorem is complete. □

It is well known that there exists a unique solution of the Volterra integral equation (1.3), that is, the Volterra integral equation (1.2) extended for $t > 0$; see [49, 54]. Now, we will find the explicit solution of the Volterra integral equation (1.3).

Theorem 2.3 *The solution of the Volterra integral equation* (1.3) *is given by the following expression*:

$$y(t) = I(t) - \sqrt{\frac{2}{\pi}} J(t), \quad t > 0, \tag{2.17}$$

with

$$I(t) = \sum_{n=0}^{+\infty} \frac{(\lambda^{2/3} t)^{3n}}{(3n)!}, \tag{2.18}$$

$$J(t) = \sum_{n=0}^{+\infty} \frac{(2\lambda^{2/3} t)^{\frac{3(2n+1)}{2}}}{(3(2n+1))!!}, \tag{2.19}$$

being series with infinite radii of convergence and we use the definition

$$(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1.$$

for the compactness expression.

Proof By using the Adomian method [55, 56] we propose, for the solution of the Volterra integral equation (1.3), the series of expansion functions given by

$$y(t) = \sum_{n=0}^{+\infty} y_n(t)$$

and we obtain the following recurrence expansions:

$$y_0(t) = 1, \quad y_n(t) = -\frac{2\lambda}{\sqrt{\pi}} \int_0^t y_{n-1}(\tau)\sqrt{t-\tau}\,d\tau, \quad \forall n \geq 1.$$

Then, following [49], we obtain (2.17) where $I(t)$ and $J(t)$ are given by (2.18) and (2.19), respectively, and the result holds.

The solution of the Volterra integral equation (1.3) is the key for obtaining the solution of the non-classical heat conduction problem given by

$$u_t(x,t) - u_{xx}(x,t) = -\lambda \int_0^t u_x(0,\tau)\,d\tau, \quad x > 0, t > 0, \tag{2.20}$$

$$u(0,t) = 0, \quad t > 0, \tag{2.21}$$

$$u(x,0) = h_0 > 0, \quad x > 0, \tag{2.22}$$

with a parameter $\lambda \in \mathbb{R}$. Then the solution of the problem above is given by

$$u(x,t) = h_0 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) - \lambda \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right) U(\tau)\,d\tau, \tag{2.23}$$

where $U(t)$ is given by

$$U(t) = \frac{h_0}{\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}\,d\tau \tag{2.24}$$

and g is the solution of the Volterra integral equation (1.3). Moreover, the heat flux on $x = 0$ is given by

$$u_x(0,t) = U'(t) = \frac{h_0}{\sqrt{\pi t}} - h_0\lambda \int_0^t g(\tau)\,d\tau, \quad t > 0.$$

For the complete proof see [49]. □

3 Dependence of the solution with respect to λ

From now on, we will consider that the solution to the singular ordinary differential equation of the third order with an integral boundary condition (1.1) or equivalently the solution of the Volterra integral equation (1.2) depends also on the parameter $\lambda \in \mathbb{R}$. From now on, without loss of generality, we will consider the Volterra integral equation (1.3)

We consider that $t \mapsto g_\lambda(t)$ be the solution of the Volterra integral equation (1.2) for the parameter λ. For $\varepsilon \in (0,1)$ be a fixed real number and $T > 0$, let consider the parameter λ such that

$$|\lambda| \leq \lambda_{\varepsilon,T} = \frac{3\sqrt{\pi}}{4} \frac{\varepsilon}{T^{3/2}}, \tag{3.1}$$

and we define the norm

$$\|g\|_T = \max_{0\le t\le T}\big|g(t)\big|.$$

Therefore, we obtain the following dependence results.

Theorem 3.1 *We have the boundedness*:

$$\|g_\lambda\|_T \le \frac{1}{1-\varepsilon}, \quad \forall\lambda : |\lambda| \le \lambda_{\varepsilon,T}. \tag{3.2}$$

Moreover, the application $\lambda \mapsto g_\lambda(t)$ *defined from* $[-\lambda_{\varepsilon,T}, \lambda_{\varepsilon,T}]$, *to* $\mathcal{C}([0,T))$ *is Lipschitzian*.

Proof From the Volterra integral equation (1.2) we obtain

$$\begin{aligned}\big|g_\lambda(t)\big| &\le 1 + \frac{2|\lambda|}{\sqrt{\pi}}\|g_\lambda\|_t \int_0^t \sqrt{t-\tau}\,d\tau \\ &\le 1 + \frac{4}{3\sqrt{\pi}}\lambda_{\varepsilon,T}T^{3/2}\|g_\lambda\|_T\end{aligned}$$

and by using (3.1) follows (3.2). Moreover, consider $g_i(t)$ the solution of the Volterra integral equation (1.2) for λ_i ($i = 1,2$), such that

$$|\lambda_i| \le \lambda_{\varepsilon,T}.$$

Then we have

$$\begin{aligned}\big|g_2(t) - g_1(t)\big| &\le \frac{2}{\sqrt{\pi}}|\lambda_2 - \lambda_1|\|g_1\|_t \int_0^t \sqrt{t-\tau}\,d\tau + \frac{2|\lambda_2|}{\sqrt{\pi}}\|g_1 - g_2\|_t \int_0^t \sqrt{t-\tau}\,d\tau \\ &\le \frac{4T^{3/2}}{3\sqrt{\pi}}\big[|\lambda_2 - \lambda_1|\|g_1\|_T + |\lambda_2|\|g_2 - g_1\|_T\big].\end{aligned}$$

Therefore, we get

$$\|g_2 - g_1\|_T \le \frac{4}{3\sqrt{\pi}}\frac{T^{3/2}}{(1-\varepsilon)^2}|\lambda_2 - \lambda_1|; \tag{3.3}$$

thus the result holds. □

Now, we obtain the dependence of the solution to the non-classical heat conduction problem (2.20)–(2.22) with respect to the parameter λ. We consider that U_λ and u_λ are given, respectively, by

$$U_\lambda(t) = \frac{h_0}{\sqrt{\pi}}\int_0^t \frac{g_\lambda(\tau)}{\sqrt{t-\tau}}\,d\tau \tag{3.4}$$

and

$$u_\lambda(x,t) = h_0\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right) - \lambda\int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)U_\lambda(\tau)\,d\tau. \tag{3.5}$$

Then we obtain the following results.

Theorem 3.2 *We have the boundedness*:

$$\|U_\lambda\|_T \le \frac{2h_0}{\sqrt{\pi}}\frac{T^{1/2}}{1-\varepsilon}, \quad \forall\lambda : |\lambda| \le \lambda_{\varepsilon,T}. \tag{3.6}$$

Moreover, the application $\lambda \mapsto U_\lambda(t)$, *from* $[-\lambda_{\varepsilon,T}, \lambda_{\varepsilon,T}]$ *to* $\mathcal{C}([0,T])$ *is Lipschitzian. We have also the following boundedness*:

$$\|u_\lambda\|_{[0,+\infty[\times[0,T]} \le h_0\left(1+\frac{3\varepsilon}{2(1-\varepsilon)}\right) \quad \forall\lambda : |\lambda| \le \lambda_{\varepsilon,T}, \tag{3.7}$$

the estimates

$$\|u_\lambda - u_0\|_{[0,+\infty[\times[0,T]} \le \frac{2h_0}{\sqrt{\pi}}\frac{T^{3/2}}{1-\varepsilon}|\lambda|, \quad \forall\lambda : |\lambda| \le \lambda_{\varepsilon,T}, \tag{3.8}$$

and find that the application $\lambda \mapsto u_\lambda(x,t)$, *from* $[-\lambda_{\varepsilon,T}, \lambda_{\varepsilon,T}]$ *to* $\mathcal{C}([0,+\infty[\times[0,T])$ *is Lipschitzian.*

Proof From (2.24), we have

$$\left|U_\lambda(t)\right| \le \frac{h_0}{\sqrt{\pi}}\|g_\lambda\|_t \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \le \frac{2h_0}{\sqrt{\pi}}\frac{T^{1/2}}{1-\varepsilon};$$

thus (3.6) holds. Consider now $U_i(t)$ given by (3.4), for λ_i $(i=1,2)$ satisfying $|\lambda_i| \le \lambda_{\varepsilon,T}$. We have

$$\left|U_2(t)-U_1(t)\right| \le \frac{2h_0T^{1/2}}{\sqrt{\pi}}\|g_2-g_1\|_T \le \frac{8h_0T^2}{3\pi(1-\varepsilon)^2}|\lambda_2-\lambda_1| \tag{3.9}$$

thus the application $\lambda \mapsto U_\lambda$ is Lipschitzian.

From (3.5), we have

$$\left|u_\lambda(x,t)\right| \le h_0 + t|\lambda|\|U_\lambda\|_t \le h_0\left(1+\frac{3\varepsilon}{2(1-\varepsilon)}\right), \quad \forall x\in[0,+\infty[, \forall t\in[0,T],$$

thus (3.7) holds.

From (3.5) also, we have

$$\left|u_\lambda(x,t)-u_0(x,t)\right| \le t|\lambda|\|U_\lambda\|_t \le \frac{2h_0}{\sqrt{\pi}}\frac{T^{3/2}}{1-\varepsilon}|\lambda|$$

thus (3.8) holds.

Consider now $u_i(x,t)$ given by (3.5) for λ_i $(i=1,2)$ satisfying $|\lambda_i| \le \lambda_{\varepsilon,T}$. Then we have

$$\begin{aligned}\left|u_2(x,t)-u_1(x,t)\right| &\le t|\lambda_2-\lambda_1|\|U_1\|_t + t|\lambda_2|\|U_2-U_1\|_t \\ &\le T|\lambda_2-\lambda_1|\|U_1\|_T + T|\lambda_2|\|U_2-U_1\|_T \\ &\le \frac{2h_0T^{1/2}}{\sqrt{\pi}(1-\varepsilon)}\left(T+\frac{\varepsilon\pi}{1-\varepsilon}\right)|\lambda_2-\lambda_1| \quad \forall x\in[0,+\infty[, \forall t\in[0,T],\end{aligned}$$

thus

$$\|u_2 - u_1\|_{[0,+\infty[\times[0,T]} \leq \frac{2h_0 T^{1/2}}{\sqrt{\pi}(1-\varepsilon)}\left(\frac{\varepsilon\pi}{1-\varepsilon} + T\right)|\lambda_2 - \lambda_1|$$

and the result holds. □

Conclusion We have obtained the equivalence between a family of singular ordinary differential equations of the third order with two initial conditions and an integral boundary condition (1.1), and the Volterra integral equation (1.2) with a parameter $\lambda \in \mathbb{R}$. We have also given the explicit solution of these equations which can be extended for all $t > 0$, and then some non-classical heat conduction problems can be solved explicitly, for any real parameter λ. Finally, we have established the dependence of the family of singular boundary problems of the third order with respect to the parameter λ.

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Authors' contributions
The authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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