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Analysis of segregated boundary-domain integral equations for BVPs with non-smooth coefficients on Lipschitz domains

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Abstract

Segregated direct boundary-domain integral equations (BDIEs) based on a parametrix and associated with the Dirichlet and Neumann boundary value problems for the linear stationary diffusion partial differential equation with a variable Hölder-continuous coefficients on Lipschitz domains are formulated. The PDE right-hand sides belong to the Sobolev (Bessel potential) space $H^{s-2}(\Omega)$ or $\widetilde{H}^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, when neither strong classical nor weak canonical co-normal derivatives are well defined. Equivalence of the BDIEs to the original BVP, BDIE solvability, solution uniqueness/non-uniqueness, and the Fredholm property and invertibility of the BDIE operators are analysed in appropriate Sobolev spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible; however, some finite-dimensional perturbations are constructed leading to invertibility of the perturbed (stabilised) operators.

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1 Introduction

Many applications in science and engineering can be modelled by boundary-value problems (BVPs) for partial differential equations with variable coefficients. Reduction of the BVPs with arbitrarily variable coefficients to explicit boundary integral equations is usually not possible, since the fundamental solution needed for such reduction is generally not available in an analytical form (except for some special dependence of the coefficients on coordinates). Using a parametrix (Levi function) introduced in [20, 25] as a substitute of a fundamental solution, it is possible however to reduce such a BVP to a system of boundary-domain integral equations, BDIEs, (see e.g. [38, Sect. 18], [43, 44], where the Dirichlet, Neumann, and Robin problems for some PDEs were reduced to *indirect* BDIEs). However, many questions about their equivalence to the original BVP, solvability, solution uniqueness, and invertibility of corresponding integral operators remained open for rather long time.

In [3, 5, 6, 8, 30], the 3D mixed (Dirichlet–Neumann) boundary value problem (BVP) for the stationary diffusion PDE *with infinitely smooth variable coefficient on a domain with an* *infinitely smooth boundary and a square-integrable right-hand side* was reduced to either segregated or united direct boundary-domain integral or integro-differential equations, some of which coincide with those formulated in [29]. Such BVPs appear, for example, in electrostatics, stationary heat transfer, and other diffusion problems for inhomogeneous media.

For a function from the Sobolev space $H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, a classical co-normal derivative in the sense of traces may not exist. However, the generalised co-normal derivative can be defined in the weak sense, associated with the first Green identity and with an extension of the corresponding second-order PDE right-hand side to $\widetilde{H}^{s-2}(\Omega)$ (see [27, Lemma 4.3], [31, Definition 3.1]). Since the extension is non-unique, the co-normal derivative operator appears to be also non-unique and non-linear in u unless a linear relation between u and the PDE right-hand side extension is enforced. This creates some difficulties in formulating the boundary-domain integral equations.

These difficulties are addressed in this paper presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann boundary value problems, on Lipschitz domains, for the divergent-type PDE with a non-smooth Hölder–Lipschitz variable scalar coefficient and a general right-hand side from $H^{s-2}(\Omega)$, extended when necessary to $\widetilde{H}^{s-2}(\Omega)$. This needed a non-trivial generalisation of the third Green identity and its co-normal derivative for such functions, which essentially extends the approach implemented in [3, 5, 6, 8, 30] for the right-hand side from $L_2(\Omega)$, with smooth coefficient and smooth domain boundary. Equivalence of the BDIEs to the original BVP, BDIE solvability, solution uniqueness/non-uniqueness, and the Fredholm properties and invertibility of the BDIE operators are analysed in the Sobolev (Bessel potential) spaces. It is shown that the BDIE operators for the Neumann BVP are not invertible, and appropriate finite-dimensional perturbations are constructed leading to invertibility of the perturbed (stabilised) operators. Some preliminary results in this direction for the infinitely smooth coefficient and domains were presented in [33].

Note that our analysis is mainly aimed not at the boundary-value problems, the properties of which are well known nowadays, but rather at the BDIE systems per se. The analysis is interesting not only in its own rights but is also to be used further on for analysis of convergence and stability of BDIE-based numerical methods for PDEs; see, for example, [16, 29, 34, 35, 46–48, 52, 53].

2 Spaces, co-normal derivatives and boundary value problems

Let $\Omega = \Omega_+$ be a bounded open n-dimensional region of $\mathbb{R}^n$, $n \geq 3$, and let $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}_+$ denote the corresponding exterior domain. For simplicity, we assume that their common boundary $\partial\Omega$ is a simply connected closed Lipschitz surface. Let Ω_0 denote Ω_+, Ω_- or $\mathbb{R}^n$.

In what follows, $\mathcal{D}(\Omega_0) := C^\infty_{\mathrm{comp}}(\Omega_0)$ and $\mathcal{D}(\overline{\Omega_0}) := \{r_{\Omega_0} g : g \in \mathcal{D}(\mathbb{R}^n)\}$. Here and further on, r_{Ω_0} denotes the restriction operator on Ω_0; we will also use the equivalent notation $g|_{\Omega_0} := r_{\Omega_0} g$. Further, $H^s(\Omega_0) = H^s_2(\Omega_0)$ and $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ are the Bessel potential spaces, where s is a real number (see, e.g., [18, 26, 27]). We recall that H^s coincide with the Sobolev–Slobodetski spaces W^s_2 for non-negative s. By $\widetilde{H}^s(\Omega_0)$ we denote the closure of $\mathcal{D}(\Omega_0)$ in $H^s(\mathbb{R}^n)$. It is a subspace of $H^s(\mathbb{R}^n)$, and for Lipschitz domains, $\widetilde{H}^s(\Omega_0) = \{g : g \in H^s(\mathbb{R}^n), \operatorname{supp} g \subset \overline{\Omega_0}\}$. By $H^s(\Omega_0)$ and $\widetilde{H}^s_\bullet(\Omega_0)$ we denote the spaces of restrictions on Ω_0 of distributions from $H^s(\mathbb{R}^n)$ and $\widetilde{H}^s(\Omega_0)$, respectively:

$$H^s(\Omega_0) := \{ r_{\Omega_0} g : g \in H^s(\mathbb{R}^n) \},$$
$$\widetilde{H}^s_\bullet(\Omega_0) := r_{\Omega_0} \widetilde{H}^s(\Omega_0) := \{ r_{\Omega_0} g : g \in \widetilde{H}^s(\Omega_0) \} \subset H^s(\Omega_0),$$

endowed by the corresponding infimum norms and the Hilbert structure defined with the help of orthogonal projections; see [27, p. 77] for $H^s(\Omega_0)$. Note that the space $\widetilde{H}^s_\bullet(\Omega_0)$ coincides with the one denoted as $L^p_{s,z}(\Omega_0)$ in [41, Eq. (5.2)] and [40, Eq. (2.212)] for $p = 2$.

Let us introduce the subspace $H^s_{\partial\Omega_0} := \{g : g \in H^s(\mathbb{R}^n), \operatorname{supp} g \subset \partial\Omega_0\}$ of $H^s(\mathbb{R}^n)$ (and of $\widetilde{H}^s(\Omega_0)$). By $\mathring{H}^s(\Omega_0)$ we denote the closure of $\mathcal{D}(\Omega_0)$ in $H^s(\Omega_0)$.

Definition 2.1 Let $\mathring{E}_{\Omega_0}$ denote the operator of extension of functions $g \in H^s(\Omega_0)$, $s \geq 0$, to the whole $\mathbb{R}^n$ by zero outside Ω_0. By, e.g., [27, Lemma 3.32 and Theorem 3.33] (see also [31, Theorem 2.7]) the operator $\mathring{E}_{\Omega_0} : H^s(\Omega_0) \to \widetilde{H}^s(\Omega_0)$ is continuous if $0 \leq s < \frac{1}{2}$, and we extend it also to the range $-\frac{1}{2} < s < \frac{1}{2}$ defining it for $-\frac{1}{2} < s < 0$ as (cf. the proof of [31, Theorem 2.16])

$$\langle \mathring{E}_{\Omega_0} g, v \rangle_{\Omega_0} := \langle g, \mathring{E}_{\Omega_0} v \rangle_{\Omega_0}, \quad \forall g \in H^s(\Omega), \forall v \in H^{-s}(\Omega). \tag{2.1}$$

Remark 2.2 Note the following known or easily deduced results:

1. There hold the continuous embeddings $\widetilde{H}^s_\bullet(\Omega_0) \hookrightarrow \mathring{H}^s(\Omega_0) \hookrightarrow H^s(\Omega_0)$; see [42, Eq. (2.123)].
2. $\widetilde{H}^s_\bullet(\Omega_0) = \mathring{H}^s(\Omega_0)$ for any $s > 1/2$ such that $s - \frac{1}{2}$ is non-integer; see, e.g., [27, Theorem 3.3].
3. $\mathring{H}^s(\Omega_0) = H^s(\Omega_0)$ for any $s \leq 1/2$; see [31, Theorem 2.12].
4. $\widetilde{H}^s_\bullet(\Omega_0) = \mathring{H}^s(\Omega_0) = H^s(\Omega_0)$ for any $s < 1/2$ such that $s - \frac{1}{2}$ is non-integer; see, e.g., [31, Lemma 2.15].
5. For any $s \in \mathbb{R}$, there evidently exists an extension from $\widetilde{H}^s_\bullet(\Omega_0)$ to $\widetilde{H}^s(\Omega_0)$, and for any $s \geq -1/2$, this extension is unique; see, e.g., [27, Lemma 3.39], [31, Theorem 2.10(i)].
6. By [31, Theorem 2.16], for any $s \in (-1/2, 1/2)$, the extension from $\widetilde{H}^s_\bullet(\Omega_0) = \mathring{H}^s(\Omega_0) = H^s(\Omega_0)$ to $\widetilde{H}^s(\Omega_0)$ is unique and is given by the operator $\mathring{E}_{\Omega_0}$.

Remark 2.3 Due to Remark 2.2(5), for $s \geq -1/2$, the space $\widetilde{H}^s_\bullet(\Omega_0)$ is isometrically isomorphic to the space $\widetilde{H}^s(\Omega_0)$, and sometimes these spaces are identified. Particularly, if $g_1, g_2 \in \widetilde{H}^s_\bullet(\Omega_0)$, then denoting by $\tilde{g}_1, \tilde{g}_2 \in \widetilde{H}^s(\Omega_0)$ the unique distributions such that $g_i = r_{\Omega_0} \tilde{g}_i$ in Ω_0, we have $\|g_i\|_{\widetilde{H}^s_\bullet(\Omega_0)} = \|\tilde{g}_i\|_{\widetilde{H}^s(\Omega_0)}$ and $(g_1, g_2)_{\widetilde{H}^s_\bullet(\Omega_0)} = (\tilde{g}_1, \tilde{g}_2)_{\widetilde{H}^s(\Omega_0)}$. Moreover, if $s \in (-1/2, 1/2)$, then by Remark 2.2(6), $\tilde{g}_i = \mathring{E}_{\Omega_0} g_i$ hence implying $\|g_i\|_{\widetilde{H}^s_\bullet(\Omega_0)} = \|\mathring{E}_{\Omega_0} g_i\|_{\widetilde{H}^s(\Omega_0)}$.

There is no such isomorphism for $s < -1/2$ since in such a case the extension from $\widetilde{H}^s_\bullet(\Omega_0)$ to $\widetilde{H}^s(\Omega_0)$ is not unique. However, due to the definition of the spaces, there is still an isometric isomorphism between the space $\widetilde{H}^s_\bullet(\Omega_0)$ and the quotient space $\widetilde{H}^s(\Omega_0)/H^s_{\partial\Omega_0}$.

Definition of the space $\widetilde{H}^s_\bullet(\Omega_0)$, Remark 2.2, and Remark 2.3 imply the following assertion.

Corollary 2.4 *The following restriction operators are isomorphisms*:

$$r_{\Omega_0}:\widetilde{H}^s(\Omega_0)\to\widetilde{H}^s_\bullet(\Omega_0),\quad -\frac{1}{2}\le s,\tag{2.2}$$

$$r_{\Omega_0}:\widetilde{H}^s(\Omega_0)\to H^s(\Omega_0)=\widetilde{H}^s_\bullet(\Omega_0),\quad -\frac{1}{2}<s<\frac{1}{2},\tag{2.3}$$

$$r_{\Omega_0}:\widetilde{H}^s(\Omega_0)/H^s_{\partial\Omega_0}\to\widetilde{H}^s_\bullet(\Omega_0),\quad s<-\frac{1}{2}.\tag{2.4}$$

The inverse to the operator (2.3) *is* $r^{-1}_{\Omega_0}=\mathring{E}_{\Omega_0}$; *see Definition* 2.1.

Definition 2.5 For a non-negative integer m and $0<\theta\le 1$, let $C^{m,\theta}(\overline{\Omega}_0)$ denote the Hölder–Lipschitz space in the closed domain $\overline{\Omega}_0$. Similar to [32, Definition 3.1], $g\in C^\mu_+(\overline{\Omega}_0)$ for $\mu\ge 0$ means that

$g\in L_\infty(\Omega_0)$ when $\mu=0$;

$g\in C^{\mu-1,1}(\overline{\Omega}_0)$ when μ is a positive integer;

$g\in C^{m,\theta+\epsilon}(\overline{\Omega}_0)$ for some $\epsilon>0$ when $\mu=m+\theta$, where m is a non-negative integer, and $0<\theta<1$.

Employing this definition, Theorem 7.2 from Sect. 7 can be reformulated as follows.

Theorem 2.6 *Let* Ω_0 *be an open set in* $\mathbb{R}^n$, $\sigma\in\mathbb{R}$, $v\in H^\sigma(\Omega_0)$, *and* $g\in C^{|\sigma|}_+(\overline{\Omega}_0)$. *Then* g *is a multiplier in* $H^\sigma(\Omega_0)$, *i.e.*, $gv\in H^\sigma(\Omega_0)$ *for every* $v\in H^\sigma(\Omega_0)$, *and the corresponding norm estimate holds*.

Let us denote $\partial_j:=\partial_{x_j}:=\partial/\partial x_j$ $(j=1,2,\dots,n)$, $\nabla=(\partial_1,\partial_2,\dots,\partial_n)$. Let

$$0<a_{\min}\le a(x)\le a_{\max}<\infty\quad\text{for almost every } x\in\Omega_\pm.\tag{2.5}$$

We consider the scalar elliptic differential equation, which can be written in the following strong form if u and a are sufficiently smooth:

$$Au(x):=A(x,\nabla)u(x):=\nabla\cdot\big(a(x)\nabla u(x)\big)=f(x),\quad x\in\Omega_\pm,\tag{2.6}$$

where u is an unknown function and f is a given function in $\Omega_\pm$.

For $u\in H^s(\Omega_\pm)$, $1/2<s<3/2$, and $a\in C^{|s-1|}_+(\overline{\Omega}_\pm)$, the partial differential operator A is understood in the sense of distributions:

$$\langle Au,v\rangle_{\Omega_\pm}:=-\mathcal{E}_{\Omega_\pm}(u,v),\quad\forall v\in\mathcal{D}(\Omega_\pm),\tag{2.7}$$

where

$$\mathcal{E}_{\Omega_\pm}(u,v):=\langle a\nabla u,\nabla v\rangle_{\Omega_\pm}:=\sum_{i=1}^n\langle a\partial_i u,\partial_i v\rangle_{\Omega_\pm},$$

and the duality brackets $\langle g,\cdot\rangle_{\Omega_\pm}$ denote value of a linear functional (distribution) g extending the usual L_2 dual product. If $s=1$, then

$$\mathcal{E}_{\Omega_\pm}(u,v)=\int_{\Omega_\pm}a(x)\nabla u(x)\cdot\nabla v(x)\,dx.$$

Since the set $\mathcal{D}(\Omega_\pm)$ is dense in $\widetilde{H}^{2-s}(\Omega_\pm)$, (2.7) defines, due to Theorem 2.6 (see, e.g., [32, Theorem 3.4]), the continuous linear operator $A : H^s(\Omega_\pm) \to H^{s-2}(\Omega_\pm) = [\widetilde{H}^{2-s}(\Omega_\pm)]^*$, where

$$\langle Au, v\rangle_{\Omega_\pm} := -\mathcal{E}_{\Omega_\pm}(u,v), \quad \forall\, u \in H^s(\Omega_\pm),\ v \in \widetilde{H}^{2-s}(\Omega_\pm). \tag{2.8}$$

Let us also consider the operator $\check{A}_{\Omega_\pm} : H^s(\Omega_\pm) \to \widetilde{H}^{s-2}(\Omega_\pm) = [H^{2-s}(\Omega_\pm)]^*$ (see [31, Eq. (3.5)], [32, Eq. (5.1)]) defined by

$$\begin{aligned}\langle \check{A}_{\Omega_\pm} u, v\rangle_{\Omega_\pm} &:= -\check{\mathcal{E}}_{\Omega_\pm}(u,v) := -\big\langle \mathring{E}_{\Omega_\pm}(a\nabla u), \nabla v\big\rangle_{\Omega_\pm} \\ &= -\big\langle \mathring{E}_{\Omega_\pm}(a\nabla u), \nabla v_e\big\rangle_{\mathbb{R}^n} = \big\langle \nabla \cdot \mathring{E}_{\Omega_\pm}(a\nabla u), v_e\big\rangle_{\mathbb{R}^n} \\ &= \big\langle \nabla \cdot \mathring{E}_{\Omega_\pm}(a\nabla u), v\big\rangle_{\Omega_\pm}, \quad \forall\, u \in H^s(\Omega_\pm),\ v \in H^{2-s}(\Omega_\pm),\end{aligned} \tag{2.9}$$

which is evidently continuous. Here $v_e \in H^{2-s}(\mathbb{R}^n)$ is such that $r_{\Omega_\pm} v_e = v$. Evidently, weak definition (2.9) can be also written (in the strong-looking form) as

$$\check{A}_{\Omega_\pm} u = \nabla \cdot \mathring{E}_{\Omega_\pm} r_{\Omega_\pm}[a\nabla u]. \tag{2.10}$$

For any $u \in H^s(\Omega_\pm)$, the functional $\check{A}_{\Omega_\pm} u$ belongs to $\widetilde{H}^{s-2}(\Omega_\pm)$ and is a specific extension of the functional $Au \in H^{s-2}(\Omega_\pm)$; recall that the functional $Au \in H^{s-2}(\Omega_\pm)$ is defined on $\widetilde{H}^{2-s}(\Omega_\pm)$, whereas the functional $\check{A}_{\Omega_\pm} u$ is defined on $H^{2-s}(\Omega_\pm)$.

Remark 2.7 Note also that Definition 2.1 for $\mathring{E}_{\Omega_\pm}$ and definition (2.9) imply that

$$\begin{aligned}&\langle \check{A}_{\Omega_\pm} u, v\rangle_{\Omega_\pm} = -\check{\mathcal{E}}_{\Omega_\pm}(u,v) = -\check{\mathcal{E}}_{\Omega_\pm}(v,u) = \langle u, \check{A}_{\Omega_\pm} v\rangle_{\Omega_\pm}, \\ &\quad \forall\, u \in H^s(\Omega_\pm),\ v \in H^{2-s}(\Omega_\pm),\ 1/2 < s < 3/2.\end{aligned}$$

From the trace theorem (see, e.g., [11, 12, 26, 27]) for $u \in H^s(\Omega_\pm)$, $1/2 < s < 3/2$, it follows that $\gamma^\pm u \in H^{s-\frac{1}{2}}(\partial\Omega)$, where $\gamma^\pm = \gamma^\pm_{\partial\Omega}$ is the trace operator on $\partial\Omega$ from $\Omega_\pm$. If $\gamma^+ u = \gamma^- u$, then we will sometimes write just γu. Let also $\gamma^{-1} := \gamma_r^{-1} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\mathbb{R}^n)$ denote a (non-unique) continuous right inverse to the trace operator γ, i.e., $\gamma\gamma^{-1} w = w$ for any $w \in H^{s-\frac{1}{2}}(\partial\Omega)$. Hence also $\gamma^\pm\gamma^{-1} w = w$ for any $w \in H^{s-\frac{1}{2}}(\partial\Omega)$.

For $u \in H^s(\Omega_\pm)$, $s > \frac{3}{2}$, and $a \in C(\overline{\Omega}_\pm)$, we denote by $T^{c\pm}$ the corresponding classical (strong) co-normal derivative operators on $\partial\Omega$ in the sense of traces:

$$T^{c\pm} u(x) := a(x)\nu(x) \cdot \gamma^\pm \nabla u(x) = a(x)\partial_\nu u(x), \quad x \in \partial\Omega, \tag{2.11}$$

where $\nu(x) = \nu^+(x)$ is the outward to Ω_+ unit normal vector at the point $x \in \partial\Omega$, and we will sometimes write $T^c u(x)$ if $T^{c+} u(x) = T^{c-} u(x)$. However, the classical co-normal derivative is, generally, not well defined if $u \in H^s(\Omega_\pm)$, $1/2 < s < 3/2$, (see an example in [33, Appendix A] of a function from $H^1(\Omega)$, where the classical normal derivative does not exist at boundary points).

Inspired by the first Green identity for smooth functions, we can define *the generalised co-normal derivative* (cf., e.g., [27, Lemma 4.3]), [31, Definition 3.1], [32, Definition 5.2]).

Definition 2.8 Let $1/2 < s < 3/2$, $u \in H^s(\Omega_\pm)$, $a \in C_+^{|s-1|}(\overline{\Omega}_\pm)$, and $r_{\Omega_\pm} Au = r_{\Omega_\pm}\tilde{f}_\pm$ for some $\tilde{f}_\pm \in \widetilde{H}^{s-2}(\Omega_\pm)$. Then the *generalised co-normal derivatives* $T^\pm(\tilde{f}_\pm; u) \in H^{s-\frac{3}{2}}(\partial\Omega)$ are defined in the weak form as

$$\begin{aligned}\pm\langle T^\pm(\tilde{f}_\pm; u), w\rangle_{\partial\Omega} &:= \langle \tilde{f}_\pm, \gamma^{-1}w\rangle_{\Omega_\pm} + \check{\mathcal{E}}_{\Omega_\pm}(u, \gamma^{-1}w) \\ &= \langle \tilde{f}_\pm - \check{A}_{\Omega_\pm}u, \gamma^{-1}w\rangle_{\Omega_\pm}, \quad \forall\, w \in H^{\frac{3}{2}-s}(\partial\Omega),\end{aligned} \tag{2.12}$$

i.e.,

$$T^\pm(\tilde{f}_\pm, u) := \pm(\gamma^{-1})^*(\tilde{f}_\pm - \check{A}_{\Omega_\pm}u). \tag{2.13}$$

If $a \equiv 1$, then $A = \Delta$, and $T^\pm(\tilde{f}_\pm; u)$ become *generalised normal derivatives* denoted as $T^\pm_\Delta(\tilde{f}_\pm; u)$.

The operator $(\gamma^{-1})^* : H^{-t}(\mathbb{R}^n) \to H^{-t+\frac{1}{2}}(\partial\Omega)$ is dual to $\gamma^{-1} : H^{t-\frac{1}{2}}(\partial\Omega) \to H^t(\mathbb{R}^n)$ and is defined as $\langle(\gamma^{-1})^*\psi, w\rangle_{\partial\Omega} := \langle\psi, \gamma^{-1}w\rangle_{\mathbb{R}^n}$ for any $w \in H^{t-\frac{1}{2}}$, $\psi \in H^{-t}(\mathbb{R}^n)$, $1/2 < t < 3/2$. In (2.13) it was employed for $t = 2 - s$.

Theorem 2.9 (Lemma 4.3 in [27], Theorem 3.2 in [31], and Theorem 5.3 in [32]) *Under the hypotheses of Definition* 2.8, *the generalised co-normal derivatives* $T^\pm u(\tilde{f}_\pm; u)$ *are independent of* (*non-unique*) *choice of the operator* γ^{-1}, *and we have the estimate*

$$\|T^\pm(\tilde{f}_\pm; u)\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \le C_1\|u\|_{H^s(\Omega_\pm)} + C_2\|\tilde{f}_\pm\|_{\widetilde{H}^{s-2}(\Omega_\pm)} \tag{2.14}$$

and the first Green identity in the form

$$\begin{aligned}\pm\langle T^\pm(\tilde{f}_\pm; u), \gamma^\pm v\rangle_{\partial\Omega} &= \langle\tilde{f}_\pm, v\rangle_{\Omega_\pm} + \check{\mathcal{E}}_{\Omega_\pm}(u, v) \\ &= \langle\tilde{f}_\pm - \check{A}_{\Omega_\pm}u, v\rangle_{\Omega_\pm}, \quad \forall\, v \in H^{2-s}(\Omega_\pm).\end{aligned} \tag{2.15}$$

As follows from Definition 2.8, the generalized co-normal derivative is nonlinear with respect to u for fixed $\tilde{f}_\pm$ but still linear with respect to the couple $(\tilde{f}_\pm, u)$, i.e., for any complex numbers α_1 and α_2,

$$\begin{aligned}\alpha_1 T^\pm(\tilde{f}_{1\pm}; u_1) + \alpha_2 T^+(\tilde{f}_{2\pm}; u_2) &= T^\pm(\alpha_1\tilde{f}_{1\pm}; \alpha_1 u_1) + T^\pm(\alpha_2\tilde{f}_{2\pm}; \alpha_2 u_2) \\ &= T^\pm(\alpha_1\tilde{f}_{1\pm} + \alpha_2\tilde{f}_{2\pm}; \alpha_1 u_1 + \alpha_2 u_2).\end{aligned}$$

Let us also define some subspaces of $H^s(\Omega_\pm)$; see [11, 15, 31, 32].

Definition 2.10 Let $s \in \mathbb{R}$, and let $A_* : H^s(\Omega_\pm) \to \mathcal{D}^*(\Omega_\pm)$ be a linear operator. For $t \in \mathbb{R}$, we introduce the space

$$H^{s,t}(\Omega_\pm; A_*) := \{g : g \in H^s(\Omega_\pm), A_*g \in \widetilde{H}^t_\bullet(\Omega_\pm)\}$$

endowed with the norm $\|g\|_{H^{s,t}(\Omega_\pm;A_*)} := (\|g\|^2_{H^s(\Omega_\pm)} + \|A_*g\|^2_{\widetilde{H}^t_\bullet(\Omega_\pm)})^{1/2}$ and the corresponding inner product.

Definition 2.11 Let Ω_0 be either Ω_+ or Ω_-. By Remark 2.3, if $g \in H^{s,t}(\Omega_0;A_*)$ for some $s \in \mathbb{R}$ and $t \geq -\frac{1}{2}$, then there exists a *unique* distribution $\tilde{f} \in \widetilde{H}^t(\Omega_0)$ such that $r_{\Omega_0}\tilde{f} = A_* g$, and hence $\tilde{f} = \tilde{A}_{*\Omega_0} g$, where $\tilde{A}_{*\Omega_0} := r^{-1}_{\Omega_0} A_*$. The operator $\tilde{A}_{*\Omega_0} : H^{s,t}(\Omega_0;A_*) \to \widetilde{H}^t(\Omega_0)$, which is continuous by Corollary 2.4, is called the *canonical* extension of the operator $A_* : H^{s,t}(\Omega_0;A_*) \to \widetilde{H}^t_\bullet(\Omega_0)$, and moreover, if $-\frac{1}{2} < t < \frac{1}{2}$, then $\tilde{A}_{*\Omega_0} = \mathring{E}_{\Omega_0} A_*$.

We will mostly use the operators A or Δ as A_* in the definition. Note that since $Au = a\Delta u + \nabla a \cdot \nabla u$, for $1/2 < s < 3/2$, we have $H^{s,-\frac{1}{2}}(\Omega_0;A) = H^{s,-\frac{1}{2}}(\Omega_0;\Delta)$ if $a \in C^{\frac{3}{2}}_+(\overline{\Omega_0})$, with equivalent norms.

Let us now define the *canonical* conormal derivative; see [32, Definition 6.5].

Definition 2.12 For $u \in H^{s,-\frac{1}{2}}(\Omega_\pm;A)$ and $a \in C^{|s-1|}_+(\overline{\Omega}_\pm)$, $1/2 < s < 3/2$, we define the *canonical co-normal derivatives* $T^\pm u \in H^{s-\frac{3}{2}}(\partial\Omega)$ as

$$\begin{aligned} \pm\langle T^\pm u, w\rangle_{\partial\Omega} &:= \langle \tilde{A}_{\Omega_\pm} u, \gamma^{-1} w\rangle_{\Omega_\pm} + \check{\mathcal{E}}_{\Omega_\pm}(u, \gamma^{-1} w) = \langle \tilde{A}_{\Omega_\pm} u - \check{A}_{\Omega_\pm} u, \gamma^{-1} w\rangle_{\Omega_\pm} \\ &= \langle (\gamma^{-1})^* (\tilde{A}_{\Omega_\pm} u - \check{A}_{\Omega_\pm} u), w\rangle_{\partial\Omega} \quad \forall\, w \in H^{\frac{3}{2}-s}(\partial\Omega), \end{aligned} \tag{2.16}$$

i.e,

$$T^\pm u := \pm(\gamma^{-1})^* (\tilde{A}_{\Omega_\pm} u - \check{A}_{\Omega_\pm} u). \tag{2.17}$$

If $a \equiv 1$, then $T^\pm u$ becomes the *canonical normal derivative* denoted as $T^\pm_\Delta u$.

Theorem 2.13 (Theorem 3.9 in [31] and Theorem 6.6 in [32]) *Under the hypotheses of Definition* 2.12, *the canonical co-normal derivatives $T^\pm u$ are independent of* (*non-unique*) *choice of the operator γ^{-1}, the operators $T^\pm : H^{s,-\frac{1}{2}}(\Omega_\pm;A) \to H^{s-\frac{3}{2}}(\partial\Omega)$ are continuous, and the first Green identity holds in the form*

$$\begin{aligned} \pm\langle T^\pm u, \gamma^\pm v\rangle_{\partial\Omega} &= \langle \tilde{A}_{\Omega_\pm} u, v\rangle_{\Omega_\pm} + \check{\mathcal{E}}_{\Omega_\pm}(u, v) \\ &= \langle \tilde{A}_{\Omega_\pm} u - \check{A}_{\Omega_\pm} u, v\rangle_{\Omega_\pm}, \quad \forall\, v \in H^{2-s}(\Omega_\pm). \end{aligned} \tag{2.18}$$

The canonical co-normal derivatives in Definition 2.12 are completely defined by the function u and operator A only and do not depend explicitly on the right-hand sides $\tilde{f}_\pm$, unlike the generalised co-normal derivatives defined in (2.15), whereas the operators $T^\pm$ are linear in u. Note that the canonical co-normal derivatives coincide with the classical co-normal derivatives $T^\pm u = T^{c\pm} u$ if the latter do exist (see [32, Corollaries 6.11 and 6.14]), which is generally not the case for the generalised conormal derivatives even for smooth functions u, unless $\tilde{f}_\pm = \tilde{A}_{\Omega_\pm} u$ is chosen. Thus the canonical conormal derivative is a continuous extension of the classical conormal derivative.

Let $1/2 < s < 3/2$ and $a \in C^{|s-1|}_+(\overline{\Omega}_\pm)$. If $u \in H^{s,-\frac{1}{2}}(\Omega_\pm;A)$, then Definitions 2.8 and 2.12 imply that the generalised co-normal derivative for arbitrary extensions $\tilde{f}_\pm \in \widetilde{H}^{s-2}(\Omega_\pm)$ of the distributions $r_{\Omega_\pm} Au$ can be expressed as

$$T^\pm(\tilde{f}_\pm; u) = T^\pm u \pm (\gamma^{-1})^* (\tilde{f}_\pm - \tilde{A}_{\Omega_\pm} u). \tag{2.19}$$

If $u \in H^s(\Omega_\pm)$ and $v \in H^{2-s,-\frac{1}{2}}(\Omega_\pm;A)$, then swapping over the roles of u and v in (2.18), we obtain the first Green identity for v:

$$\pm\langle T^\pm v, \gamma^\pm u\rangle_{\partial\Omega} = \check{\mathcal{E}}_{\Omega_\pm}(v,u) + \langle \tilde{A}_{\Omega_\pm} v, u\rangle_{\Omega_\pm}. \tag{2.20}$$

If, in addition, $r_{\Omega_\pm}Au = r_{\Omega_\pm}\tilde{f}_\pm$, where $\tilde{f}_\pm \in \widetilde{H}^{s-2}(\Omega_\pm)$, then subtracting (2.20) from (2.15) and taking into account that $\check{\mathcal{E}}_{\Omega_\pm}(u,v) = \check{\mathcal{E}}_{\Omega_\pm}(v,u)$ by Remark 2.7, we obtain the following second Green identity:

$$\langle \tilde{f}_\pm, v\rangle_{\Omega_\pm} - \langle \tilde{A}_{\Omega_\pm} v, u\rangle_{\Omega_\pm} = \pm\langle T^\pm(\tilde{f}_\pm;u), \gamma^\pm v\rangle_{\partial\Omega} \mp \langle T^\pm v, \gamma^\pm u\rangle_{\partial\Omega}.$$

If, finally, $u \in H^{s,-\frac{1}{2}}(\Omega_\pm;A)$ and $v \in H^{2-s,-\frac{1}{2}}(\Omega_\pm;A)$, then we arrive at the familiar form of the second Green identity for the canonical extension $\tilde{A}$ of the operator A and the canonical co-normal derivatives

$$\langle \tilde{A}_{\Omega_\pm} u, v\rangle_{\Omega_\pm} - \langle \tilde{A}_{\Omega_\pm} v, u\rangle_{\Omega_\pm} = \pm\langle T^\pm u, \gamma^\pm v\rangle_{\partial\Omega} \mp \langle T^\pm v, \gamma^\pm u\rangle_{\partial\Omega}. \tag{2.21}$$

3 Parametrix and potential type operators on Lipschitz domains

Recall that unless stated otherwise, we will assume that $\Omega = \Omega_+$.

We will say that a function $P(x,y)$ of two variables $x, y \in \mathbb{R}^n$ is a parametrix (the Levi function) for the operator $A(x,\partial_x)$ in $\mathbb{R}^n$ if (see, e.g., [19, 20, 25, 29, 38, 43, 44])

$$A(x,\nabla_x)P(x,y) = \delta(x-y) + R(x,y), \tag{3.1}$$

where $\delta(\cdot)$ is the Dirac distribution, and $R(x,y)$ possesses a weak (integrable) singularity at $x = y$, i.e.,

$$R(x,y) = \mathcal{O}\big(|x-y|^{-\varkappa}\big) \quad \text{with } \varkappa < n. \tag{3.2}$$

Let $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ denote the area of the unit sphere in $\mathbb{R}^n$. It is well known that the function

$$P_\Delta(x,y) = \frac{-1}{(n-2)\omega_n|x-y|^{n-2}}, \quad x,y \in \mathbb{R}^n, \tag{3.3}$$

is the fundamental solution of the Laplace equation, i.e., $\Delta_x P_\Delta(x,y) = \Delta_y P_\Delta(x,y) = \delta(x-y)$.

It is easy to see that for the operator $A(x,\partial_x)$ given by the left-hand side in (2.6), the function

$$P(x,y) = \frac{1}{a(y)}P_\Delta(x,y) = \frac{-1}{(n-2)\omega_n a(y)|x-y|^{n-2}}, \quad x,y \in \mathbb{R}^n, \tag{3.4}$$

is a parametrix, whereas the corresponding remainder function is

$$\begin{aligned} R(x,y) &= \nabla a(x) \cdot \nabla_x P(x,y) = \frac{1}{a(y)}\nabla a(x) \cdot \nabla_x P_\Delta(x,y) \\ &= \frac{(x-y)\cdot\nabla a(x)}{\omega_n a(y)|x-y|^n}, \quad x,y \in \mathbb{R}^n, \end{aligned} \tag{3.5}$$

and if $a \in C^1_+(\mathbb{R}^n)$, then it satisfies estimate (3.2) a.e. with $\varkappa = n - 1$. Note also that

$$A(y, \nabla_y)P(x, y) = \delta(x - y) + R_*(x, y), \tag{3.6}$$

where

$$\begin{aligned} R_*(x, y) &= -\nabla_y \cdot \big(P(x, y)\nabla a(y)\big) \\ &= \frac{\Delta(\ln a(y))}{(n-2)\omega_n |x - y|^{n-2}} - \frac{(x - y) \cdot \nabla a(y)}{\omega_n a(y)|x - y|^n}, \quad x, y \in \mathbb{R}^n. \end{aligned} \tag{3.7}$$

Evidently, the parametrix $P(x, y)$ given by (3.4) is related to the fundamental solution to the operator $A(y, \nabla_x) := a(y)\Delta_x$ with "frozen" coefficient $a(y)$, and $A(y, \nabla_x)P(x, y) = \delta(x - y)$.

Note that parametrix (3.4) and remainders (3.5) and (3.7) are not smooth enough for the corresponding potential operators to be directly treated as in [27], which thus need some additional consideration.

For $g \in \mathcal{D}(\mathbb{R}^n)$ and sufficiently smooth coefficient a, the parametrix-based volume potential operator and the remainder potential operator corresponding to parametrix (3.4) and remainders (3.5) and (3.7) for $y \in \mathbb{R}^n$ are

$$\mathbf{P}g(y) := \langle P(\cdot, y), g\rangle_{\mathbb{R}^n} = \int_{\mathbb{R}^n} P(x, y)g(x)\, dx, \tag{3.8}$$

$$\mathbf{R}g(y) := \langle R(\cdot, y), g\rangle_{\mathbb{R}^n} = \int_{\mathbb{R}^n} R(x, y)g(x)\, dx, \tag{3.9}$$

$$\mathbf{R}_*g(y) := \langle R_*(\cdot, y), g\rangle_{\mathbb{R}^n} = \int_{\mathbb{R}^n} R_*(x, y)g(x)\, dx, \tag{3.10}$$

and from (3.1)–(3.10) we obtain,

$$\mathbf{P}Ag = g + \mathbf{R}g, \qquad A\mathbf{P}g = g + \mathbf{R}_*g \quad \text{in } \mathbb{R}^n. \tag{3.11}$$

For the function g defined on a domain $\Omega_+ \subset \mathbb{R}^n$, e.g., $g \in \mathcal{D}(\overline{\Omega}_+)$, the corresponding potentials for $y \in \Omega_+$ are

$$\mathcal{P}g(y) := \langle P(\cdot, y), g\rangle_{\Omega_+} = \int_{\Omega_+} P(x, y)g(x)\, dx, \tag{3.12}$$

$$\mathcal{R}g(y) := \langle R(\cdot, y), g\rangle_{\Omega_+} = \int_{\Omega_+} R(x, y)g(x)\, dx, \tag{3.13}$$

$$\mathcal{R}_*g(y) := \langle R_*(\cdot, y), g\rangle_{\Omega_+} = \int_{\Omega_+} R_*(x, y)g(x)\, dx. \tag{3.14}$$

From definitions (3.4), (3.5), and (3.7) we can obtain representations of the parametrix-based potential operators in terms of their counterparts for $a = 1$ (i.e., associated with the Laplace operator Δ; see, e.g., [21]), which we equip with the subscript Δ (see [3]):

$$\mathbf{P}g = \frac{1}{a}\mathbf{P}_\Delta g, \qquad \mathbf{R}g = -\frac{1}{a}\nabla \cdot \mathbf{P}_\Delta(g\nabla a), \qquad \mathbf{R}_*g = -\nabla \cdot \left(\frac{\nabla a}{a}\mathbf{P}_\Delta g\right), \tag{3.15}$$

$$\mathcal{P}g = \frac{1}{a}\mathcal{P}_\Delta g, \qquad \mathcal{R}g = -\frac{1}{a}\nabla \cdot \mathcal{P}_\Delta(g\nabla a), \qquad \mathcal{R}_*g = -\nabla \cdot \left(\frac{\nabla a}{a}\mathcal{P}_\Delta g\right). \tag{3.16}$$

Hence

$$\Delta(a\mathbf{P}g) = g \quad \text{in } \mathbb{R}^n, \qquad \Delta(a\mathcal{P}g) = g \quad \text{in } \Omega. \tag{3.17}$$

Employing relations (3.16) and the well-known properties of the operator $\mathbf{P}_\Delta$ as the pseudo-differential operator of order −2 together with Theorem 2.6, definitions (3.8)–(3.10) can be extended to $g \in H^s(\mathbb{R}^n)$, $g \in \widetilde{H}^s(\Omega)$ and lower-smoothness coefficient a. For $g \in \widetilde{H}^s(\Omega)$ and $g \in H^s(\Omega)$, the potentials $\mathcal{P}$, $\mathcal{R}$, $\mathcal{R}_*$ defined on functions (or distributions) having support on $\overline{\Omega}$ are understood as

$$\mathcal{P}g := r_\Omega \mathbf{P}g, \qquad \mathcal{R}g := r_\Omega \mathbf{R}g, \qquad \mathcal{R}_*g := r_\Omega \mathbf{R}_*g, \qquad g \in \widetilde{H}^s(\Omega),\ s \in \mathbb{R}; \tag{3.18}$$

$$\mathcal{P}g := r_\Omega \mathbf{P}\mathring{E}_\Omega g, \quad \mathcal{R}g := r_\Omega \mathbf{R}\mathring{E}_\Omega g, \quad \mathcal{R}_*g := r_\Omega \mathbf{R}_*\mathring{E}_\Omega g, \quad g \in H^s(\Omega),\ s > -\frac{1}{2}. \tag{3.19}$$

To prove mapping properties of the parametrix-based volume potential operators in Sobolev spaces, we first provide some well-known results for the classical Newtonian volume potential associated with the Laplace operator.

Lemma 3.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. The following operators are continuous*:

$$\mu\mathbf{P}_\Delta : H^{s-2}(\mathbb{R}^n) \to H^s(\mathbb{R}^n), \quad s \in \mathbb{R},\ \forall\, \mu \in \mathcal{D}(\mathbb{R}^n); \tag{3.20}$$

$$\mathcal{P}_\Delta : \widetilde{H}^{s-2}(\Omega) \to H^s(\Omega), \quad s \in \mathbb{R}; \tag{3.21}$$

$$\mathcal{P}_\Delta : H^{s-2}(\Omega) \to H^s(\Omega), \quad \frac{3}{2} < s < \frac{5}{2}; \tag{3.22}$$

$$\mathcal{P}_\Delta : \widetilde{H}^{s-2}(\Omega) \to H^{s,-\frac{1}{2}}(\Omega;\Delta), \quad s \geq \frac{3}{2}; \tag{3.23}$$

$$\gamma^+\mathcal{P}_\Delta : \widetilde{H}^{s-2}(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \frac{1}{2} < s < \frac{3}{2}; \tag{3.24}$$

$$\gamma^+\mathcal{P}_\Delta : H^s(\Omega) \to H^1(\partial\Omega), \quad -\frac{1}{2} < s; \tag{3.25}$$

$$T^+_\Delta\mathcal{P}_\Delta : \widetilde{H}^s(\Omega) \to L_2(\partial\Omega), \quad -\frac{1}{2} < s; \tag{3.26}$$

$$T^+_\Delta\mathcal{P}_\Delta : H^s(\Omega) \to L_2(\partial\Omega), \quad -\frac{1}{2} < s. \tag{3.27}$$

If $\frac{1}{2} < s < \frac{3}{2}$, $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, and $\tilde{f}_0 \in \widetilde{H}^{s-2}(\Omega)$ is such that $r_\Omega\tilde{f}_0 = r_\Omega\tilde{f}$, then there exist constants $C_0, C_1 > 0$ such that

$$\left\| T^+_\Delta(\tilde{f}_0; \mathcal{P}_\Delta\tilde{f}) \right\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \leq C_1\|\tilde{f}\|_{\widetilde{H}^{s-2}(\Omega)} + C_2\|\tilde{f}_0\|_{\widetilde{H}^{s-2}(\Omega)}. \tag{3.28}$$

Proof Operator (3.20) and hence (3.21) are continuous since $\mathbf{P}_\Delta$ is a pseudo-differential operator of order −2. The continuity of operator (3.22) follows from the first relation in (3.19) for $\mathcal{P}_\Delta$ and $\mathbf{P}_\Delta$ and from (3.21). Since $\Delta\mathcal{P}_\Delta g = g$ in Ω, the continuity of operator (3.21) implies that of operator (3.23).

The continuity of operator (3.24) is implied by that of operator (3.21) and the trace theorem for Lipschitz domains; see, e.g., [11, Lemma 3.6] and [27, Theorem 3.38]. The continuity of operator (3.25) follows from that of operator (3.22) and from, e.g., [54], [31, Lemma 2.5] for $-\frac{1}{2} < s < \frac{1}{2}$, and then by the embedding argument for $s \geq \frac{1}{2}$.

The continuity of operators (3.26) and (3.27) is implied by that of (3.21) and (3.22), respectively, and by [31, Corollary 3.14] since $s + 2 > \frac{3}{2}$ in both cases. Estimate (3.28) follows from the continuity of operator (3.21), relation $\Delta\mathcal{P}_\Delta\tilde{f} = \tilde{f}$, and estimate (2.14). □

Now the following mapping properties of the parametrix-based operators can be obtained.

Theorem 3.2 *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$. *The following operators are continuous*:

$$\mu\mathbf{P}: H^{s-2}(\mathbb{R}^n) \to H^s(\mathbb{R}^n), \quad s \in \mathbb{R},\ a \in C_+^{|s|}(\mathbb{R}^n),\ \forall\, \mu \in \mathcal{D}(\mathbb{R}^n); \tag{3.29}$$

$$\mathcal{P}: \widetilde{H}^{s-2}(\Omega) \to H^s(\Omega), \quad s \in \mathbb{R},\ a \in C_+^{|s|}(\overline{\Omega}); \tag{3.30}$$

$$\mathcal{P}: H^{s-2}(\Omega) \to H^s(\Omega), \quad \frac{3}{2} < s < \frac{5}{2},\ a \in C_+^{s}(\overline{\Omega}); \tag{3.31}$$

$$\mathcal{P}: \widetilde{H}^{s-2}(\Omega) \to H^{s,-\frac{1}{2}}(\Omega;A), \quad \frac{3}{2} \leq s,\ a \in C_+^{s}(\overline{\Omega}); \tag{3.32}$$

$$\mu\mathbf{R}: H^{s-1}(\mathbb{R}^n) \to H^s(\mathbb{R}^n), \quad s \in \mathbb{R},\ a \in C_+^{|s-1|+1}(\mathbb{R}^n),\ \forall\, \mu \in \mathcal{D}(\mathbb{R}^n); \tag{3.33}$$

$$\mathcal{R}: H^{s-1}(\Omega) \to H^s(\Omega), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^{|s-1|+1}(\overline{\Omega}); \tag{3.34}$$

$$\mathcal{R}: H^{s}(\Omega) \to H^s(\Omega), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^{s}(\overline{\Omega}); \tag{3.35}$$

$$\mathcal{R}: H^{s}(\Omega) \to H^{s,-\frac{1}{2}}(\Omega;A), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^{\frac{3}{2}}(\overline{\Omega}); \tag{3.36}$$

$$\mu\mathbf{R}_*: H^{s}(\mathbb{R}^n) \to H^{s+1}(\mathbb{R}^n), \quad s \in \mathbb{R},\ a \in C_+^{|s+2|+1}(\mathbb{R}^n),\ \forall\, \mu \in \mathcal{D}(\mathbb{R}^n); \tag{3.37}$$

$$\mathcal{R}_*: \widetilde{H}^{s}(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R},\ a \in C_+^{|s+2|+1}(\overline{\Omega}); \tag{3.38}$$

$$\mathcal{R}_*: \widetilde{H}^{s}(\Omega) \to H^{\sigma}(\Omega), \quad -\frac{3}{2} < s,\ a \in C_+^{\frac{3}{2}}(\overline{\Omega}),\ \textit{for some}\ \sigma > -\frac{1}{2}; \tag{3.39}$$

$$\gamma^+\mathcal{P}: \widetilde{H}^{s-2}(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^{s}(\overline{\Omega}); \tag{3.40}$$

$$\gamma^+\mathcal{P}: H^{s}(\Omega) \to H^{1}(\partial\Omega), \quad -\frac{1}{2} < s,\ a \in C_+^{\frac{3}{2}}(\overline{\Omega}); \tag{3.41}$$

$$\gamma^+\mathcal{R}: H^{s}(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^{s}(\overline{\Omega}); \tag{3.42}$$

$$T^+\mathcal{P}: \widetilde{H}^{s}(\Omega) \to L_2(\partial\Omega), \quad -\frac{1}{2} < s,\ a \in C_+^{\frac{3}{2}}(\overline{\Omega}); \tag{3.43}$$

$$T^+\mathcal{P}: H^{s}(\Omega) \to L_2(\partial\Omega), \quad -\frac{1}{2} < s,\ a \in C_+^{\frac{3}{2}}(\overline{\Omega}); \tag{3.44}$$

$$T^+\mathcal{R}: H^{s}(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad \frac{1}{2} < s < \frac{3}{2},\ a \in C_+^{\frac{3}{2}}(\overline{\Omega}). \tag{3.45}$$

Moreover, operators (3.35), (3.36), (3.42), *and* (3.45) *are compact.*

If $\frac{1}{2} < s < \frac{3}{2}$, $a \in C^s_+(\overline{\Omega})$, $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, *and* $\tilde{f}_0 \in \widetilde{H}^{s-2}(\Omega)$ *is such that* $r_\Omega \tilde{f}_0 = r_\Omega A\mathcal{P}\tilde{f}$, *then there exist constants* $C_0, C_1 > 0$ *such that*

$$\left\| T^+(\tilde{f}_0; \mathcal{P}\tilde{f})\right\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \le C_1 \|\tilde{f}\|_{\widetilde{H}^{s-2}(\Omega)} + C_2 \|\tilde{f}_0\|_{\widetilde{H}^{s-2}(\Omega)}. \tag{3.46}$$

Proof The continuity of operators (3.29)–(3.31) is implied by the first relations in (3.15) and (3.16) and by the continuity of operators (3.20)–(3.22) together with Theorem 2.6.

The continuity of operators (3.30) and (3.31) and Remark 2.2(4) imply that of operator (3.32) for $s > \frac{3}{2}$. Let us now prove (3.32) for $s = \frac{3}{2}$. For $g \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$, we have, $\mathcal{P}g \in H^{\frac{3}{2}}(\Omega)$ due to (3.30), whereas

$$A\mathcal{P}g = \nabla \cdot \left(a\nabla\left[\frac{1}{a}\mathcal{P}_\Delta g\right]\right) = g - \nabla \cdot \left[(\nabla \ln a)\mathcal{P}_\Delta g\right] \quad \text{in } \Omega, \tag{3.47}$$

where we have taken into account that $\Delta\mathcal{P}_\Delta g = g$. The first term in the right-hand side of (3.47) belongs to $\widetilde{H}^{-\frac{1}{2}}_\bullet(\Omega)$, whereas the second term belongs to $H^\sigma(\Omega) = \widetilde{H}^\sigma_\bullet(\Omega)$ for some $\sigma \in (-1/2, 1/2)$ (cf. item 4 in Remark 2.2) since $\nabla a \in C^{\frac{1}{2}}_+(\overline{\Omega})$ and $a \ge a_{\min} > 0$, which completes the proof of the continuity of operator (3.32).

The continuity of operator (3.33) follows from the second relation in (3.15) together with Theorem 2.6 and the continuity of operator (3.20). Indeed, let us take arbitrary $\mu \in \mathcal{D}(\mathbb{R}^n)$, let B_μ be a ball such that $\operatorname{supp}\mu \subset B_\mu$, and let $\mu_1 \in \mathcal{D}(\mathbb{R}^n)$ be such that $\mu_1 = 1$ in B_μ. Then for any $g \in H^{s-1}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|\mu\mathbf{R}g\|_{H^s(\mathbb{R}^n)} &= \left\| \frac{\mu}{a}\nabla \cdot \left(\mu_1 \mathbf{P}_\Delta(g\nabla a)\right)\right\|_{H^s(\mathbb{R}^n)} \le c_1 \left\| \nabla \cdot \left(\mu_1 \mathbf{P}_\Delta(g\nabla a)\right)\right\|_{H^s(\mathbb{R}^n)} \\ &\le c_2 \left\| \mu_1 \mathbf{P}_\Delta(g\nabla a)\right\|_{H^{s+1}(\mathbb{R}^n)} \le c_3 \|g\nabla a\|_{H^{s-1}(\mathbb{R}^n)} \le c_4 \|g\|_{H^{s-1}(\mathbb{R}^n)},\end{aligned} \tag{3.48}$$

where c_i are positive constants (depending on μ, μ_1, and a), and we took into account that $C^{|s-1|+1}_+(\mathbb{R}^n) \subset C^{|s|}_+(\mathbb{R}^n)$ since $|s| \le |s-1| + 1$.

To prove the continuity of operator (3.34), we similarly employ the second relation in (3.16) together with Theorem 2.6 and the continuity of operator (3.22). Then we obtain for any $g \in H^{s-1}(\Omega)$, $1/2 < s < 3/2$, and some positive constants c_i:

$$\begin{aligned}\|\mathcal{R}g\|_{H^s(\Omega)} &= \left\| \frac{1}{a}\nabla \cdot \mathcal{P}_\Delta(g\nabla a)\right\|_{H^s(\Omega)} \le c_1 \left\| \nabla \cdot \mathcal{P}_\Delta(g\nabla a)\right\|_{H^s(\Omega)} \\ &\le c_2 \left\| \mathcal{P}_\Delta(g\nabla a)\right\|_{H^{s+1}(\Omega)} \le c_3 \|g\nabla a\|_{H^{s-1}(\Omega)} \le c_4 \|g\|_{H^{s-1}(\Omega)}.\end{aligned} \tag{3.49}$$

Let us prove the continuity and compactness of operator (3.35). For $1 \le s < \frac{3}{2}$, we have $s = |s-1| + 1$, and then the continuity of operator (3.34) implies the continuity and compactness of (3.35). For $\frac{1}{2} < s < 1$, we need a sharper estimate of the norm $\|g\nabla a\|_{H^{s-1}(\Omega)}$. First, by Definition 2.5 the inclusion $a \in C^s_+(\overline{\Omega})$ implies that there exists $t \in (s, 1)$ such that $a \in C^{0,t}(\overline{\Omega}) = B^t_{\infty,\infty}(\Omega) = F^t_{\infty,\infty}(\Omega)$; see, e.g., Proposition in [45, Sect. 2.1.2], and hence $\nabla a \in F^{t-1}_{\infty,\infty}(\Omega)$. Then, by Theorems 1 from [45, Sect. 4.4.3] we have

$$\begin{aligned}\|g\nabla a\|_{F^{t-1}_{2,\infty}(\Omega)} &\le C\|\nabla a\|_{F^{t-1}_{\infty,\infty}(\Omega)} \|g\|_{H^\sigma(\Omega)} \\ &\le C\|a\|_{C^{0,t}(\overline{\Omega})} \|g\|_{H^\sigma(\Omega)}, \quad \forall\, \sigma \in (1-t, s).\end{aligned} \tag{3.50}$$

On the other hand, by (3.49), item (ii) of Proposition from [45, Sect. 2.2.1], and (3.50) we obtain

$$\|\mathcal{R}g\|_{H^s(\Omega)} \le c_3\|g\nabla a\|_{H^{s-1}(\Omega)} = c_3\|g\nabla a\|_{F^{s-1}_{2,2}(\Omega)}$$
$$\le C_1\|g\nabla a\|_{F^{t-1}_{2,\infty}(\Omega)} \le C_1 C\|a\|_{C^{0,t}(\overline{\Omega})}\|g\|_{H^\sigma(\Omega)}.$$

Thus the operator $\mathcal{R}: H^\sigma(\Omega) \to H^s(\Omega)$ is continuous, which implies the continuity and, by the Rellich compact embedding theorem, also the compactness of operator (3.35) for $\frac{1}{2} < s < 1$.

Let us prove the continuity of operator (3.36). Since $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, by Definition 2.5 there exists $\epsilon > 0$ such that $a \in C^{1,\frac{1}{2}+\epsilon}(\overline{\Omega})$, and let us choose any $\sigma \in (\frac{1}{2}, \min\{s, \frac{1}{2}+\epsilon\})$. By the continuity of (3.34) the operator $\mathcal{R}: H^\sigma(\Omega) \to H^s(\Omega)$ is continuous. Now let us prove that the operator $A\mathcal{R}: H^\sigma(\Omega) \to \widetilde{H}_\bullet^{-\frac{1}{2}}(\Omega)$ is continuous as well. Indeed, for some positive constants c_i, we have

$$\begin{aligned}
\|A\mathcal{R}g\|_{\widetilde{H}_\bullet^{-\frac{1}{2}}(\Omega)} &\le \|A\mathcal{R}g\|_{\widetilde{H}_\bullet^{\sigma-1}(\Omega)} \\
&\le c_0\|A\mathcal{R}g\|_{H^{\sigma-1}(\Omega)} \\
&= c_0\left\|\nabla\cdot\left[a\nabla\left(\frac{1}{a}\nabla\cdot\mathcal{P}_\Delta(g\nabla a)\right)\right]\right\|_{H^{\sigma-1}(\Omega)} \\
&= c_0\left\|-\nabla\cdot\left[(\nabla\ln a)\nabla\cdot\mathcal{P}_\Delta(g\nabla a)\right] + \Delta\left(\nabla\cdot\mathcal{P}_\Delta(g\nabla a)\right)\right\|_{H^{\sigma-1}(\Omega)} \\
&\le c_1\left\|-(\nabla\ln a)\nabla\cdot\mathcal{P}_\Delta(g\nabla a) + (g\nabla a)\right\|_{H^\sigma(\Omega)} \\
&\le c_2\|a\|_{C^{1,\frac{1}{2}+\epsilon}}\left\|\mathcal{P}_\Delta(g\nabla a)\right\|_{H^{\sigma+1}(\Omega)} + c_1\|g\nabla a\|_{H^\sigma(\Omega)} \\
&\le c_3\|g\nabla a\|_{H^{\sigma-1}(\Omega)} + c_1\|g\nabla a\|_{H^\sigma(\Omega)} \le c_4\|g\nabla a\|_{H^\sigma(\Omega)} \\
&\le c_5\|a\|_{C^{1,\frac{1}{2}+\epsilon}}\|g\|_{H^\sigma(\Omega)}.
\end{aligned}$$

Hence we proved the continuity of the operator $H^\sigma(\Omega) \to H^{s,-\frac{1}{2}}(\Omega; A)$, which implies that of operator (3.36) and by the Rellich compact embedding theorem also its compactness.

The continuity of operator (3.37) is implied by the last relation in (3.15), the continuity of operator (3.20), and Theorem 2.6 in the chain of inequalities analogous to (3.48). Similarly, the continuity of operator (3.38) is implied by the last relation in (3.16), the continuity of operator (3.21), and Theorem 2.6. The continuity of operator (3.39) is implied by that of (3.38) since $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$ implies that there exists $\epsilon > 0$ such that $a \in C^{1,1/2+\epsilon}(\overline{\Omega})$, and we can take $\sigma \in (\frac{3}{2}, \min\{s+1, \frac{3}{2}+\epsilon\})$.

The continuity of operator (3.40) is implied by that of operator (3.30) and the trace theorem for Lipschitz domains; see, e.g., [11, Lemma 3.6], [27, Theorem 3.38]. The continuity of operator (3.41) for $-\frac{1}{2} < s < -\frac{1}{2}+\epsilon$ with any sufficiently small $\epsilon > 0$ follows from that of operator (3.31) together with, e.g., [54], [31, Lemma 2.5] and then by the embedding argument for all $s > -\frac{1}{2}$. Similarly, the continuity of operators (3.43) and (3.44) is implied by that of (3.30) and (3.31), respectively, and by [31, Corollary 3.14] since $s + 2 > \frac{3}{2}$ in the both cases.

The continuity and compactness of operators (3.42) and (3.45) are implied by those of operators (3.35) and (3.36), the trace theorem for Lipschitz domains, and Theorem 2.9.

Estimate (3.46) follows from the continuity of operator (3.30) and estimate (2.14). □

The parametrix-based single- and double-layer surface potential operators are defined as

$$Vg(y) := -\int_{\partial\Omega} P(x,y)\psi(x)\,dS_x, \quad y \notin \partial\Omega, \tag{3.51}$$

$$Wg(y) := -\int_{\partial\Omega} \bigl[T^c\bigl(x,n(x),\partial_x\bigr)P(x,y)\bigr]\varphi(x)\,dS_x, \quad y \notin \partial\Omega, \tag{3.52}$$

where the integrals are understood as duality forms if ψ and φ are not integrable. Particularly, for $\psi \in H^{\frac{1}{2}-s}(\partial\Omega)$ and $\varphi \in H^{\frac{1}{2}-s}(\partial\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, we have

$$\begin{aligned} V\psi(y) &:= -\bigl\langle \gamma P(\cdot,y), \psi\bigr\rangle_{\partial\Omega} = -\bigl\langle P(\cdot,y), \gamma^*\psi\bigr\rangle_{\mathbb{R}^n} \\ &= -\mathbf{P}\gamma^*\psi(y) = -\frac{1}{a(y)}\mathbf{P}_\Delta\gamma^*\psi(y), \end{aligned} \tag{3.53}$$

$$\begin{aligned} W\varphi(y) &:= -\bigl\langle T^c P(\cdot,y), \varphi\bigr\rangle_{\partial\Omega} = -\bigl\langle P(\cdot,y), T^{c*}\varphi\bigr\rangle_{\mathbb{R}^n} \\ &= -\mathbf{P}T^{c*}\varphi(y) = -\frac{1}{a(y)}\mathbf{P}_\Delta T^{c*}\varphi(y), \end{aligned} \tag{3.54}$$

where $\gamma^*\psi$ and $T^{c*}\varphi$ are well defined for any $\psi \in H^{-1}(\partial\Omega)$, $\varphi \in L_2(\partial\Omega)$, and $a \in L_\infty(\partial\Omega)$, in the sense of distributions, as

$$\bigl\langle \gamma^*\psi, \phi\bigr\rangle_{\mathbb{R}^n} := \langle \psi, \gamma\phi\rangle_{\partial\Omega}, \quad \forall\, \phi \in \mathcal{D}\bigl(\mathbb{R}^n\bigr), \quad \text{and}$$

$$\bigl\langle T^{c*}\varphi, \phi\bigr\rangle_{\mathbb{R}^n} := \bigl\langle \varphi, T^c\phi\bigr\rangle_{\partial\Omega} = \bigl\langle \varphi, aT^c_\Delta\phi\bigr\rangle_{\partial\Omega}, \quad \forall\, \phi \in \mathcal{D}\bigl(\mathbb{R}^n\bigr),$$

which evidently implies that $\operatorname{supp}\gamma^*\psi \subset \partial\Omega$ and $\operatorname{supp} T^{c*}\varphi \subset \partial\Omega$. Moreover,

$$\gamma^* : H^{\frac{1}{2}-s}(\partial\Omega) \to H^{-s}_{\partial\Omega}, \qquad T^{c*} : H^{\frac{1}{2}-s}(\partial\Omega) \to H^{-s-1}_{\partial\Omega}, \quad \frac{1}{2} < s < \frac{3}{2}, \tag{3.55}$$

are the continuous operators adjoint, respectively, to the continuous trace operator $\gamma : H^s_{\mathrm{loc}}(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\partial\Omega)$ and to the continuous classical conormal derivative operator $T^c : H^{s+1}_{\mathrm{loc}}(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\partial\Omega)$; for the continuity of T^c and T^{c*}, it is also assumed that $a \in C_+^{s-\frac{1}{2}}(\partial\Omega)$.

When $a = 1$, formulas (3.51) and (3.52) define the corresponding harmonic potentials, which we denote as V_Δ and W_Δ, respectively. From definitions (3.51) and (3.52), similar to (3.15)–(3.16), we have (cf. [3])

$$Vg = \frac{1}{a}V_\Delta g, \qquad Wg = \frac{1}{a}W_\Delta(ag). \tag{3.56}$$

Hence

$$\Delta(aVg) = 0, \qquad \Delta(aWg) = 0 \quad \text{in } \Omega_\pm. \tag{3.57}$$

We will mainly need the restrictions of the layer potentials to Ω, i.e., $r_\Omega V$ and $r_\Omega W$, but will often omit the restriction operator r_Ω if this is clear from the context.

The mapping properties and jump relations for the single- and double-layer potentials are well known for the case $a = const$ and were extended to the case of infinitely smooth boundary and variable coefficient $a(x)$ in [3, 5]. Before proving the corresponding properties for the parametrix-based potentials on Lipschitz domains, we further collect the following well-known mapping and jump properties for the harmonic potentials on Lipschitz domains.

Theorem 3.3 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*

(i) *If $\frac{1}{2} \le s \le \frac{3}{2}$, then the following operators are continuous for any $\mu \in \mathcal{D}(\mathbb{R}^n)$:*

$$\mu V_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\mathbb{R}^n), \tag{3.58}$$

$$r_\Omega W_\Delta : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega), \qquad \mu r_{\Omega_-} W_\Delta : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\overline{\Omega}_-). \tag{3.59}$$

(ii) *If $\frac{1}{2} < s < \frac{3}{2}$, then the following operators are continuous:*

$$\gamma^\pm V_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \qquad \gamma^\pm W_\Delta : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \tag{3.60}$$

$$T_\Delta^\pm V_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \qquad T_\Delta^\pm W_\Delta : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \tag{3.61}$$

(iii) *If $\frac{1}{2} < s < \frac{3}{2}$, then, for any $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$ and $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$, the following jump properties hold:*

$$\gamma^+ V_\Delta\psi - \gamma^- V_\Delta\psi = 0, \qquad \gamma^+ W_\Delta\varphi - \gamma^- W_\Delta\varphi = -\varphi, \tag{3.62}$$

$$T_\Delta^+ V_\Delta\psi - T_\Delta^- V_\Delta\psi = \psi, \qquad T_\Delta^+ W_\Delta\varphi - T_\Delta^- W_\Delta\varphi = 0. \tag{3.63}$$

Proof Items (i) and (ii) follow, e.g., from [11, Theorem 1(i,ii) and Remark], [22–24, 51] (see also [27, Theorem 6.12]) if we take into account that the canonical co-normal derivative operators in (3.61) are well defined since $\Delta V = 0$ and $\Delta W = 0$ in $\Omega_\pm$. The jump properties of item (iii) for $s = 1$ are implied, e.g., by [11, Lemma 4.1]; see also [27, Theorem 6.11]. Hence they evidently hold if $1 \le s < \frac{3}{2}$ and by the density argument also if $\frac{1}{2} < s < 1$. □

Theorem 3.3 implies the following assertion.

Corollary 3.4 *Let $\partial\Omega$ be a compact Lipschitz boundary, and let $\frac{1}{2} < s < \frac{3}{2}$. The following operators are continuous:*

$$\mathcal{V}_\Delta := \gamma^+ V_\Delta = \gamma^- V_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \tag{3.64}$$

$$\mathcal{W}_\Delta := \frac{1}{2}\left(\gamma^+ W_\Delta + \gamma^- W_\Delta\right) : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \tag{3.65}$$

$$\mathcal{W}'_\Delta := \frac{1}{2}\left(T_\Delta^+ V_\Delta + T_\Delta^- V_\Delta\right) : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \tag{3.66}$$

$$\mathcal{L}_\Delta := T_\Delta^+ W_\Delta = T_\Delta^- W_\Delta : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega). \tag{3.67}$$

Employing relations (3.56), Theorem 3.3, and Theorem 2.6, we obtain the following mapping properties for the parametrix-based potentials on Lipschitz domains.

Theorem 3.5 *Let Ω be a bounded Lipschitz domain.*

(i) *The following operators are continuous if* $\frac{1}{2} \le s \le \frac{3}{2}$,

$$\mu V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\mathbb{R}^n), \quad a \in C_+^s(\mathbb{R}^n), \forall \mu \in \mathcal{D}(\mathbb{R}^n); \tag{3.68}$$

$$r_\Omega W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega), \quad a \in C_+^s(\overline{\Omega}); \tag{3.69}$$

$$\mu r_{\Omega_-} W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\overline{\Omega}_-), \quad a \in C_+^s(\overline{\Omega}_-), \forall \mu \in \mathcal{D}(\mathbb{R}^n). \tag{3.70}$$

(ii) *The following operators are continuous if* $\frac{1}{2} < s \le \frac{3}{2}$ *and* $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$:

$$r_\Omega V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s,-\frac{1}{2}}(\Omega; A); \tag{3.71}$$

$$\mu r_{\Omega_-} V : H^{s-\frac{3}{2}}(\partial\Omega) \to H_{\mathrm{loc}}^{s,-\frac{1}{2}}(\Omega_-; A), \quad \forall \mu \in \mathcal{D}(\mathbb{R}^n); \tag{3.72}$$

$$r_\Omega W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s,-\frac{1}{2}}(\Omega; A); \tag{3.73}$$

$$\mu r_{\Omega_-} W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s,-\frac{1}{2}}(\Omega_-; A), \quad \forall \mu \in \mathcal{D}(\mathbb{R}^n). \tag{3.74}$$

(iii) *The following operators are continuous if* $\frac{1}{2} < s < \frac{3}{2}$:

$$\gamma^\pm V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad a \in C_+^s(\overline{\Omega}_\pm); \tag{3.75}$$

$$\gamma^\pm W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad a \in C_+^s(\overline{\Omega}_\pm); \tag{3.76}$$

$$T^\pm V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad a \in C_+^{\frac{3}{2}}(\overline{\Omega}_\pm); \tag{3.77}$$

$$T^\pm W : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad a \in C_+^{\frac{3}{2}}(\overline{\Omega}_\pm). \tag{3.78}$$

Proof Relations (3.56), Theorem 3.3(i), and Theorem 2.6 immediately imply the continuity of operators (3.68) and (3.69). Further, if $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, then there exists $\epsilon > 0$ such that $a \in C^{1,\frac{1}{2}+\epsilon}(\overline{\Omega})$. For $\frac{1}{2} < s \le \frac{3}{2}$, $g \in H^{s-\frac{3}{2}}(\Omega)$, and any $\sigma \in (\frac{1}{2}, \min\{s, \frac{1}{2}+\epsilon\})$, we have

$$\begin{aligned}
\|AVg\|_{H^{\sigma-1}(\Omega)} &= \left\| \nabla \cdot \left(a \nabla \left[\frac{1}{a} V_\Delta g \right] \right) \right\|_{H^{\sigma-1}(\Omega)} = \left\| \nabla \cdot \left[(\nabla \ln a) V_\Delta g \right] \right\|_{H^{\sigma-1}(\Omega)} \\
&\le \left\| (\nabla \ln a) V_\Delta g \right\|_{H^\sigma} \le C \|a\|_{C^{1,\frac{1}{2}+\epsilon}(\overline{\Omega})} \| V_\Delta g \|_{H^\sigma(\Omega)} \\
&\le C \|a\|_{C^{1,\frac{1}{2}+\epsilon}(\overline{\Omega})} \| V_\Delta g \|_{H^s(\Omega)},
\end{aligned}$$

where we have taken into account that $\Delta V_\Delta g = 0$ in Ω. Hence, along with continuity of operator in (3.58), this implies $AVg \in H^{\sigma-1}(\Omega)$ and thus, by Remark 2.2(4), $r_\Omega AVg \in \widetilde{H}_\bullet^{\sigma-1}(\Omega) \subset \widetilde{H}_\bullet^{-\frac{1}{2}}(\Omega)$ with the corresponding norm estimate, from which the continuity of operator (3.71) follows. The continuity of operator (3.73) is proved in a similar fashion.

The continuity of operators (3.70), (3.72), and (3.74) immediately follows from the continuity of their counterparts for the interior domain.

The continuity of operators (3.75) and (3.76) for the potential traces is implied by the continuity of operators (3.68)–(3.70) and the trace theorem, whereas the continuity of operators (3.77) and (3.78) for the potential co-normal derivatives is implied by the continuity of operators (3.71)–(3.74) and Theorem 2.9. □

Now we can prove the jump properties for the parametrix-based potentials on Lipschitz domains.

Theorem 3.6 *Let $\partial\Omega$ be a compact Lipschitz boundary, $\frac{1}{2} < s < \frac{3}{2}$, $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$, and $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$. Then*

$$\gamma^+ V\psi - \gamma^- V\psi = 0, \qquad \gamma^+ W\varphi - \gamma^- W\varphi = -\varphi, \quad \text{if } a \in C^s_+(\mathbb{R}^n); \tag{3.79}$$

$$T^+ V\psi - T^- V\psi = \psi, \qquad T^+ W\varphi - T^- W\varphi = (\partial_\nu a)\varphi, \quad \text{if } a \in C^{\frac{3}{2}}_+(\mathbb{R}^n). \tag{3.80}$$

Proof Relations (3.56) and (3.62) along with Theorem 2.6 immediately imply jump relations (3.79).

To prove the first jump relation in (3.80), we generalise to the parametrix-based potentials the arguments from the proof of Lemma 4.1 in [11]. Let $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$. From (3.53) we obtain, in the sense of distributions,

$$\begin{aligned} AV\psi &= -A\left(\frac{1}{a}\mathbf{P}_\Delta \gamma^*\psi\right) = -\gamma^*\psi + \nabla\cdot\left(\frac{\nabla a}{a}\mathbf{P}_\Delta\gamma^*\psi\right) \\ &= -\gamma^*\psi - \nabla\cdot\big((\nabla a)V\psi\big) \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{3.81}$$

where we have taken into account that $\Delta\mathbf{P}_\Delta\gamma^*\psi = \gamma^*\psi$. Then, since the operator A is formally self-adjoint, for any test function $\phi \in \mathcal{D}(\mathbb{R}^n)$, we obtain

$$\int_{\mathbb{R}^n} V\psi(y)A\phi(y)\,dy = \langle AV\psi, \phi\rangle_{\mathbb{R}^n} = -\langle\psi, \gamma\phi\rangle_{\partial\Omega} - \big\langle\nabla\cdot\big((\nabla a)V\psi\big), \phi\big\rangle_{\mathbb{R}^n}. \tag{3.82}$$

Note that, for $a \in C^{\frac{3}{2}}_+(\mathbb{R}^n)$ and $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$ with $\frac{1}{2} < s < \frac{3}{2}$, the continuity of operator (3.68) and Theorem 2.6 imply that $V\psi \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$ and $(\nabla a)V\psi \in H^{\frac{1}{2}+\epsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\epsilon \in (0,1)$. Hence, from the second Green identity (2.21) with $v = V\psi$ and $u = \phi$, along with (3.81), we have

$$\begin{aligned} &\int_{\Omega_\pm} V\psi(y)A\phi(y)\,dy - \langle\tilde{A}_{\Omega_\pm}V\psi, \phi\rangle_{\Omega_\pm} \\ &\quad = \int_{\Omega_\pm} V\psi(y)A\phi(y)\,dy - \langle\mathring{E}_{\Omega_\pm} r_{\Omega_\pm} AV\psi, \phi\rangle_{\Omega_\pm} \\ &\quad = \int_{\Omega_\pm} V\psi(y)A\phi(y)\,dy + \big\langle\mathring{E}_{\Omega_\pm} r_{\Omega_\pm}\nabla\cdot\big((\nabla a)V\psi\big), \phi\big\rangle_{\Omega_\pm} \\ &\quad = \pm\big\langle T^\pm\phi, \gamma^\pm V\psi\big\rangle_{\partial\Omega} \mp \big\langle T^\pm V\psi, \gamma^\pm\phi\big\rangle_{\partial\Omega}. \end{aligned} \tag{3.83}$$

Here we employed that $r_{\Omega_\pm}\gamma^*\psi = 0$ since $\operatorname{supp}\gamma^*\psi \subset \partial\Omega$. Let us take into account that $\gamma^+\phi = \gamma^-\phi = \gamma\phi$ and $T^+\phi = T^-\phi = T^c\phi$ due to smoothness of ϕ, whereas $\gamma^+ V\psi =$

$\gamma^- V\psi = \gamma V\psi$ by the first relation in (3.79). Moreover, we also have

$$\big\langle \mathring{E}_{\Omega_\pm} r_{\Omega_\pm} \nabla \cdot \big((\nabla a)V\psi\big), \phi\big\rangle_{\Omega_\pm} = \big\langle r_{\Omega_\pm} \nabla \cdot \big((\nabla a)V\psi\big), \mathring{E}_{\Omega_\pm}\phi\big\rangle_{\Omega_\pm}$$
$$= \pm\big\langle (\partial_\nu a)\gamma^\pm V\psi, \gamma\phi\big\rangle_{\partial\Omega} - \big\langle (\nabla a)V\psi, \nabla\phi\big\rangle_{\Omega_\pm}.$$

Then summing up (3.83) for Ω_+ and Ω_-, we obtain

$$\int_{\mathbb{R}^n} V\psi(y)A\phi(y)\,dy = -\big\langle T^+V\psi - T^-V\psi, \gamma\phi\big\rangle_{\partial\Omega} + \big\langle (\nabla a)V\psi, \nabla\phi\big\rangle_{\mathbb{R}^n}. \tag{3.84}$$

Comparing (3.84) and (3.82), we obtain $\langle T^+V\psi - T^-V\psi, \gamma\phi\rangle_{\partial\Omega} = \langle \psi, \gamma\phi\rangle_{\partial\Omega}$ for arbitrary $\phi \in \mathcal{D}(\mathbb{R}^n)$, which implies the first jump relation in (3.80).

Let us similarly prove the second jump relation in (3.80). Let $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$. From (3.54) we obtain, in the sense of distributions,

$$AW\varphi = -A\left(\frac{1}{a}\mathbf{P}_\Delta T^{c*}\varphi\right) = -T^{c*}\varphi + \nabla \cdot \left(\frac{\nabla a}{a}\mathbf{P}_\Delta T^{c*}\varphi\right)$$
$$= -T^{c*}\varphi - \nabla \cdot \big((\nabla a)W\varphi\big) \quad \text{in } \mathbb{R}^n, \tag{3.85}$$

where we have taken into account that $\Delta\mathbf{P}_\Delta T^{c*}\varphi = T^{c*}\varphi$. Then for any test function $\phi \in \mathcal{D}(\mathbb{R}^n)$, we obtain

$$\int_{\mathbb{R}^n} W\varphi(y)A\phi(y)\,dy = \langle AW\varphi, \phi\rangle_{\mathbb{R}^n} = -\big\langle T^{c*}\varphi + \nabla \cdot \big((\nabla a)W\varphi\big), \phi\big\rangle_{\mathbb{R}^n}$$
$$= -\big\langle \varphi, T^c\phi\big\rangle_{\partial\Omega} + \big\langle (\nabla a)W\varphi, \nabla\phi\big\rangle_{\mathbb{R}^n}. \tag{3.86}$$

Note that for $a \in C_+^{\frac{3}{2}}(\mathbb{R}^n)$ and $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$ with $\frac{1}{2} < s < \frac{3}{2}$, the continuity of operators (3.69) and (3.70) and Theorem 2.6 imply that $r_{\Omega_+}W\varphi \in H^s(\Omega_+)$, $r_{\Omega_-}W\varphi \in H^s_{\mathrm{loc}}(\overline{\Omega}_-)$, and $(\nabla a)r_{\Omega_+}W\varphi \in H^{\frac{1}{2}+\epsilon}(\Omega_+)$, $(\nabla a)r_{\Omega_-}W\varphi \in H^{\frac{1}{2}+\epsilon}_{\mathrm{loc}}(\overline{\Omega}_-)$ for some $\epsilon \in (0,1)$. Hence from the second Green identity (2.21) for $v = W\varphi$ and $u = \phi$, along with (3.85), we have

$$\int_{\Omega_\pm} W\varphi(y)A\phi(y)\,dy - \langle \tilde{A}_{\Omega_\pm}W\varphi, \phi\rangle_{\Omega_\pm}$$
$$= \int_{\Omega_\pm} W\varphi(y)A\phi(y)\,dy - \langle \mathring{E}_{\Omega_\pm} r_{\Omega_\pm} AW\varphi, \phi\rangle_{\Omega_\pm}$$
$$= \int_{\Omega_\pm} W\varphi(y)A\phi(y)\,dy + \big\langle \mathring{E}_{\Omega_\pm} r_{\Omega_\pm} \nabla \cdot \big((\nabla a)W\varphi\big), \phi\big\rangle_{\Omega_\pm}$$
$$= \pm\big\langle T^\pm\phi, \gamma^\pm W\varphi\big\rangle_{\partial\Omega} \mp \big\langle T^\pm W\varphi, \gamma^\pm\phi\big\rangle_{\partial\Omega}. \tag{3.87}$$

Here we employed that $r_{\Omega_\pm}T^{c*}\varphi = 0$ since $\operatorname{supp} T^{c*}\varphi \subset \partial\Omega$. Let us also take into account that $\gamma^+\phi = \gamma^-\phi = \gamma\phi$ and $T^+\phi = T^-\phi = T^c\phi$ due to smoothness of ϕ, whereas $\gamma^+W\varphi - \gamma^-W\varphi = -\varphi$ by the second relation in (3.79). Moreover, we also have

$$\big\langle \mathring{E}_{\Omega_\pm} r_{\Omega_\pm} \nabla \cdot \big((\nabla a)W\varphi\big), \phi\big\rangle_{\Omega_\pm} = \big\langle r_{\Omega_\pm} \nabla \cdot \big((\nabla a)W\varphi\big), \mathring{E}_{\Omega_\pm}\phi\big\rangle_{\Omega_\pm}$$
$$= \pm\big\langle (\partial_\nu a)\gamma^\pm W\varphi, \gamma\phi\big\rangle_{\partial\Omega} - \big\langle (\nabla a)W\varphi, \nabla\phi\big\rangle_{\Omega_\pm}.$$

Then summing up (3.87) for Ω_+ and Ω_-, we obtain

$$\int_{\mathbb{R}^n} W\varphi(y)A\phi(y)\,dy - \langle(\partial_\nu a)\varphi, \gamma\phi\rangle_{\partial\Omega} - \langle(\nabla a)W\varphi, \nabla\phi\rangle_{\mathbb{R}^n}$$
$$= -\langle T^c\phi, \varphi\rangle_{\partial\Omega} - \langle T^+W\varphi - T^-W\varphi, \gamma\phi\rangle_{\partial\Omega}. \tag{3.88}$$

Comparing (3.88) and (3.86), we obtain $\langle T^+W\varphi - T^-W\varphi, \gamma\phi\rangle_{\partial\Omega} = \langle(\partial_\nu a)\varphi, \gamma\phi\rangle_{\partial\Omega}$ for arbitrary $\phi \in \mathcal{D}(\mathbb{R}^n)$, which implies the second jump relation in (3.80). □

Theorem 3.5(iii) and the first relation in (3.79) imply the following assertion.

Corollary 3.7 *Let $\partial\Omega$ be a compact Lipschitz boundary, and let $\frac{1}{2} < s < \frac{3}{2}$. The following operators are continuous*:

$$\mathcal{V} := \gamma^+V = \gamma^-V : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad a \in C_+^s(\overline{\Omega}_\pm); \tag{3.89}$$

$$\mathcal{W} := \frac{1}{2}\left(\gamma^+W + \gamma^-W\right) : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad a \in C_+^s(\overline{\Omega}_\pm); \tag{3.90}$$

$$\mathcal{W}' := \frac{1}{2}\left(T^+V + T^-V\right) : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad a \in C_+^{\frac{3}{2}}(\overline{\Omega}_\pm); \tag{3.91}$$

$$\mathcal{L} := \frac{1}{2}\left(T^+W + T^-W\right) : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega), \quad a \in C_+^{\frac{3}{2}}(\overline{\Omega}_\pm). \tag{3.92}$$

For the case of smooth boundary, the boundary operators defined in Corollary 3.7 (see [27, Eq. (7.3)] for the fundamental solution-based potentials on Lipschitz domains) correspond to the boundary integral (pseudo-differential) operators of direct surface values of the single-layer potential, the double-layer potential $\mathcal{W}$, and the co-normal derivatives of the single-layer potential $\mathcal{W}'$ and of the double-layer potential (see [3, Eq. (3.6)-(3.8)]) for the parametrix-based potentials on smooth domains. See also [27, Theorems 7.3, 7.4] about integral representations on Lipschitz domains of the boundary operators associated with the layer potentials based on fundamental solutions.

If $a = 1$, then we equip the operators defined in Corollary 3.7 with subscript Δ. Then, under the hypotheses of Corollary 3.7, we have (see [3, Eq. (3.10)–(3.13)] for the potentials on smooth domains)

$$\mathcal{V}g = \frac{1}{a}\mathcal{V}_\Delta g, \qquad \mathcal{W}g = \frac{1}{a}\mathcal{W}_\Delta(ag), \tag{3.93}$$

$$\mathcal{W}'g = \mathcal{W}'_\Delta g - \frac{\partial_\nu a}{a}\mathcal{V}_\Delta g, \qquad \mathcal{L}g = \mathcal{L}_\Delta(ag) - \frac{\partial_\nu a}{a}\mathcal{W}_\Delta(ag). \tag{3.94}$$

Indeed, relations (3.93) immediately follow from (3.89), (3.90), and (3.56). Further, $T^+Vg = T^+(\frac{1}{a}V_\Delta g)$. Let $\{v_k\} \subset \mathcal{D}(\overline{\Omega})$ be a sequence such that $\|v_k - V_\Delta g\|_{H^{s,-\frac{1}{2}}(\Omega;\Delta)} \to 0$ as $k \to \infty$, which implies that also $\|\frac{1}{a}v_k - Vg\|_{H^{s,-\frac{1}{2}}(\Omega;A)} \to 0$ as $k \to \infty$. Then (see [32, Lemma 6.10])

$$T^+Vg = \lim_{k\to\infty} T^c\left(\frac{1}{a}v_k\right) = \lim_{k\to\infty} aT^c_\Delta\left(\frac{1}{a}v_k\right) = \lim_{k\to\infty}\left(\partial_\nu v_k - \frac{\partial_\nu a}{a}\gamma^+v_k\right)$$
$$= T^+_\Delta V_\Delta g - \frac{\partial_\nu a}{a}\gamma^+V_\Delta g.$$

Similarly, $T^- Vg = T^-_\Delta V_\Delta g - \frac{\partial_\nu a}{a}\gamma^- V_\Delta g$, which, together with (3.91), implies the first relation in (3.94). The second relation in (3.94) is proved by similar arguments.

Employing definitions (3.89)–(3.92), the jump properties (3.79)–(3.80) can be re-written for $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$ and $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$ with $\frac{1}{2} < s < \frac{3}{2}$ as follows:

$$\gamma^\pm V\psi = \mathcal{V}\psi, \qquad \gamma^\pm W\varphi = \mp\frac{1}{2}\varphi + \mathcal{W}\varphi \quad \text{if } a \in C^s_+(\mathbb{R}^n); \tag{3.95}$$

$$T^\pm V\psi = \pm\frac{1}{2}\psi + \mathcal{W}'\psi, \qquad T^\pm W\varphi = \pm\frac{1}{2}(\partial_\nu a)\varphi + \mathcal{L}\varphi \quad \text{if } a \in C^{\frac{3}{2}}_+(\mathbb{R}^n). \tag{3.96}$$

4 The third Green identity and integral relations

In this section, we apply some limiting procedures to obtain the parametrix-based third Green identity.

Theorem 4.1 *Let Ω be a bounded Lipschitz domain, $u \in H^s(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, and $a \in C^s_+(\overline{\Omega})$.*

(i) *The following* generalised third Green identity *holds*:

$$u + \mathcal{R}u + W\gamma^+ u = \mathcal{P}\check{A}_\Omega u \quad \textit{in } \Omega, \tag{4.1}$$

where, by (2.9) *and* (2.10),

$$\begin{aligned}\mathcal{P}\check{A}_\Omega u(y) &:= \langle \check{A}_\Omega u, P(\cdot,y)\rangle_\Omega = -\check{\mathcal{E}}_\Omega(u, P(\cdot,y)) = -\langle \mathring{E}_\Omega(a\nabla u), \nabla P(\cdot,y)\rangle_\Omega \\ &= \frac{1}{a(y)}\nabla\cdot\mathcal{P}_\Delta\mathring{E}_\Omega(a\nabla u)(y), \quad \textit{a.e. } y \in \Omega,\end{aligned} \tag{4.2}$$

and, particularly, if $s = 1$, *then*

$$\mathcal{P}\check{A}_\Omega u(y) = -\int_\Omega a(x)\nabla u(x)\cdot\nabla_x P(x,y)\,dx, \quad \textit{a.e. } y \in \Omega. \tag{4.3}$$

(ii) *Moreover, if $Au = r_\Omega\tilde{f}$ in Ω, where $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, then the* generalised third Green identity *takes the form*

$$u + \mathcal{R}u - VT^+(\tilde{f}; u) + W\gamma^+ u = \mathcal{P}\tilde{f} \quad \textit{in } \Omega. \tag{4.4}$$

Proof (i) Let first $u \in \mathcal{D}(\overline{\Omega})$. For $y \in \Omega$, let $B_\epsilon(y) \subset \Omega$ be a ball centred in y with sufficiently small radius ϵ, and let $\Omega_\epsilon := \Omega \setminus \overline{B_\epsilon}(y)$. For any fixed y, evidently, $P(\cdot,y) = \frac{1}{a(y)}P_\Delta(\cdot,y) \in \mathcal{D}(\overline{\Omega_\epsilon}) \subset H^{1,0}(A;\Omega_\epsilon)$ and has the coinciding classical and canonical co-normal derivatives on $\partial\Omega_\epsilon$. Then from the first Green identity (2.20) applied to Ω_ϵ with $v = P(\cdot,y)$ we obtain

$$\begin{aligned}&-\langle T^+P(\cdot,y), \gamma^+ u\rangle_{\partial B_\epsilon(y)} - \langle T^+P(\cdot,y), \gamma^+ u\rangle_{\partial\Omega} + \langle R(\cdot,y), u\rangle_{\Omega_\epsilon} \\ &\quad = -\langle \nabla P(\cdot,y), a\nabla u\rangle_{\Omega_\epsilon}.\end{aligned} \tag{4.5}$$

Since

$$\begin{aligned}&\lim_{\epsilon\to 0}\langle T^+P(\cdot,y), \gamma^+ u\rangle_{\partial B_\epsilon(y)} \\ &\quad = \frac{1}{a(y)}\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(y)}\left[\partial_{\nu(x)}P_\Delta(x,y)\right]a(x)\gamma^+ u(x)\,dS(x) = -u(y),\end{aligned}$$

by passing to the limits as $\epsilon \to 0$ equation (4.5) reduces to the third Green identity (4.1) for any $u \in \mathcal{D}(\overline{\Omega})$. Taking into account the denseness of $\mathcal{D}(\overline{\Omega})$ in $H^s(\Omega)$ and the mapping properties of the volume potentials (3.30) and (3.35) in Theorem 3.2 and of the double-layer potential (3.69) in Theorem 3.5(i), we obtain that (4.1)–(4.2) also hold for any $u \in H^s(\Omega)$ with $\frac{1}{2} < s < \frac{3}{2}$ in the sense of $H^s(\Omega)$, which also implies (4.3) for $s = 1$.

(ii) Let $\{u_k\} \in \mathcal{D}(\overline{\Omega})$ be a sequence converging to u in $H^s(\Omega)$. By (4.2), (4.3), and (2.18) we have

$$\begin{aligned}\mathcal{P}\check{A}_\Omega u_k(y) &= -\lim_{\epsilon\to 0}\int_{\Omega_\epsilon} a(x)\nabla u_k(x)\cdot\nabla_x P(x,y)\,dx = -\lim_{\epsilon\to 0}\check{\mathcal{E}}_{\Omega_\epsilon}\big(u_k, P(\cdot,y)\big)\\ &= \lim_{\epsilon\to 0}\bigg[\int_{\Omega_\epsilon}(\tilde{A}_\Omega u_k)(x)P(x,y)\,dx - \int_{\partial B_\epsilon(y)} P(x,y)T^+u_k(x)\,dS(x)\\ &\quad - \int_{\partial\Omega} P(x,y)T^+u_k(x)\,dS(x)\bigg] = \mathcal{P}\tilde{A}_\Omega u_k(y) + VT^+u_k(y). \end{aligned}\tag{4.6}$$

Let now $\tilde{f}_k := \widetilde{E}^{s-2}_\Omega r_\Omega(\tilde{A}_\Omega u_k - \check{A}_\Omega u) + \tilde{f}$, where $\widetilde{E}^{s-2}_\Omega : H^{s-2}(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is a (non-unique) continuous extension operator, which exists by [31, Theorem 2.16]. Since $r_\Omega\check{A}_\Omega u = r_\Omega\tilde{f}$, we obtain $r_\Omega\tilde{f}_k = r_\Omega\tilde{A}_\Omega u_k = r_\Omega\check{A}_\Omega u_k$. Hence

$$r_\Omega\tilde{A}_\Omega u_k - r_\Omega\check{A}_\Omega u = r_\Omega\check{A}_\Omega(u_k - u) \to 0 \quad \text{in } H^{s-2}(\Omega),$$

and $\tilde{f}_k \to \tilde{f}$ in $\widetilde{H}^{s-2}(\Omega)$ as $k \to \infty$. Then by (4.6), (3.53), and (2.19) we obtain

$$\begin{aligned}\mathcal{P}\check{A}_\Omega u_k &= \mathcal{P}\tilde{A}_\Omega u_k + VT^+u_k = \mathcal{P}\tilde{A}_\Omega u_k + VT^+(\tilde{f}_k; u_k) - V\big(\gamma^{-1}\big)^*(\tilde{f}_k - \tilde{A}_\Omega u_k)\\ &= \mathcal{P}\tilde{A}_\Omega u_k + VT^+(\tilde{f}_k; u_k) + \mathcal{P}(\tilde{f}_k - \tilde{A}_\Omega u_k) = VT^+(\tilde{f}_k; u_k) + \mathcal{P}\tilde{f}_k,\end{aligned}$$

where we took into account that $\gamma^*(\gamma^{-1})^*(\tilde{f}_k - \tilde{A}_\Omega u_k) = \tilde{f}_k - \tilde{A}_\Omega u_k$ by [31, Corollary 2.11] since $\tilde{f}_k - \tilde{A}_\Omega u_k \in H^{s-2}_{\partial\Omega}$. Passing to the limits as $k \to \infty$, we obtain $\mathcal{P}\check{A}_\Omega u(y) = \mathcal{P}\tilde{f} + VT^+(\tilde{f}; u)$, which by substitution into (4.1) gives (4.4). □

For some functions $\tilde{f}$, Ψ, Φ, let us consider a more general "indirect" integral relation associated with (4.4):

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}\tilde{f} \quad \text{in } \Omega. \tag{4.7}$$

The following lemma extends Lemma 4.1 from [3], where the corresponding assertion was proved for $\tilde{f} \in L_2(\Omega)$, $s = 1$, $a \in C^\infty(\Omega)$, and the infinitely smooth boundary.

Lemma 4.2 *Let* $\frac{1}{2} < s < \frac{3}{2}$ *and* $a \in C^s_+(\overline{\Omega})$. *Let* $u \in H^s(\Omega)$, $\Psi \in H^{s-\frac{3}{2}}(\partial\Omega)$, $\Phi \in H^{s-\frac{1}{2}}(\partial\Omega)$, *and* $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ *satisfy* (4.7). *Then*

$$Au = r_\Omega\tilde{f} \quad \textit{in } \Omega, \tag{4.8}$$

$$V\big(\Psi - T^+(\tilde{f}; u)\big) - W\big(\Phi - \gamma^+u\big) = 0 \quad \textit{in } \Omega. \tag{4.9}$$

Proof Subtracting (4.7) from identity (4.1), we obtain

$$V\Psi - W\big(\Phi - \gamma^+u\big) = \mathcal{P}[\check{A}_\Omega u - \tilde{f}] \quad \text{in } \Omega. \tag{4.10}$$

Multiplying equality (4.10) by a, applying the Laplace operator Δ, and taking into account (3.57) and (3.17), we get $r_\Omega\tilde{f} = r_\Omega(\check{A}_\Omega u) = Au$ in Ω. This means that $\tilde{f}$ is an extension of the distribution $Au \in H^{s-2}(\Omega)$ to $\widetilde{H}^{s-2}(\Omega)$, and u satisfies (4.8). Then (2.15) implies

$$\begin{aligned}\mathcal{P}[\check{A}_\Omega u - \tilde{f}](y) = \langle \check{A}_\Omega u - \tilde{f}, P(\cdot, y)\rangle_\Omega = -\langle T^+(\tilde{f}; u), P(\cdot, y)\rangle_{\partial\Omega} \\ = VT^+(\tilde{f}; u)(y), \quad y \in \Omega. \end{aligned} \tag{4.11}$$

Substituting (4.11) into (4.10) leads to (4.9). □

For $\frac{1}{2} < s < \frac{3}{2}$, $a \in C^s_+(\overline{\Omega})$, and $g \in H^{s-1}(\Omega)$, let us introduce the operator A^∇_Ω as

$$A^\nabla_\Omega g := -\nabla \cdot \mathring{E}_\Omega(g\nabla a). \tag{4.12}$$

Lemma 4.3 *Let* $\frac{1}{2} < s < \frac{3}{2}$.

(i) *If* $a \in C^{|s-1|+1}_+(\overline{\Omega})$, *then the following operator is continuous*:

$$A^\nabla_\Omega : H^{s-1}(\Omega) \to \widetilde{H}^{s-2}(\Omega). \tag{4.13}$$

(ii) *If* $a \in C^s_+(\overline{\Omega})$, *then the following operator is continuous and compact*:

$$A^\nabla_\Omega : H^{s}(\Omega) \to \widetilde{H}^{s-2}(\Omega). \tag{4.14}$$

Proof (i) If $a \in C^{|s-1|+1}_+(\overline{\Omega})$, then $\nabla a \in C^{|s-1|}_+(\overline{\Omega})$, and by Theorem 2.6, ∇a is a multiplier in $H^{s-1}(\Omega)$, which implies the continuity of operator (4.13).

(ii) For $1 \le s < \frac{3}{2}$, we have $s = |s-1| + 1$, which by item (i) implies the continuity of operator (4.13) and thus the continuity and compactness of operator (4.14).

For $\frac{1}{2} < s < 1$, we need an estimate of the norm $\|g\nabla a\|_{H^{s-1}(\Omega)}$. First, by Definition 2.5 the inclusion $a \in C^s_+(\overline{\Omega})$ implies that there exists $t \in (s, 1)$ such that $a \in C^{0,t}(\overline{\Omega}) = B^t_{\infty,\infty}(\Omega) = F^t_{\infty,\infty}(\Omega)$ (see, e.g., Proposition in [45, Sect. 2.1.2]) and hence $\nabla a \in F^{t-1}_{\infty,\infty}(\Omega)$. Then, by Theorem 1 from [45, Sect. 4.4.3],

$$\begin{aligned}\|g\nabla a\|_{F^{t-1}_{2,\infty}(\Omega)} &\le C\|\nabla a\|_{F^{t-1}_{\infty,\infty}(\Omega)}\|g\|_{H^\sigma(\Omega)} \\ &\le C\|a\|_{C^{0,t}(\overline{\Omega})}\|g\|_{H^\sigma(\Omega)}, \quad \forall\, \sigma \in (1-t, s).\end{aligned} \tag{4.15}$$

On the other hand, by (3.49), item (ii) of Proposition from [45, Sect. 2.2.1], and (4.15) we obtain

$$\begin{aligned}\|A^\nabla_\Omega g\|_{\widetilde{H}^{s-2}(\Omega)} &\le c_3\|g\nabla a\|_{H^{s-1}(\Omega)} = c_3\|g\nabla a\|_{F^{s-1}_{2,2}(\Omega)} \\ &\le C_1\|g\nabla a\|_{F^{t-1}_{2,\infty}(\Omega)} \le C_1 C\|a\|_{C^{0,t}(\overline{\Omega})}\|g\|_{H^\sigma(\Omega)}.\end{aligned}$$

Thus the operator $A^\nabla_\Omega : H^\sigma(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is continuous, which implies the continuity and, by the Rellich compact embedding theorem, also the compactness of operator (4.14) for $\frac{1}{2} < s < 1$. □

In accordance with notation (2.10), let us also denote

$$\check{\Delta}_\Omega g := \nabla \cdot \mathring{E}_\Omega r_\Omega \nabla g.$$

Let us now discuss the trace and two forms of the co-normal derivative associated with equation (4.7).

Lemma 4.4

(i) *Under the hypotheses of Lemma* 4.2,

$$\gamma^+ u + \gamma^+ \mathcal{R}u - \mathcal{V}\Psi - \frac{1}{2}\Phi + \mathcal{W}\Phi = \gamma^+ \mathcal{P}\tilde{f} \quad on\ \partial\Omega, \tag{4.16}$$

$$\begin{aligned} &T^+(\tilde{f}; u) + T^+_\Delta\big(\mathring{A}^\nabla_\Omega u; a\mathcal{R}u\big) - \frac{1}{2}\Psi - \mathcal{W}'_\Delta \Psi + \mathcal{L}_\Delta(a\Phi) \\ &\quad = T^+_\Delta(\tilde{f}; \mathcal{P}_\Delta \tilde{f}) \quad on\ \partial\Omega. \end{aligned} \tag{4.17}$$

(ii) *If, moreover,* $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, *then*

$$T^+(\tilde{f}; u) + T^+\mathcal{R}u - \frac{1}{2}\Psi - \mathcal{W}'\Psi + T^+ W\Phi = T^+(\tilde{f} + \mathring{E}\mathcal{R}_*\tilde{f}; \mathcal{P}\tilde{f}) \quad on\ \partial\Omega, \tag{4.18}$$

where $\mathcal{R}_*$ *is defined in* (3.14) *and* (3.16).

Proof (i) Equation (4.16) is implied by (4.7) and (3.95).

To prove (4.17), let us first multiply (4.7) by a to obtain

$$-V_\Delta \Psi + W_\Delta(a\Phi) = \mathcal{P}_\Delta \tilde{f} - au - a\mathcal{R}u \quad \text{in } \Omega. \tag{4.19}$$

Since $\Delta\{-V_\Delta \Psi + W_\Delta(a\Phi)\} = 0$, for the both sides of (4.19) the canonical co-normal derivative T^+_Δ is well defined,

$$-T^+_\Delta V_\Delta \Psi + T^+_\Delta W_\Delta(a\Phi) = T^+_\Delta(\mathcal{P}_\Delta \tilde{f} - au - a\mathcal{R}u), \tag{4.20}$$

and by (2.17)

$$T^+_\Delta(\mathcal{P}_\Delta \tilde{f} - au - a\mathcal{R}u) = -\big(\gamma^{-1}\big)^* \check{\Delta}_\Omega(\mathcal{P}_\Delta \tilde{f} - au - a\mathcal{R}u), \tag{4.21}$$

because by (4.19),

$$\tilde{\Delta}_\Omega(\mathcal{P}_\Delta \tilde{f} - au - a\mathcal{R}u) = \mathring{E}_\Omega \Delta(\mathcal{P}_\Delta \tilde{f} - au - a\mathcal{R}u) = \mathring{E}_\Omega \Delta\big\{-V_\Delta \Psi + W_\Delta(a\Phi)\big\} = 0.$$

Note that, by the second equality in (3.16),

$$\Delta(a\mathcal{R}u) = -\nabla \cdot \big[\Delta \mathcal{P}_\Delta(u\nabla a)\big] = -\nabla \cdot (u\nabla a) = r_\Omega \mathring{A}^\nabla_\Omega u \quad \text{in } \Omega, \tag{4.22}$$

which implies that $\mathring{A}^\nabla_\Omega u \in \widetilde{H}^{s-2}(\Omega)$ is an extension of $\Delta(a\mathcal{R}u) \in H^{s-2}(\Omega)$. Further (see (2.10)),

$$\check{\Delta}_\Omega(au) = \nabla \cdot \mathring{E}_\Omega r_\Omega \nabla(ua) = \nabla \cdot \mathring{E}_\Omega r_\Omega(u\nabla a) + \nabla \cdot \mathring{E}_\Omega r_\Omega(a\nabla u)$$

$$= -A^{\nabla}_{\Omega}u + \check{A}_{\Omega}u \quad \text{in } \mathbb{R}^n.$$

Then

$$-\check{\Delta}_{\Omega}(\mathcal{P}_{\Delta}\tilde{f} - au - a\mathcal{R}u) = \tilde{f} - \check{\Delta}_{\Omega}\mathcal{P}_{\Delta}\tilde{f} - (\tilde{f} - \check{A}_{\Omega}u) - \left[A^{\nabla}_{\Omega}u - \check{\Delta}_{\Omega}(a\mathcal{R}u)\right] \quad \text{in } \mathbb{R}^n,$$

and by (2.13)

$$T^+_{\Delta}(\mathcal{P}_{\Delta}\tilde{f} - au - a\mathcal{R}u) = T^+_{\Delta}(\tilde{f};\mathcal{P}_{\Delta}\tilde{f}) - T^+(\tilde{f};u) - T^+_{\Delta}\left(A^{\nabla}_{\Omega}u;a\mathcal{R}u\right).$$

Substituting this in (4.20), we obtain

$$T^+(\tilde{f};u) + T^+_{\Delta}\left(A^{\nabla}_{\Omega}u;a\mathcal{R}u\right) - T^+_{\Delta}V_{\Delta}\Psi + T^+_{\Delta}W_{\Delta}(a\Phi) = T^+_{\Delta}(\tilde{f};\mathcal{P}_{\Delta}\tilde{f}) \quad \text{on } \partial\Omega.$$

Taking into account jump relation (3.63) and (3.66) with (3.67), we arrive at (4.17).

(ii) To prove (4.18), let us first remark that

$$A\mathcal{P}\tilde{f} = \tilde{f} + \mathcal{R}_*\tilde{f} \quad \text{in } \Omega, \tag{4.23}$$

which implies, due to (4.8), that $A(\mathcal{P}\tilde{f} - u) = \mathcal{R}_*\tilde{f}$ in Ω, where $\mathcal{R}_*$ is defined in (3.14) and (3.16), and since $a \in C^{\frac{3}{2}}_+(\overline{\Omega})$, we obtain by (3.39) that $\mathcal{R}_*\tilde{f} \in H^{\sigma}(\Omega)$ for some $\sigma > -\frac{1}{2}$. Then $A(\mathcal{P}\tilde{f} - u)$ can be canonically extended to $\tilde{A}(\mathcal{P}\tilde{f} - u) = \mathring{E}_{\Omega}\mathcal{R}_*\tilde{f} \in \widetilde{H}^{\sigma}(\Omega) \subset \widetilde{H}^{s-2}(\Omega)$. This implies that there exists a canonical co-normal derivative of $(\mathcal{P}\tilde{f} - u)$, for which, due to (2.17) and (2.13), we have

$$\begin{aligned} T^+(\mathcal{P}\tilde{f} - u) &= \left(\gamma^{-1}\right)^*\left[\tilde{A}(\mathcal{P}\tilde{f} - u) - \check{A}_{\Omega}\mathcal{P}\tilde{f} + \check{A}_{\Omega}u\right] \\ &= \left(\gamma^{-1}\right)^*[\mathring{E}_{\Omega}\mathcal{R}_*\tilde{f} - \check{A}_{\Omega}\mathcal{P}\tilde{f} + \check{A}_{\Omega}u] \\ &= \left(\gamma^{-1}\right)^*[\tilde{f} + \mathring{E}_{\Omega}\mathcal{R}_*\tilde{f} - \check{A}_{\Omega}\mathcal{P}\tilde{f} + \check{A}_{\Omega}u - \tilde{f}] \\ &= T^+(\tilde{f} + \mathring{E}_{\Omega}\mathcal{R}_*\tilde{f}, \mathcal{P}\tilde{f}) - T^+(\tilde{f}, u), \end{aligned} \tag{4.24}$$

where $\tilde{f} + \mathring{E}_{\Omega}\mathcal{R}_*\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ is an extension of $A\mathcal{P}\tilde{f}$ due to (4.23). From (4.7) we have $\mathcal{P}\tilde{f} - u = \mathcal{R}u - V\Psi + W\Phi$ in Ω. Substituting this in the left-hand side of (4.24) and taking into account jump relation (3.96), we arrive at (4.18). □

Note that, unlike (4.17), the co-normal derivative form (4.18) of relation (4.7) is written without referring to the corresponding constant-coefficient potentials.

Remark 4.5 Let $\frac{1}{2} < s < \frac{3}{2}$ and $\tilde{f} \in \widetilde{H}^{-1/2}(\Omega) \subset \widetilde{H}^{s-2}(\Omega)$.

(i) Then evidently $\mathcal{P}_{\Delta}\tilde{f} \in H^{s,-1/2}(\Omega, \Delta)$ and

$$T^+_{\Delta}(\tilde{f};\mathcal{P}_{\Delta}\tilde{f}) = T^+_{\Delta}\mathcal{P}_{\Delta}\tilde{f}. \tag{4.25}$$

(ii) Furthermore, if the hypotheses of Lemma 4.2 are satisfied and $\tilde{f} \in \widetilde{H}^{-1/2}(\Omega)$, then (4.8) implies that $u \in H^{s,-1/2}(\Omega, A)$ and $T^+(\tilde{f};u) = T^+(\tilde{A}u;u) = T^+u$. Henceforth, (4.17) takes the simpler form

$$T^+u + T^+_{\Delta}\left(A^{\nabla}_{\Omega}u;a\mathcal{R}u\right) - \frac{1}{2}\Psi - \mathcal{W}'_{\Delta}\Psi + \mathcal{L}^+_{\Delta}(a\Phi) = T^+_{\Delta}\mathcal{P}_{\Delta}\tilde{f} \quad \text{on } \partial\Omega. \tag{4.26}$$

If, in addition, $au \in H^{s,-1/2}(\Omega, \Delta)$, then by (4.22)

$$\Delta(a\mathcal{R}u) = r_\Omega A^\nabla_\Omega u = -\nabla \cdot (u\nabla a) = Au - \Delta(au) \in \widetilde{H}^{-\frac{1}{2}}_\bullet(\Omega).$$

Hence the canonical co-normal derivative $T^+_\Delta(a\mathcal{R}u)$ is well defined, and by (2.13), (2.17), (3.16), and (4.22)

$$\begin{aligned} &T^+_\Delta\big(A^\nabla_\Omega u; a\mathcal{R}u\big) \\ &\quad = \big(\gamma^{-1}\big)^*\big[A^\nabla_\Omega u - \check{\Delta}_\Omega(a\mathcal{R}u)\big] \\ &\quad = \big(\gamma^{-1}\big)^*\big[A^\nabla_\Omega u - \tilde{\Delta}(a\mathcal{R}u)\big] + T^+_\Delta(a\mathcal{R}u) \\ &\quad = \big(\gamma^{-1}\big)^*\big[-\nabla \cdot \mathring{E}_\Omega(u\nabla a) + \mathring{E}_\Omega\nabla \cdot (u\nabla a)\big] + T^+_\Delta(a\mathcal{R}u) \\ &\quad = \big(\gamma^{-1}\big)^*[\check{A}u - \tilde{A}u] + \big(\gamma^{-1}\big)^*\big[-\check{\Delta}(au) + \tilde{\Delta}(au)\big] + T^+_\Delta(a\mathcal{R}u) \\ &\quad = -T^+u + T^+_\Delta(au) + T^+_\Delta(a\mathcal{R}u). \end{aligned} \tag{4.27}$$

This reduces (4.26) to the relation

$$T^+_\Delta(au) + T^+_\Delta(a\mathcal{R}u) - \frac{1}{2}\Psi - \mathcal{W}'_\Delta\Psi + \mathcal{L}^+_\Delta(a\Phi) = T^+_\Delta\mathcal{P}_\Delta\tilde{f} \quad \text{on } \partial\Omega \tag{4.28}$$

with only *canonical* normal derivatives associated with the Laplace operator involved.

(iii) If the hypotheses of Lemma 4.2 are satisfied and, moreover, $\tilde{f} \in \widetilde{H}^{-1/2}(\Omega)$, and $a \in C^{\frac{3}{2}}_+(\overline{\Omega})$, then, by (3.32) and (3.39), $\mathcal{P}\tilde{f} \in H^{s,-\frac{1}{2}}(\Omega; A)$ and $\mathcal{R}_*\tilde{f} \in \widetilde{H}^{-1/2}(\Omega)$, implying $T^+(\tilde{f} + \mathring{E}_\Omega\mathcal{R}_*\tilde{f}; \mathcal{P}\tilde{f}) = T^+(\mathcal{P}\tilde{f})$. Henceforth, (4.18) reduces to the relation

$$T^+u + T^+\mathcal{R}u - \frac{1}{2}\Psi - \mathcal{W}'\Psi + T^+W\Phi = T^+\mathcal{P}\tilde{f} \quad \text{on } \partial\Omega$$

with only *canonical* co-normal derivatives associated with the operator A involved.

Remark 4.6 (i) Let the hypotheses of Lemma 4.2 be satisfied and suppose that a sequence $\{\tilde{f}_j\} \in \widetilde{H}^{-\frac{1}{2}}(\Omega)$ converges to $\tilde{f}$ in $\widetilde{H}^{s-2}(\Omega)$. By the continuity of operators (3.30) and (3.34), estimate (2.14), and relation (4.25) for $\tilde{f}_j$, we obtain that

$$T^+_\Delta(\tilde{f}; \mathcal{P}_\Delta\tilde{f}) = \lim_{j\to\infty} T^+_\Delta\mathcal{P}_\Delta\tilde{f}_j \quad \text{in } H^{s-\frac{3}{2}}(\partial\Omega);$$

see also Theorem 7.1.

(ii) If, moreover, $a \in C^{\frac{3}{2}}_+(\overline{\Omega})$, then, similarly,

$$T^+(\tilde{f} + \mathring{E}_\Omega\mathcal{R}_*\tilde{f}, \mathcal{P}\tilde{f}) = \lim_{j\to\infty} T^+(\tilde{f}_j + \mathring{E}_\Omega\mathcal{R}_*\tilde{f}_j, \mathcal{P}\tilde{f}_j) = \lim_{j\to\infty} T^+\mathcal{P}\tilde{f}_j.$$

Lemma 4.4 and the third Green identity (4.4) imply the following assertion.

Corollary 4.7 *Let Ω be a bounded Lipschitz domain, and let $\frac{1}{2} < s < \frac{3}{2}$, $a \in C^s_+(\overline{\Omega})$, $u \in H^s(\Omega)$, and $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ be such that $Au = r_\Omega\tilde{f}$ in Ω.*

(i) *Then*

$$\frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+(\tilde{f};u) + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}\tilde{f} \quad \text{on } \partial\Omega, \tag{4.29}$$

$$\frac{1}{2}T^+(\tilde{f},u) + T^+_\Delta\big(A^\nabla_\Omega u;a\mathcal{R}u\big) - \mathcal{W}'_\Delta T^+(\tilde{f},u) + \mathcal{L}_\Delta\big(a\gamma^+u\big)$$
$$= T^+_\Delta(\tilde{f};\mathcal{P}_\Delta\tilde{f}) \quad \text{on } \partial\Omega. \tag{4.30}$$

(ii) *If, moreover,* $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, *then*

$$\frac{1}{2}T^+(\tilde{f},u) + T^+\mathcal{R}u - \mathcal{W}'T^+(\tilde{f},u) + T^+W\gamma^+u$$
$$= T^+(\tilde{f} + \mathring{E}_\Omega\mathcal{R}_*\tilde{f},\mathcal{P}\tilde{f}) \quad \text{on } \partial\Omega, \tag{4.31}$$

where $\mathcal{R}_*$ *is defined in* (3.14) *in* (3.16).

Let us extend to Lipschitz domains and $s \in (\frac{1}{2},\frac{3}{2})$ Lemma 4.2(i,ii) from [3], which is proved there for smooth domains and $s = 1$.

Lemma 4.8 *Let* Ω *be a bounded simply connected Lipschitz domain, and let* $a \in C^s_+(\overline{\Omega})$ *with* $\frac{1}{2} < s < \frac{3}{2}$.

(i) *If* $\Psi^* \in H^{s-\frac{3}{2}}(\partial\Omega)$ *and* $r_\Omega V\Psi^* = 0$, *then* $\Psi^* = 0$.
(ii) *If* $\Phi^* \in H^{s-\frac{1}{2}}(\partial\Omega)$ *and* $r_\Omega W\Phi^* = 0$, *then* $\Phi^* = 0$.

Proof To prove (i), let us multiply equation $r_\Omega V\Psi^* = 0$ by a, which by the first relation in (3.56) reduces it to $r_\Omega V_\Delta\Psi^* = 0$ in Ω. Taking the trace of this equation on $\partial\Omega$ and using the first relation in (3.95) (for the case $a = 1$), by Theorem 7.3 we obtain item (i).

Similarly, multiplying the equation $r_\Omega W\Phi^* = 0$ by a, the second relation in (3.56) reduces it to $r_\Omega W_\Delta(a\Phi^*) = 0$ in Ω. Taking the trace of this equation on $\partial\Omega$ and using the first jump relation in (3.95) (for the case $a = 1$), we obtain $-\frac{1}{2}\hat{\Phi}^* + \mathcal{W}_\Delta\hat{\Phi}^* = 0$ on $\partial\Omega$, where $\hat{\Phi}^* = a\Phi^*$. Since this equation for $\hat{\Phi}^*$ is uniquely solvable (see Theorem 7.3), by condition (2.5) this implies item (ii). □

Theorem 4.9 *Let* Ω *be a bounded simply connected Lipschitz domain, and let* $a \in C^s_+(\overline{\Omega})$ *with* $\frac{1}{2} < s < \frac{3}{2}$. *Let* $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$. *A function* $u \in H^s(\Omega)$ *is a solution of PDE* $Au = r_\Omega\tilde{f}$ *in* Ω *if and only if it is a solution of boundary-domain integro-differential equation* (4.4).

Proof If $u \in H^s(\Omega)$ solves PDE $Au = r_\Omega\tilde{f}$ in Ω, then by Theorem 4.1(ii) it satisfies (4.4). On the other hand, if u solves the boundary-domain integro-differential equation (4.4), then using Lemma 4.2 for $\Psi = T^+(\tilde{f};u)$ and $\Phi = \gamma^+u$ completes the proof. □

5 Segregated BDIE systems for the Dirichlet problem

For $\frac{1}{2} < s < \frac{3}{2}$, let us consider the *Dirichlet problem:*
Find a function $u \in H^s(\Omega)$ *satisfying the equations*

$$Au = f \quad \text{in } \Omega, \tag{5.1}$$

$$\gamma^+u = \varphi_0 \quad \text{on } \partial\Omega, \tag{5.2}$$

where $f \in H^{s-2}(\Omega)$ and $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$.

Equation (5.1) is understood in the distributional sense (2.7), and the Dirichlet boundary condition (5.2) is understood in the trace sense. The following uniqueness assertion is well known for $s = 1$ and follows from the first Green identity; hence it also holds for $1 \le s < 3/2$.

Theorem 5.1 *Let $a \in C_+^{|s-1|}(\overline{\Omega})$ with $1 \le s < \frac{3}{2}$. The Dirichlet problem* (5.1)–(5.2) *has at most one solution in $H^s(\Omega)$.*

5.1 BDIE formulations and equivalence to the Dirichlet problem

Let $\frac{1}{2} < s < \frac{3}{2}$. In this section, we reduce the Dirichlet problem (5.1)–(5.2) to three different *segregated* boundary-domain integral equation (BDIE) systems. Two of these formulations, for $s = 1$ and infinitely smooth coefficients and infinitely smooth boundary, were analysed in [33].

Let $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ be an extension of $f \in H^{s-2}(\Omega)$ (i.e., $f = r_\Omega \tilde{f}$), which always exists; see [31, Lemma 2.15 and Theorem 2.16]. Let us substitute into (4.4), (4.29), (4.30), and (4.31) the generalised co-normal derivative and the trace of the function u as

$$T^+(\tilde{f}; u) = \psi, \qquad \gamma^+ u = \varphi_0,$$

where φ_0 is the known right-hand side of the Dirichlet boundary condition (5.2), and $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$ is a new unknown function that will be regarded as formally *segregated* from u. Thus we will look for the unknown couple $(u, \psi) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$.

BDIE system (D1). Let $a \in C_+^s(\overline{\Omega})$. To reduce the Dirichlet BVP (5.1)–(5.2) to the BDIE system (D1), we will use equation (4.4) in Ω and equation (4.29) on $\partial\Omega$. Then we arrive at the following system of the boundary-domain integral equations, (D1), which is similar to the corresponding system in [33]:

$$u + \mathcal{R}u - V\psi = \mathcal{F}_1^{D1} \quad \text{in } \Omega, \tag{5.3}$$

$$\gamma^+\mathcal{R}u - \mathcal{V}\psi = \mathcal{F}_2^{D1} \quad \text{on } \partial\Omega, \tag{5.4}$$

where

$$\mathcal{F}^{D1} = \begin{bmatrix} \mathcal{F}_1^{D1} \\ \mathcal{F}_2^{D1} \end{bmatrix} = \begin{bmatrix} F_0^D \\ \gamma^+ F_0^D - \varphi_0 \end{bmatrix} \quad \text{and} \quad F_0^D := \mathcal{P}\tilde{f} - W\varphi_0 \quad \text{in } \Omega. \tag{5.5}$$

Note that, for $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, we have the inclusion $F_0^D \in H^s(\Omega)$ due to the mapping properties of the Newtonian (volume) and layer potentials; see (3.30) and (3.69). Hence $\mathcal{F}^{D1} \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$.

BDIE system (D2$_\Delta$). Let $a \in C_+^s(\overline{\Omega})$. To obtain a segregated BDIE system of *the second kind*, we will use equation (4.4) in Ω and equation (4.30) on $\partial\Omega$. Then we arrive at the following BDIE system (D2$_\Delta$):

$$u + \mathcal{R}u - V\psi = \mathcal{F}_1^{D2\Delta} \quad \text{in } \Omega, \tag{5.6}$$

$$\frac{1}{2}\psi + T_\Delta^+\big(A_\Omega^\nabla u; a\mathcal{R}u\big) - \mathcal{W}'_\Delta\psi = \mathcal{F}_2^{D2\Delta} \quad \text{on } \partial\Omega, \tag{5.7}$$

where

$$\mathcal{F}^{D2\Delta} = \begin{bmatrix} \mathcal{F}_1^{D2\Delta} \\ \mathcal{F}_2^{D2\Delta} \end{bmatrix} = \begin{bmatrix} \mathcal{P}\tilde{f} - W\varphi_0 \\ T_\Delta^+(\tilde{f}; \mathcal{P}_\Delta\tilde{f}) - \mathcal{L}_\Delta(a\varphi_0) \end{bmatrix}. \tag{5.8}$$

Due to the mapping properties of the operators involved in (5.11), we have $\mathcal{F}^{D2\Delta} \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$.

BDIE system (D2). Let the coefficient be smoother than in the first two cases, $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$. Now we will use equation (4.4) in Ω and equation (4.31) on $\partial\Omega$. Then we arrive at another BDIE system of *the second kind*, (D2), which is similar to the corresponding system in [33]:

$$u + \mathcal{R}u - V\psi = \mathcal{F}_1^{D2} \quad \text{in } \Omega, \tag{5.9}$$

$$\frac{1}{2}\psi + T^+\mathcal{R}u - \mathcal{W}'\psi = \mathcal{F}_2^{D2} \quad \text{on } \partial\Omega, \tag{5.10}$$

where

$$\mathcal{F}^{D2} = \begin{bmatrix} \mathcal{F}_1^{D2} \\ \mathcal{F}_2^{D2} \end{bmatrix} = \begin{bmatrix} \mathcal{P}\tilde{f} - W\varphi_0 \\ T^+(\tilde{f} + \mathring{E}_\Omega r_\Omega \mathcal{R}_*\tilde{f}; \mathcal{P}\tilde{f}) - T^+W\varphi_0 \end{bmatrix}. \tag{5.11}$$

Due to the mapping properties of the operators involved in (5.11), we have $\mathcal{F}^{D2} \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$.

Let us prove that BVP (5.1)–(5.2) in Ω is equivalent to each of the three systems of BDIEs, (D1), (D2$_\Delta$), and (D2).

Theorem 5.2 *Let $a \in C_+^s(\overline{\Omega})$ with $\frac{1}{2} < s < \frac{3}{2}$. Let $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$, $f \in H^{s-2}(\Omega)$, and $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ be such that $r_\Omega\tilde{f} = f$.*

(i) *If a function $u \in H^s(\Omega)$ solves the Dirichlet BVP* (5.1)–(5.2), *then the couple $(u, \psi) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$, where*

$$\psi = T^+(\tilde{f}; u) \quad \text{on } \partial\Omega \tag{5.12}$$

solves the BDIE systems (D1), (D2$_\Delta$) *and, if $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, then also the BDIE system* (D2).

(ii) *Vice versa, if a a couple $(u, \psi) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$ solves one of the BDIE systems,* (D1), (D2$_\Delta$), *or* (D2) (*if $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$*), *then this solution solves the other BDIE systems, whereas u solves the Dirichlet BVP, and ψ satisfies* (5.12).

Proof (i) Let $u \in H^s(\Omega)$ be a solution to BVP (5.1)–(5.2). Setting ψ by (5.12) evidently implies $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$. Then it immediately follows from Theorem 4.9 and relations (4.29)–(4.31) that the couple (u, ψ) solves systems (D1), (D2)$_\Delta$, and, if $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, then also (D2), with the right-hand sides (5.5), (5.8), and (5.11), respectively, which completes the proof of item (i).

(ii) Let now a couple $(u, \psi) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$ solve BDIE system (5.3)–(5.4). Taking the trace of equation (5.3) on $\partial\Omega$, and subtracting equation (5.4) from it, we obtain

$$\gamma^+u = \varphi_0 \quad \text{on } \partial\Omega, \tag{5.13}$$

i.e., u satisfies the Dirichlet condition (5.2). Equation (5.3) and Lemma 4.2 with $\Psi = \psi$ and $\Phi = \varphi_0$ imply that u is a solution of PDE (5.1), and

$$V\Psi^* - W\Phi^* = 0 \quad \text{in } \Omega,$$

where $\Psi^* = \psi - T^+(\tilde{f}; u)$ and $\Phi^* = \varphi_0 - \gamma^+ u$. Due to equation (5.13), $\Phi^* = 0$. Then Lemma 4.8(i) implies $\Psi^* = 0$, i.e., condition (5.12). Thus u obtained from solution of BDIE system (D1) solves the Dirichlet problem and hence, by item (i) of the theorem, (u, ψ) solves also BDIE system (D2$_\Delta$) and, if $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, then also (D2).

Let now a couple $(u, \psi) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$ solve BDIE system (5.6)–(5.7). Lemma 4.2 for equation (5.6) implies that u is a solution of PDE (5.1), and equation (4.9) holds for $\Psi = \psi$ and $\Phi = \varphi_0$, whereas Corollary 4.7 gives equation (4.30). Multiplication of (4.9) by a reduces it to

$$V_\Delta\bigl(\psi - T^+(\tilde{f}; u)\bigr) - W_\Delta\bigl(a(\varphi_0 - \gamma^+ u)\bigr) = 0 \quad \text{in } \Omega. \tag{5.14}$$

Subtracting (4.30) from equation (5.7) and taking into account (5.14) give

$$\psi - T^+(\tilde{f}; u) = 0 \quad \text{on } \partial\Omega, \tag{5.15}$$

that is, equation (5.12) is proved. Equations (5.14) and (5.15) give $W_\Delta \Phi^* = 0$ in Ω, where $\Phi^* = a(\varphi_0 - \gamma^+ u)$. Then Lemma 4.8(ii) implies $\Phi^* = 0$ on $\partial\Omega$. This means that u satisfies the Dirichlet condition (5.2). Thus u obtained from solution of BDIE system (D2$_\Delta$) solves the Dirichlet problem, and hence, by item (i) of the theorem, the couple (u, ψ) solves also BDIE system (D1) and, if $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, then also (D2).

Let, finally, $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, and let a couple $(u, \psi) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$ solve BDIE system (5.9)–(5.10). Lemma 4.2 for equation (5.9) implies that u is a solution of PDE (5.1), and equation (4.9) holds for $\Psi = \psi$ and $\Phi = \varphi_0$, whereas Corollary 4.7 gives equation (4.31). Subtracting (4.31) from equation (5.10) and adding to it the canonical co-normal derivative T^+ of equation (4.9) lead to (5.12). Equations (4.9) and (5.12) imply $W\Phi^* = 0$ in Ω, where $\Phi^* = \varphi_0 - \gamma^+ u$. Then by Lemma 4.8(ii) we deduce $\Phi^* = 0$ on $\partial\Omega$. This means that u satisfies the Dirichlet condition (5.2). Thus u obtained from solution of BDIE system (D2) solves the Dirichlet problem, and hence, by item (i) of the theorem, the couple (u, ψ) solves also BDIE systems (D1) and (D2$_\Delta$). □

5.2 Properties of BDIE system operators for the Dirichlet problem

BDIE systems (D1), (D2$_\Delta$), and (D2) can be written as

$$\mathfrak{D}^1 \mathcal{U}^D = \mathcal{F}^{D1}, \qquad \mathfrak{D}^{2\Delta} \mathcal{U}^D = \mathcal{F}^{D2\Delta}, \quad \text{and} \quad \mathfrak{D}^2 \mathcal{U}^D = \mathcal{F}^{D2},$$

respectively. Here $\mathcal{U}^D := (u, \psi)^\top \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$,

$$\mathfrak{D}^1 := \begin{bmatrix} I - \mathcal{R} & -V \\ \gamma^+\mathcal{R} & -\mathcal{V} \end{bmatrix},$$

$$\mathfrak{D}^{2\Delta} := \begin{bmatrix} I + \mathcal{R} & -V \\ T_\Delta^+(A_\Omega^\nabla; a\mathcal{R}) & \frac{1}{2}I - \mathcal{W}'_\Delta \end{bmatrix}, \qquad \mathfrak{D}^2 := \begin{bmatrix} I + \mathcal{R} & -V \\ T^+\mathcal{R} & \frac{1}{2}I - \mathcal{W}' \end{bmatrix},$$

whereas $\mathcal{F}^{D1}$, $\mathcal{F}^{D2\Delta}$, and $\mathcal{F}^{D2}$ are given by (5.5), (5.8), and (5.11), respectively. Note that

$$T^+_\Delta\big(A^\nabla_\Omega; a\mathcal{R}\big)u := \big(\gamma^{-1}\big)^*\big(A^\nabla_\Omega u - \check{\Delta}_\Omega(a\mathcal{R}u)\big). \tag{5.16}$$

Let $\frac{1}{2} < s < \frac{3}{2}$. The operators

$$\mathfrak{D}^1 : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \quad \text{if } a \in C^s_+(\overline{\Omega}), \tag{5.17}$$

$$\mathfrak{D}^{2\Delta} : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \quad \text{if } a \in C^s_+(\overline{\Omega}), \tag{5.18}$$

$$\mathfrak{D}^2 : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \quad \text{if } a \in C^{\frac{3}{2}}_+(\overline{\Omega}), \tag{5.19}$$

are continuous due to the mapping properties of the operators constituting them (see Sect. 3), whereas for the right-hand sides of the BDIE systems, we have the inclusions $\mathcal{F}^{D1} \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$, $\mathcal{F}^{D2\Delta} \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$, and $\mathcal{F}^{D2} \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$.

Theorem 5.3 *Let Ω be a bounded simply connected Lipschitz domain, and let $\frac{1}{2} < s < \frac{3}{2}$. Operators* (5.17)–(5.19) *are Fredholm operators with zero index.*

Proof The continuity of operators has been already proved.

To prove the Fredholm property of operator (5.17), let us consider the operator

$$\mathfrak{D}^1_0 := \begin{bmatrix} I & -V \\ 0 & -\mathcal{V} \end{bmatrix} : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega). \tag{5.20}$$

As a result of compactness properties of the operators $\mathcal{R}$ and $\gamma^+\mathcal{R}$ given by (3.35) and (3.42) in Theorem 3.2, operator (5.20) is a compact perturbation of operator (5.17). The operator $\mathfrak{D}^1_0$ is an upper triangular matrix operator with the following scalar diagonal invertible operators:

$$I : H^s(\Omega) \to H^s(\Omega),$$

$$\mathcal{V} : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega),$$

where the invertibility of the operator $\mathcal{V}$ is implied by the invertibility of operator $\mathcal{V}_\Delta$ in (7.4) and by the first relation in (3.93). This implies that operator (5.20) is invertible. Thus (5.17) is a Fredholm operator with zero index.

The operator

$$\mathfrak{D}^2_0 := \begin{bmatrix} I & -V \\ 0 & \frac{1}{2}I - \mathcal{W}'_\Delta \end{bmatrix} : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \tag{5.21}$$

is a compact perturbation of operator (5.18). Indeed, the operators $\mathcal{R} : H^s(\Omega) \to H^s(\Omega)$ is compact due to Theorem 3.2. The compactness of the operator $T^+_\Delta(A^\nabla_\Omega; a\mathcal{R}) : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$, defined by (5.16), follows from that of the operator $A^\nabla_\Omega : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ given by Lemma 4.3(ii) and of the operator $\mathcal{R} : H^s(\Omega) \to H^s(\Omega)$, i.e., operator (3.35) in Theorem 3.2.

Consider the diagonal operators of the upper triangular matrix operator $\mathfrak{D}_0^2$. The operator $I : H^s(\Omega) \to H^s(\Omega)$ is evidently invertible, whereas the invertibility of the operator $\frac{1}{2}I - \mathcal{W}'_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$ is stated by Theorem 7.3. This implies that operator (5.21) is invertible, and hence operator (5.18) is Fredholm with zero index.

Operator (5.21) is also a compact perturbation of operator (5.19). Indeed, the operators $\mathcal{R} : H^s(\Omega) \to H^s(\Omega)$ and $T^+\mathcal{R} : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$ are compact due to Theorem 3.2. From the first representation in (3.94), for $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, the operator $\mathcal{W}'_\Delta - \mathcal{W}' = \frac{\partial_\nu a}{a}\mathcal{V}_\Delta$: $H^{s-\frac{3}{2}}(\partial\Omega) \to H^\sigma(\partial\Omega)$, where $\sigma = \min\{\frac{1}{2}, s - \frac{1}{2}\}$, is continuous, which implies that the operator $\mathcal{W}'_\Delta - \mathcal{W}' : H^{s-\frac{3}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$ is compact. Since operator (5.21) is invertible, this implies that operator (5.19) is Fredholm with zero index. □

Theorem 5.4 *Let* Ω *be a bounded simply connected Lipschitz domain*, $\frac{1}{2} < s < \frac{3}{2}$, *and* $\sigma = \max\{1, s\}$. *The following operators are continuously invertible*:

$$\mathfrak{D}^1 : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \quad \text{if } a \in C_+^\sigma(\overline{\Omega}), \tag{5.22}$$

$$\mathfrak{D}^{2\Delta} : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \quad \text{if } a \in C_+^\sigma(\overline{\Omega}), \tag{5.23}$$

$$\mathfrak{D}^2 : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \quad \text{if } a \in C_+^{\frac{3}{2}}(\overline{\Omega}). \tag{5.24}$$

Proof First, let $1 \le s < \frac{3}{2}$. Then $\sigma = s$, and the injectivity of operators (5.22)–(5.24) is implied by the equivalence Theorem 5.2(ii) and the BVP uniqueness Theorem 5.1. Indeed, consider, for example, the injectivity of operator (5.22). For the homogeneous equation $\mathfrak{D}^1\mathcal{U}^D = 0$, its zero right-hand side $\mathcal{F}^{D1} = 0$ can be represented as in (5.5) in terms of $\tilde{f} = 0$ and $\varphi_0 = 0$. Then, by Theorem 5.2(ii), $\mathcal{U}^D = (u, T^+(0; u))^\top$, where u is a solution of the Dirichlet problem (5.1)–(5.2) with the right-hand sides $f = 0$ and $\varphi_0 = 0$, which has only the trivial solution $u = 0$ due to Theorem 5.1. The arguments for the injectivity of operators (5.23) and (5.22) are similar.

Since, by Theorem 5.3, operators (5.22)–(5.24) are Fredholm with zero index, this implies their invertibility for $1 \le s < \frac{3}{2}$.

Let now $\frac{1}{2} < s \le 1$. Then $\sigma = 1$, i.e., $a \in C_+^1(\overline{\Omega})$ for operators (5.22)–(5.23), and $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$ for operator (5.24). Hence, for a fixed function a satisfying the corresponding conditions in (5.22)–(5.23), all these operators are continuous for $\frac{1}{2} < s \le 1$. By Theorem 5.3 they are also Fredholm with zero index. Since, as already proved, at $s = 1$, these operators are also invertible, Lemma 7.5 implies that their kernels (null-spaces) consist of only the zero element for any $s \in (\frac{1}{2}, 1]$, which implies that the operators are invertible for all s from this interval. □

Theorems 5.4 and 5.2 imply the following assertion.

Corollary 5.5 *Let* Ω *be a bounded simply connected Lipschitz domain*, $\frac{1}{2} < s < \frac{3}{2}$, $f \in H^{s-2}(\Omega)$, $\varphi_0 \in H^{s-\frac{1}{2}}(\partial\Omega)$, *and* $a \in C_+^\sigma(\overline{\Omega})$ *with* $\sigma = \max\{1, s\}$. *Then the Dirichlet problem* (5.1)–(5.2) *is uniquely solvable in* $H^s(\Omega)$. *The solution is* $u = (\mathcal{A}^D)^{-1}(f, \varphi_0)^\top$, *where the inverse operator* $(\mathcal{A}^D)^{-1} : H^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$ *to the left-hand side operator* $\mathcal{A}^D : H^s(\Omega) \to H^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ *of the Dirichlet problem* (5.1)–(5.2) *is continuous*.

Remark 5.6 For a given function $f \in H^{s-2}(\Omega)$, its extension $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ is not unique. Nevertheless, since the solution of the Dirichlet BVP (5.1)–(5.2) does not depend on this extension, equivalence Theorem 5.2(ii) implies that u in the solution of BDIE systems (D1) and (D2) does not depend on the particular choice of extension $\tilde{f}$ although ψ obviously does; see (5.12).

6 Segregated BDIE systems for the Neumann problem

Let us consider the *Neumann problem*: *Find a function* $u \in H^s(\Omega)$ *satisfying the equations*

$$Au = r_\Omega \tilde{f} \quad \text{in } \Omega, \tag{6.1}$$

$$T^+(\tilde{f}; u) = \psi_0 \quad \text{on } \partial\Omega, \tag{6.2}$$

where $\psi_0 \in H^{s-\frac{3}{2}}(\partial\Omega)$ and $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$.

Equation (6.1) in understood in the distribution sense (2.7), and the Neumann boundary condition (6.2) in the sense (2.13). The following assertion is well known and can be proved, e.g., using variational settings and the Lax–Milgram lemma.

Theorem 6.1 *Let* $s = 1$ *and* $a \in L_\infty(\Omega)$.

(i) *The homogeneous Neumann problem* (6.1)–(6.2) *admits only one linearly independent solution* $u^0 = 1$ *in* $H^1(\Omega)$.

(ii) *The non-homogeneous Neumann problem* (6.1)–(6.2) *is solvable if and only if*

$$\langle \tilde{f}, u^0 \rangle_\Omega - \langle \psi_0, \gamma^+ u^0 \rangle_{\partial\Omega} = 0. \tag{6.3}$$

Remark 6.2 Item (i) in Theorem 6.1 evidently implies that, for $1 \le s < \frac{3}{2}$ and $a \in C_+^{|s-1|}(\overline{\Omega})$, the homogeneous Neumann problem associated with (6.1)–(6.2) also admits only one linearly independent solution $u^0 = 1$ in $H^s(\Omega)$.

6.1 BDIE formulations and equivalence to the Neumann problem

Let $\frac{1}{2} < s < \frac{3}{2}$. We will explore different possibilities of reducing the Neumann problem (6.1)–(6.2) to a BDIE system. Let us represent in (4.4), (4.29), (4.30), and (4.31) the generalised co-normal derivative and the trace of the function u as

$$T^+(\tilde{f}; u) = \psi_0, \qquad \gamma^+ u = \varphi,$$

where ψ_0 is the known right-hand side of the Neumann boundary condition (6.2), and $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$ is a new unknown function that will be regarded as formally *segregated* from u. Thus we will look for the unknown couple $(u, \varphi) \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$.

BDIE system (N1$_\Delta$). Let $a \in C_+^s(\overline{\Omega})$. Using equation (4.4) in Ω and equation (4.30) on $\partial\Omega$, we arrive at the following BDIE system (N1$_\Delta$) of two equations for the couple of unknowns (u, φ):

$$u + \mathcal{R}u + W\varphi = \mathcal{F}_1^{N1\Delta} \quad \text{in } \Omega, \tag{6.4}$$

$$T_\Delta^+\bigl(A_\Omega^\nabla u; a\mathcal{R}u\bigr) + \mathcal{L}_\Delta(a\varphi) = \mathcal{F}_2^{N1\Delta} \quad \text{on } \partial\Omega, \tag{6.5}$$

where

$$\mathcal{F}^{N1\Delta} = \begin{bmatrix} \mathcal{F}_1^{N1\Delta} \\ \mathcal{F}_2^{N1\Delta} \end{bmatrix} = \begin{bmatrix} \mathcal{P}\tilde{f} + V\psi_0 \\ T_\Delta^+(\tilde{f}; \mathcal{P}_\Delta \tilde{f}) - \frac{1}{2}\psi_0 + \mathcal{W}'_\Delta \psi_0 \end{bmatrix}. \tag{6.6}$$

Due to the mapping properties of the operators involved in (6.9), we have $\mathcal{F}^{N1\Delta} \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$.

BDIE system (N1). Let the coefficient be smoother than in the previous case, $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$. Now, using equation (4.4) in Ω and equation (4.31) on $\partial\Omega$, we arrive at the following BDIE system (N1) of two equations for the couple of unknowns (u, φ), which is similar to the corresponding system in [33]:

$$u + \mathcal{R}u + W\varphi = \mathcal{F}_1^{N1} \quad \text{in } \Omega, \tag{6.7}$$

$$T^+\mathcal{R}u + T^+W\varphi = \mathcal{F}_2^{N1} \quad \text{on } \partial\Omega, \tag{6.8}$$

where

$$\mathcal{F}^{N1} = \begin{bmatrix} \mathcal{F}_1^{N1} \\ \mathcal{F}_2^{N1} \end{bmatrix} = \begin{bmatrix} \mathcal{P}\tilde{f} + V\psi_0 \\ T^+(\tilde{f} + \mathring{E}_\Omega r_\Omega \mathcal{R}_* \tilde{f}; \mathcal{P}\tilde{f}) - \frac{1}{2}\psi_0 + \mathcal{W}'\psi_0 \end{bmatrix}. \tag{6.9}$$

Due to the mapping properties of the operators involved in (6.9), we have $\mathcal{F}^{N1} \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$.

BDIE system (N2). Let again $a \in C_+^s(\overline{\Omega})$. If we use equation (4.4) in Ω and equation (4.29) on $\partial\Omega$, we arrive for the couple (u, φ) at the following BDIE system (N2) of two equations *of the second kind*, which is also similar to the corresponding system in [33]:

$$u + \mathcal{R}u + W\varphi = \mathcal{F}_1^{N2} \quad \text{in } \Omega, \tag{6.10}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{R}u + \mathcal{W}\varphi = \mathcal{F}_2^{N2}, \quad \text{on } \partial\Omega. \tag{6.11}$$

where

$$\mathcal{F}^{N2} = \begin{bmatrix} \mathcal{F}_1^{N2} \\ \mathcal{F}_2^{N2} \end{bmatrix} = \begin{bmatrix} F_0^N \\ \gamma^+ F_0^N \end{bmatrix}, \qquad F_0^N := \mathcal{P}\tilde{f} + V\psi_0 \quad \text{in } \Omega. \tag{6.12}$$

Due to the mapping properties of the operators involved in (6.12), we have $\mathcal{F}^{N2} \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$.

Theorem 6.3 *Let* $\frac{1}{2} < s < \frac{3}{2}$, $a \in C_+^s(\overline{\Omega})$, $\psi_0 \in H^{s-\frac{3}{2}}(\partial\Omega)$, *and* $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$.

(i) *If a function* $u \in H^s(\Omega)$ *solves the Neumann problem* (6.1)–(6.2), *then the couple* (u, φ) *with* $\varphi = \gamma^+u \in H^{s-\frac{1}{2}}(\partial\Omega)$ *solves BDIE systems* ($N1_\Delta$), ($N2$), *and, if* $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$, *also* ($N1$).

(ii) *Vice versa, if a couple* $(u, \varphi) \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ *solves one of the BDIE systems,* ($N1_\Delta$), ($N2$), *or* ($N1$) (*if* $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$), *then the couple solves the other two BDE systems, whereas u solves the Neumann problem* (6.1)–(6.2) *and* $\gamma^+u = \varphi$.

Proof (i) Let $u \in H^s(\Omega)$ be a solution of the Neumann problem (6.1)–(6.2). Then from Theorem 4.9 and relations (4.29)–(4.31) we see that the couple (u, φ) with $\varphi = \gamma^+u$ solves BDIE systems (N1_Δ), (N2), and (N1) with the right-hand sides (6.6), (6.12), and (6.9), respectively, which proves item (i).

(ii) Let a couple $(u, \varphi) \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N1_Δ). Lemma 4.2 for equation (6.4) implies that u is a solution of PDE (6.1), and equation (4.9) holds for $\Psi = \psi_0$

and $\Phi = \varphi$, whereas Corollary 4.7 gives equation (4.31). Multiplication of (4.9) by a reduces it to

$$V_\triangle\big(\psi_0 - T^+(\tilde{f};u)\big) - W_\triangle\big(a\big(\varphi - \gamma^+u\big)\big) = 0 \quad \text{in } \Omega. \tag{6.13}$$

Subtracting (4.31) from equation (6.5), we get $T^+(\tilde{f};u) = \psi_0$ on $\partial\Omega$, i.e., u satisfies the Neumann condition (6.2). Further, from (6.13) we derive $W_\triangle(a(\varphi - \gamma^+u)) = 0$ in Ω, whence $\gamma^+u = \varphi$ on $\partial\Omega$ by Lemma 4.8, completing item (ii) for BDIE system (N1$_\triangle$).

Let a couple $(u,\varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N1). Lemma 4.2 for equation (6.7) implies that u is a solution of PDE (6.1), and equation (4.9) holds for $\Psi = \psi_0$ and $\Phi = \varphi$, whereas Corollary 4.7 gives equation (4.31). Subtracting (4.31) from equation (6.8) gives $T^+(\tilde{f};u) = \psi_0$ on $\partial\Omega$, i.e., u satisfies the Neumann condition (6.2). Further, from (4.9) we derive $W(\gamma^+u - \varphi) = 0$ in Ω, whence $\gamma^+u = \varphi$ on $\partial\Omega$ by Lemma 4.8, completing item (ii) for BDIE system (N1).

Let now a couple $(u,\varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N2). Further, taking the trace of (6.10) on $\partial\Omega$ and comparing the result with (6.11), we easily derive that $\gamma^+u = \varphi$ on $\partial\Omega$. Lemma 4.2 for equation (6.10) implies that u is a solution of PDE (6.1), and equations (4.9) holds for $\Psi = \psi_0$ and $\Phi = \varphi$. Further, from (4.9) and relation $\gamma^+u = \varphi$ we derive

$$V\big(\psi_0 - T^+(\tilde{f};u)\big) = 0 \quad \text{in } \Omega,$$

whence $T^+(\tilde{f};u) = \psi_0$ on $\partial\Omega$ by Lemma 4.8, i.e., u solves the Neumann problem (6.1)–(6.2), which completes the proof of item (ii) for BDIE system (N2). □

6.2 Properties of BDIE system operators for the Neumann problem

BDIE systems (N1$_\triangle$), (N1), and (N2) can be written, respectively, as

$$\mathfrak{N}^{1\triangle}\mathcal{U}^N = \mathcal{F}^{N1\triangle}, \qquad \mathfrak{N}^1\mathcal{U}^N = \mathcal{F}^{N1}, \qquad \mathfrak{N}^2\mathcal{U}^N = \mathcal{F}^{N2},$$

where $\mathcal{U}^N = (u,\varphi)^\top \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$,

$$\mathfrak{N}^{1\triangle} := \begin{bmatrix} I + \mathcal{R} & W \\ T^+_\triangle(A^\nabla_\Omega; a\mathcal{R}) & \mathcal{L}_0 \end{bmatrix}, \qquad \mathfrak{N}^1 := \begin{bmatrix} I + \mathcal{R} & W \\ T^+\mathcal{R} & T^+W \end{bmatrix},$$

$$\mathfrak{N}^2 := \begin{bmatrix} I + \mathcal{R} & W \\ \gamma^+\mathcal{R} & \frac{1}{2}I + \mathcal{W} \end{bmatrix},$$

and we denoted $\mathcal{L}_0 g := \mathcal{L}_\triangle(ag)$. Let $\frac{1}{2} < s < \frac{3}{2}$. Due to the mapping properties of the potentials (see Sect. 3), the operators

$$\mathfrak{N}^{1\triangle} : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \quad \text{if } a \in C^s_+(\overline{\Omega}), \tag{6.14}$$

$$\mathfrak{N}^1 : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \quad \text{if } a \in C^{\frac{3}{2}}_+(\overline{\Omega}), \tag{6.15}$$

$$\mathfrak{N}^2 : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \quad \text{if } a \in C^s_+(\overline{\Omega}) \tag{6.16}$$

are continuous, whereas for the right-hand sides of the BDIE systems, we have the inclusions $\mathcal{F}^{N1\triangle} \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$, $\mathcal{F}^{N1} \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$, $\mathcal{F}^{N2} \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$.

Theorem 6.4 *Let Ω be a bounded simply connected Lipschitz domain, and let $\frac{1}{2} < s < \frac{3}{2}$. Operators* (6.14)–(6.16) *are Fredholm operators with zero index.*

Proof The continuity of operators is already proved.

Let us consider operator (6.14). Due to estimate (2.5) and Theorem 7.3, the operator $\mathcal{L}_0 : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$ is a Fredholm operator with zero index. Therefore the operator

$$\mathfrak{N}_0^1 := \begin{bmatrix} I & W \\ 0 & \mathcal{L}_0 \end{bmatrix} : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \tag{6.17}$$

is also Fredholm with zero index. Operator (6.14) is a compact perturbation of $\mathfrak{N}_0^1$ since the operators

$$\mathcal{R} : H^s(\Omega) \to H^s(\Omega), \tag{6.18}$$

$$T_\Delta^+\big(A_\Omega^\nabla; a\mathcal{R}\big) : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega) \tag{6.19}$$

are compact due to Theorem 3.2, as has been shown in the compactness proof related to operator (5.21). Thus operator (6.14) is Fredholm with zero index as well.

Operator (6.17) is also a compact perturbation of operator (6.15). Indeed, the operators (6.18),

$$T^+ W - \mathcal{L}_0 : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega),$$

$$T^+\mathcal{R} : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega)$$

are compact, due to relations (3.94) and (3.96) and Theorem 3.7. Thus operator (6.15) is Fredholm with zero index as well.

To analyse operator (6.16), let us consider the auxiliary operator

$$\mathfrak{N}_0^2 := \begin{bmatrix} I & W \\ 0 & \frac{1}{2}I + \mathcal{W} \end{bmatrix} : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega). \tag{6.20}$$

For any function g, we can represent $(\frac{1}{2}I + \mathcal{W})g = \frac{1}{a}(\frac{1}{2}I + \mathcal{W}_\Delta)(ag)$, which, by Theorem 7.3, implies that the operator $\frac{1}{2}I + \mathcal{W} : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ and hence operator (6.20) are Fredholm with zero index. Due to the compactness of operator (6.18), operator (6.16) is a compact perturbation of operator (6.20) and thus is Fredholm with zero index as well. □

Theorem 6.5 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, and $\sigma = \max\{1, s\}$. The following operators have one-dimensional null-spaces*, $\ker \mathfrak{N}^{1\Delta} = \ker \mathfrak{N}^1 = \ker \mathfrak{N}^2$, *in $H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$, spanned over the element $(u^0, \varphi^0) = (1, 1)$:*

$$\mathfrak{N}^{1\Delta} : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \quad \textit{if } a \in C_+^\sigma(\overline{\Omega}), \tag{6.21}$$

$$\mathfrak{N}^1 : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \quad \textit{if } a \in C_+^{\frac{3}{2}}(\overline{\Omega}), \tag{6.22}$$

$$\mathfrak{N}^2 : H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \quad \textit{if } a \in C_+^\sigma(\overline{\Omega}). \tag{6.23}$$

Proof The conditions on the coefficient a imply that, for $s = 1$, operators (6.21)–(6.23) are continuous. Then the equivalence Theorem 6.3 and Theorem 6.1(i) imply that the homogeneous BDIE systems (N1$_\Delta$), (N1), and (N2) have only one linear independent solution $\mathcal{U}^0 = (u^0, \varphi^0)^\top = (1,1)^\top$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Indeed, consider, for example, the homogeneous equation $\mathfrak{N}^{1\Delta}\mathcal{U}^N = 0$. Its zero right-hand side $\mathcal{F}^{N1\Delta} = 0$ can be represented as in (6.6) in terms of $\tilde{f} = 0$ and $\psi_0 = 0$. Then, by Theorem 6.3(ii), $\mathcal{U}^N = (u, \gamma^+ u)^\top$, where u is a solution of the Neumann problem (6.1)–(6.2) with the right-hand sides $f = 0$ and $\psi_0 = 0$, which has only the one linearly independent solution, $u = 1$, due to Theorem 6.1. This proves the theorem for $s = 1$, and then Lemma 7.5 and Theorem 6.4 complete the proof for $\frac{1}{2} < s < \frac{3}{2}$. □

Lemma 6.6 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, and $a \in C_+^\sigma(\overline{\Omega})$ with $\sigma = \max\{1, s\}$. For any couple $(\mathcal{F}_1, \mathcal{F}_2) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_*, \Phi_*) \in \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ such that*

$$\mathcal{F}_1 = \mathcal{P}\tilde{f}_* - W\Phi_* \quad \text{in } \Omega, \tag{6.24}$$

$$\mathcal{F}_2 = T_\Delta^+(\tilde{f}_*; \mathcal{P}\tilde{f}_*) - \mathcal{L}_\Delta(a\Phi_*) \quad \text{on } \partial\Omega. \tag{6.25}$$

Moreover, $(\tilde{f}_*, \Phi_*) = \mathcal{C}_*(\mathcal{F}_1, \mathcal{F}_2)$, and $\mathcal{C}_* : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator given by

$$\tilde{f}_* = \check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\mathcal{F}_2, \tag{6.26}$$

$$\Phi_* = \frac{1}{a}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\left\{-a\mathcal{F}_1 + \mathcal{P}_\Delta\left[\check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\mathcal{F}_2\right]\right\}. \tag{6.27}$$

Proof Let us first assume that there exist $(\tilde{f}_*, \Phi_*) \in \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ satisfying equations (6.24) and (6.25) and find their expressions in terms of $\mathcal{F}_1$ and $\mathcal{F}_2$. Multiplying (6.24) by a, we get

$$a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_* = -W_\Delta(a\Phi_*) \quad \text{in } \Omega. \tag{6.28}$$

Applying the Laplace operator to (6.28), we obtain

$$\Delta(a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_*) = \Delta(a\mathcal{F}_1) - \tilde{f}_* = -\Delta W_\Delta(a\Phi_*) = 0 \quad \text{in } \Omega, \tag{6.29}$$

which means

$$\Delta(a\mathcal{F}_1) = r_\Omega\tilde{f}_* \quad \text{in } \Omega \tag{6.30}$$

and $a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_* \in H^{s,0}(\Omega; \Delta)$. Applying the canonical co-normal derivative operator T_Δ^+ to both sides of equation (6.28) and taking into account that $-\tilde{\Delta} W_\Delta(a\Phi_*) = \tilde{\Delta}(a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_*) = 0$ because $W_\Delta(a\Phi_*)$ is a harmonic function in Ω, we obtain, due to (2.17) and (2.13),

$$\begin{aligned} -\mathcal{L}_\Delta(a\Phi_*) &= -T_\Delta^+ W_\Delta(a\Phi_*) \\ &= T_\Delta^+(a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_*) \end{aligned}$$

$$= \left(\gamma^{-1}\right)^*\left[\tilde{\Delta}_\Omega(a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_*) - \check{\Delta}_\Omega(a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_*)\right]$$
$$= -\left(\gamma^{-1}\right)^*\check{\Delta}_\Omega(a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_*) = T^+_\Delta(0; a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_*), \tag{6.31}$$

where (6.30) was taken into account. Substituting this into (6.25), we obtain

$$\mathcal{F}_2 = T^+_\Delta(\tilde{f}_*, a\mathcal{F}_1) \quad \text{on } \partial\Omega. \tag{6.32}$$

Due to (6.30), we can represent

$$\tilde{f}_* = \check{\Delta}_\Omega(a\mathcal{F}_1) + \tilde{f}_{1*} = \nabla\cdot\mathring{E}_\Omega\nabla(a\mathcal{F}_1) - \gamma^*\Psi_*, \tag{6.33}$$

where $\tilde{f}_{1*} \in H^{s-2}_{\partial\Omega}$, which, due to, e.g., [31, Theorem 2.10], can be in turn represented as $\tilde{f}_{1*} = -\gamma^*\Psi_*$ with some $\Psi_* \in H^{s-\frac{3}{2}}(\partial\Omega)$. Then (6.30) is satisfied, and

$$\mathcal{F}_2 = T^+_\Delta(\tilde{f}_*, a\mathcal{F}_1) = \left(\gamma^{-1}\right)^*\left[\tilde{f}_* - \check{\Delta}(a\mathcal{F}_1)\right]$$
$$= \left(\gamma^{-1}\right)^*\tilde{f}_{1*} = -\left(\gamma^{-1}\right)^*\gamma^*\Psi_* = -\Psi_*, \tag{6.34}$$

because $\langle(\gamma^{-1})^*\gamma^*\Psi_*, w\rangle_{\partial\Omega} = \langle\gamma^*\Psi_*, \gamma^{-1}w\rangle_\Omega = \langle\Psi_*, w\rangle_{\partial\Omega}$ for any $w \in H^{\frac{3}{2}-s}(\partial\Omega)$. Hence (6.33) reduces to (6.26).

Now (6.28) can be written in the form

$$W_\Delta(a\Phi_*) = \mathcal{F}_\Delta \quad \text{in } \Omega, \tag{6.35}$$

where

$$\mathcal{F}_\Delta := -a\mathcal{F}_1 + \mathcal{P}_\Delta\tilde{f}_* = -a\mathcal{F}_1 + \mathcal{P}_\Delta\left[\check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\mathcal{F}_2\right] \tag{6.36}$$

is a harmonic function in Ω due to (6.29). The trace of equation (6.35) gives

$$-\frac{1}{2}a\Phi_* + \mathcal{W}_\Delta(a\Phi_*) = \gamma^+\mathcal{F}_\Delta \quad \text{on } \partial\Omega. \tag{6.37}$$

Since the operator $-\frac{1}{2}I + \mathcal{W}_\Delta : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ is an isomorphism (see Theorem 7.3), this implies

$$\Phi_* = \frac{1}{a}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\mathcal{F}_\Delta$$
$$= \frac{1}{a}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\left\{-a\mathcal{F}_1 + \mathcal{P}_\Delta\left[\check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\mathcal{F}_2\right]\right\},$$

which coincides with (6.27).

Relations (6.26) and (6.27) can be written as $(\tilde{f}_*, \Phi_*) = \mathcal{C}_*(\mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{C}_* : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as claimed. We still have to check that the functions $\tilde{f}_*$ and Φ_*, given by (6.26) and (6.27), satisfy equations (6.24) and (6.25). Indeed, Φ_* given by (6.27) satisfies equation (6.37) with $\mathcal{F}_\Delta$ given by (6.36), and thus $\gamma^+W_\Delta(a\Phi_*) = \gamma^+\mathcal{F}_\Delta$. Since both $W_\Delta(a\Phi_*)$ and $\mathcal{F}_\Delta$ are harmonic functions belonging to the space $H^s(\Omega)$, this implies (6.35) and, by (6.26), also (6.24). Finally, (6.26) implies by (6.34) that (6.32) is satisfied, and adding (6.31) to it leads to (6.25).

Let us now prove that the operator $\mathcal{C}_*$ is unique. Indeed, let a couple $(\tilde{f}_*, \Phi_*) \in \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (6.24)–(6.25) with $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. Then (6.30) implies that $r_\Omega \tilde{f}_* = 0$ in Ω, i.e., $\tilde{f}_* \in H^{s-2}_{\partial\Omega} \subset \widetilde{H}^{s-2}(\Omega)$. Hence, (6.32) reduces to $0 = T^+_\Delta(\tilde{f}_*, 0)$ on $\partial\Omega$. By the first Green identity (2.15) this gives

$$0 = \langle T^+_\Delta(\tilde{f}_*, 0), \gamma^+ v\rangle_{\partial\Omega} = \langle \tilde{f}_*, v\rangle_\Omega \quad \forall\, v \in H^{2-s}(\Omega),$$

which implies $\tilde{f}_* = 0$ in $\mathbb{R}^n$. Finally, (6.27) gives $\Phi_* = 0$. Hence, any solution of non-homogeneous linear system (6.24)–(6.25) has only one solution, which implies the uniqueness of the operator $\mathcal{C}_*$. □

Theorem 6.7 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, and $a \in C^\sigma_+(\overline{\Omega})$ with $\sigma = \max\{1, s\}$. The co-kernel of operator* (6.14) *is spanned over the functional*

$$g^{*1\Delta} := (0, 1)^\top \tag{6.38}$$

in $[H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)]^* = \widetilde{H}^{-s}(\Omega) \times H^{\frac{3}{2}-s}(\partial\Omega)$, i.e., $g^{*1\Delta}(\mathcal{F}_1, \mathcal{F}_2) = \langle \mathcal{F}_2, \gamma^+ u^0\rangle_{\partial\Omega}$, where $u^0 = 1$.

Proof Let us consider the equation $\mathfrak{N}^{1\Delta}\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, i.e., the BDIE system $(\mathrm{N1}_\Delta)$ for $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$,

$$u + \mathcal{R}u + W\varphi = \mathcal{F}_1 \quad \text{in } \Omega, \tag{6.39}$$

$$T^+_\Delta\big(A^\nabla_\Omega u; a\mathcal{R}u\big) + \mathcal{L}_\Delta(a\varphi) = \mathcal{F}_2 \quad \text{on } \partial\Omega, \tag{6.40}$$

with arbitrary $(\mathcal{F}_1, \mathcal{F}_2) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$. By Lemma 6.6 the right-hand side of the system can be presented in the form (6.24)–(6.25), i.e., system (6.39)–(6.40) reduces to

$$u + \mathcal{R}u + W(\varphi + \Phi_*) = \mathcal{P}\tilde{f}_* \quad \text{in } \Omega, \tag{6.41}$$

$$T^+_\Delta\big(A^\nabla_\Omega u; a\mathcal{R}u\big) + \mathcal{L}_\Delta(a\varphi + a\Phi_*) = T^+_\Delta(\tilde{f}_*; \mathcal{P}\tilde{f}_*) \quad \text{on } \partial\Omega, \tag{6.42}$$

where the couple $(\tilde{f}_*, \Phi_*) \in \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ is given by (6.26)–(6.27). Up to the notations, system (6.41)–(6.42) is the same as (6.4)–(6.5) with the right-hand side given by (6.6), where $\psi_0 = 0$.

First, let $s = 1$. Then Theorems 6.1 and 6.3 imply that BDIE system (6.41)–(6.42) and hence (6.39)–(6.40) are solvable if and only if

$$\begin{aligned}
\langle \tilde{f}_*, u^0\rangle_\Omega &= \big\langle \check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\mathcal{F}_2, u^0\big\rangle_\Omega \\
&= \big\langle \nabla \cdot \mathring{E}_\Omega \nabla(a\mathcal{F}_1) + \gamma^*\mathcal{F}_2, u^0\big\rangle_{\mathbb{R}^n} \\
&= -\big\langle \mathring{E}_\Omega \nabla(a\mathcal{F}_1), \nabla u^0\big\rangle_{\mathbb{R}^n} + \big\langle \mathcal{F}_2, \gamma^+ u^0\big\rangle_{\partial\Omega} \\
&= \big\langle \mathcal{F}_2, \gamma^+ u^0\big\rangle_{\partial\Omega} = 0,
\end{aligned} \tag{6.43}$$

where we took into account that $u^0 = 1$ in $\mathbb{R}^n$. Thus the functional $g^{*1\Delta}$ defined by (6.38) generates the necessary and sufficient solvability condition of equation $\mathfrak{N}^{1\Delta}\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$. Hence $g^{*1\Delta}$ is a basis of the co-kernel of $\mathfrak{N}^{1\Delta}$ (and thus the kernel of the operator $\mathfrak{N}^{1\Delta*}$ adjoint to $\mathfrak{N}^{1\Delta}$) for $s = 1$.

Let us now choose any $s \in (\frac{1}{2}, \frac{3}{2})$. By Theorem 6.4, operator (6.14) and thus its adjoint are Fredholm with zero index. We already proved that, at $s = 1$, the kernel of the adjoint operator is spanned over $g^{*1\Delta}$. For any fixed coefficient $a \in C_+^\sigma(\overline{\Omega})$, the operator

$$\mathfrak{N}^{1\Delta} : H^{s'}(\Omega) \times H^{s'-\frac{1}{2}}(\partial\Omega) \to H^{s'}(\Omega) \times H^{s'-\frac{3}{2}}(\partial\Omega) \tag{6.44}$$

is continuous for any $s' \in (\frac{1}{2}, \sigma]$ and particularly for $s' = s$ and $s' = 1$. Then Lemma 7.5 implies that the co-kernel of operator (6.44) is the same for $s' = s$ and $s' = 1$ and is spanned over $g^{*1\Delta}$. □

Lemma 6.8 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, and $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$. For any couple $(\mathcal{F}_1, \mathcal{F}_2) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$, there exists a unique couple $(\tilde{f}_{**}, \Phi_{**}) \in \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ such that*

$$\mathcal{F}_1 = \mathcal{P}\tilde{f}_{**} - W\Phi_{**} \quad \text{in } \Omega, \tag{6.45}$$

$$\mathcal{F}_2 = T^+(\tilde{f}_{**} + \mathring{E}_\Omega \mathcal{R}_* \tilde{f}_{**}; \mathcal{P}\tilde{f}_{**}) - T^+ W\Phi_{**} \quad \text{on } \partial\Omega. \tag{6.46}$$

Moreover, $(\tilde{f}_{**}, \Phi_{**}) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$, and $\mathcal{C}_{**} : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator given by

$$\tilde{f}_{**} = \check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\big(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n a\big), \tag{6.47}$$

$$\Phi_{**} = \frac{1}{a}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1} \gamma^+\big\{-a\mathcal{F}_1 + \mathcal{P}_\Delta\big[\check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\big(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n a\big)\big]\big\}. \tag{6.48}$$

Proof Let us first assume that there exist $(\tilde{f}_{**}, \Phi_{**}) \in \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ satisfying equations (6.45) and (6.46) and prove that they are then expressed in terms of $\mathcal{F}_1$ and $\mathcal{F}_2$ by (6.47)–(6.48). Let us rewrite (6.45) as

$$\mathcal{F}_1 - \mathcal{P}\tilde{f}_{**} = -W\Phi_{**} \quad \text{in } \Omega, \tag{6.49}$$

Multiplying (6.49) by a and applying the Laplace operator to it, we obtain

$$\Delta(a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_{**}) = \Delta(a\mathcal{F}_1) - \tilde{f}_{**} = -\Delta W_\Delta(a\Phi_{**}) = 0 \quad \text{in } \Omega, \tag{6.50}$$

which means that

$$\Delta(a\mathcal{F}_1) = r_\Omega\tilde{f}_{**} \quad \text{in } \Omega \tag{6.51}$$

and $a\mathcal{F}_1 - \mathcal{P}_\Delta\tilde{f}_{**} \in H^{s,0}(\Omega; \Delta)$. By equality (6.49) and the continuity of operator (3.73) in Theorem 3.5, we also have $\mathcal{F}_1 - \mathcal{P}\tilde{f}_{**} \in H^{1,0}(\Omega; A)$, which implies that the canonical co-normal derivative $T^+(\mathcal{F}_1 - \mathcal{P}\tilde{f}_{**})$ is well defined. Applying the canonical co-normal derivative operator T^+ to both sides of equation (6.49), we obtain

$$\begin{aligned}
-T^+ W\Phi_{**} &= T^+(\mathcal{F}_1 - \mathcal{P}\tilde{f}_{**}) = T^+\big(\tilde{A}(\mathcal{F}_1 - \mathcal{P}\tilde{f}_{**}); \mathcal{F}_1 - \mathcal{P}\tilde{f}_{**}\big) \\
&= T^+\big(\mathring{E}_\Omega \nabla \cdot \big(a\nabla(\mathcal{F}_1 - \mathcal{P}\tilde{f}_{**})\big); \mathcal{F}_1 - \mathcal{P}\tilde{f}_{**}\big) \\
&= T^+\big(\mathring{E}_\Omega \Delta(a\mathcal{F}_1 - \mathcal{P}_\Delta \tilde{f}_{**}) - \mathring{E}_\Omega \nabla \cdot \big((\mathcal{F}_1 - \mathcal{P}\tilde{f}_{**})\nabla a\big); \mathcal{F}_1 - \mathcal{P}\tilde{f}_{**}\big) \\
&= T^+\big(-\mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a) - \mathring{E}_\Omega \mathcal{R}_* \tilde{f}_{**}; \mathcal{F}_1 - \mathcal{P}\tilde{f}_{**}\big),
\end{aligned} \tag{6.52}$$

where (6.50) and the third relation in (3.16) were taken into account. Substituting this into (6.46), we obtain

$$\mathcal{F}_2 = T^+\big(\tilde{f}_{**} - \mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a), \mathcal{F}_1\big) \quad \text{on } \partial\Omega. \tag{6.53}$$

Due to (6.51), we can represent

$$\tilde{f}_{**} = \check{\Delta}_\Omega(a\mathcal{F}_1) + \tilde{f}_{1*} = \nabla \cdot \mathring{E}_\Omega \nabla(a\mathcal{F}_1) - \gamma^* \Psi_{**}, \tag{6.54}$$

where $\tilde{f}_{1*} \in H^{s-2}_{\partial\Omega}$, which, due to, e.g., [31, Theorem 2.10], can be in turn represented as $\tilde{f}_{1*} = -\gamma^* \Psi_{**}$ with some $\Psi_{**} \in H^{s-\frac{3}{2}}(\partial\Omega)$. Then (6.51) is satisfied, and

$$\begin{aligned}
\mathcal{F}_2 &= T^+\big(\tilde{f}_{**} - \mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a), \mathcal{F}_1\big) = \big(\gamma^{-1}\big)^*\big[\tilde{f}_{**} - \mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a) - \check{A}\mathcal{F}_1\big] \\
&= \big(\gamma^{-1}\big)^*\big[\nabla \cdot \mathring{E}_\Omega \nabla(a\mathcal{F}_1) - \gamma^* \Psi_{**} - \mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a) - \nabla \cdot \mathring{E}_\Omega (a\nabla \mathcal{F}_1)\big] \\
&= \big(\gamma^{-1}\big)^*\big[\nabla \cdot \mathring{E}_\Omega (\mathcal{F}_1 \nabla a) - \gamma^* \Psi_{**} - \mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a)\big] = -\Psi_{**} - \big(\gamma^+ \mathcal{F}_1\big)\partial_n a,
\end{aligned} \tag{6.55}$$

because for any $w \in H^{\frac{3}{2}-s}(\partial\Omega)$,

$$\begin{aligned}
&\big\langle \big(\gamma^{-1}\big)^*\big[\nabla \cdot \mathring{E}_\Omega (\mathcal{F}_1 \nabla a) - \gamma^* \Psi_{**} - \mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a)\big], w\big\rangle_{\partial\Omega} \\
&\quad = \big\langle \nabla \cdot \mathring{E}_\Omega (\mathcal{F}_1 \nabla a) - \gamma^* \Psi_{**} - \mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a), \gamma^{-1} w\big\rangle_\Omega \\
&\quad = \big\langle \nabla \cdot \mathring{E}_\Omega (\mathcal{F}_1 \nabla a), \gamma^{-1} w\big\rangle_{\mathbb{R}^n} - \big\langle \gamma^* \Psi_{**}, \gamma^{-1} w\big\rangle_\Omega - \big\langle \mathring{E}_\Omega \nabla \cdot (\mathcal{F}_1 \nabla a), \gamma^{-1} w\big\rangle_\Omega \\
&\quad = -\big\langle \mathring{E}_\Omega (\mathcal{F}_1 \nabla a), \nabla \gamma^{-1} w\big\rangle_{\mathbb{R}^n} - \langle \Psi_{**}, w\rangle_{\partial\Omega} + \big\langle \mathcal{F}_1 \nabla a, \nabla \gamma^{-1} w\big\rangle_\Omega \\
&\qquad - \big\langle n \cdot \gamma^+(\mathcal{F}_1 \nabla a), \gamma^+ \gamma^{-1} w\big\rangle_{\partial\Omega} = -\big\langle \big(\gamma^+ \mathcal{F}_1\big)\partial_n a, w\big\rangle_{\partial\Omega} - \langle \Psi_{**}, w\rangle_{\partial\Omega}.
\end{aligned}$$

Hence (6.53) reduces to $\Psi_{**} = -\mathcal{F}_2 - (\gamma^+ \mathcal{F}_1)\partial_n a$, and (6.54) to (6.47).

Now (6.49) can be written in the form

$$W_\Delta(a\Phi_{**}) = \mathcal{F}_\Delta \quad \text{in } \Omega, \tag{6.56}$$

where

$$\mathcal{F}_\Delta := -a\mathcal{F}_1 + \mathcal{P}_\Delta \tilde{f}_{**} = -a\mathcal{F}_1 + \mathcal{P}_\Delta\big[\check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\big(\mathcal{F}_2 + \big(\gamma^+ \mathcal{F}_1\big)\partial_n a\big)\big] \tag{6.57}$$

is a harmonic function in Ω due to (6.50). The trace of equation (6.56) gives

$$-\frac{1}{2}a\Phi_{**} + \mathcal{W}_\Delta(a\Phi_{**}) = \gamma^+ \mathcal{F}_\Delta \quad \text{on } \partial\Omega. \tag{6.58}$$

Since the operator $-\frac{1}{2}I + \mathcal{W}_\Delta : H^{s-\frac{1}{2}}(\partial\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ is an isomorphism (see Theorem 7.3), this implies

$$\begin{aligned}\Phi_{**} &= \frac{1}{a}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\mathcal{F}_\Delta \\ &= \frac{1}{a}\left(-\frac{1}{2}I + \mathcal{W}_\Delta\right)^{-1}\gamma^+\left\{-a\mathcal{F}_1 + \mathcal{P}_\Delta\left[\check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*\left(\mathcal{F}_2 + \left(\gamma^+\mathcal{F}_1\right)\partial_n a\right)\right]\right\},\end{aligned}$$

which coincides with (6.48).

Relations (6.47) and (6.48) can be written as $(\tilde{f}_{**}, \Phi_{**}) = \mathcal{C}_{**}(\mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{C}_{**} : H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \to \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ is a linear continuous operator, as claimed. We still have to check that the functions $\tilde{f}_{**}$ and Φ_{**} given by (6.47) and (6.48) satisfy equations (6.45) and (6.46). Indeed, Φ_{**} given by (6.48) satisfies equation (6.58), and thus $\gamma^+W_\Delta(a\Phi_{**}) = \gamma^+\mathcal{F}_\Delta$. Since both $W_\Delta(a\Phi_{**})$ and $\mathcal{F}_\Delta$ are harmonic functions belonging to the space $H^s(\Omega)$, this implies (6.56)–(6.57) and by (6.47) also (6.45). Finally, (6.47) implies by (6.55) that (6.53) is satisfied, and adding (6.52) to it leads to (6.46).

Let us now prove that the operator $\mathcal{C}_{**}$ is unique. Indeed, let a couple $(\tilde{f}_{**}, \Phi_{**}) \in \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ be a solution of linear system (6.45)–(6.46) with $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 = 0$. Then (6.51) implies that $r_\Omega\tilde{f}_{**} = 0$ in Ω, i.e., $\tilde{f}_{**} \in H^{s-2}_{\partial\Omega} \subset \widetilde{H}^{s-2}(\Omega)$. Hence, (6.53) reduces to $0 = T^+(\tilde{f}_{**}, 0)$ on $\partial\Omega$. By the first Green identity (2.15) this gives

$$0 = \left\langle T^+(\tilde{f}_{**}, 0), \gamma^+v\right\rangle_{\partial\Omega} = \langle\tilde{f}_{**}, v\rangle_\Omega \quad \forall\, v \in H^{2-s}(\Omega),$$

which implies $\tilde{f}_{**} = 0$ in $\mathbb{R}^n$. Finally, (6.48) gives $\Phi_{**} = 0$. Hence, non-homogeneous linear system (6.45)–(6.46) has only one solution, which implies the uniqueness of the operator $\mathcal{C}_{**}$. □

Theorem 6.9 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, and $a \in C_+^{\frac{3}{2}}(\overline{\Omega})$. The co-kernel of operator* (6.15) *is spanned over the functional*

$$g^{*1} := \left(\left(\gamma^+\right)^*\partial_n a, 1\right)^\top \tag{6.59}$$

in $[H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)]^* = \widetilde{H}^{-s}(\Omega) \times H^{\frac{3}{2}-s}(\partial\Omega)$, *i.e.*,

$$g^{*1}(\mathcal{F}_1, \mathcal{F}_2) = \left\langle\left(\gamma^+\mathcal{F}_1\right)\partial_n a + \mathcal{F}_2, \gamma^+u^0\right\rangle_{\partial\Omega},$$

where $u^0 = 1$.

Proof Let us consider the equation $\mathfrak{N}^1\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$, i.e., the BDIE system (N1) for $(u, \varphi) \in H^s(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$,

$$u + \mathcal{R}u + W\varphi = \mathcal{F}_1 \quad \text{in } \Omega, \tag{6.60}$$

$$T^+\mathcal{R}u + T^+W^+\varphi = \mathcal{F}_2 \quad \text{on } \partial\Omega, \tag{6.61}$$

with arbitrary $(\mathcal{F}_1, \mathcal{F}_2) \in H^s(\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega)$. By Lemma 6.8 the right-hand side of the system has form (6.45)–(6.46), i.e., system (6.60)–(6.61) reduces to

$$u + \mathcal{R}u + W(\varphi + \Phi_{**}) = \mathcal{P}\tilde{f}_{**} \quad \text{in } \Omega, \tag{6.62}$$

$$T^+\mathcal{R}u + T^+W(\varphi + \Phi_{**}) = T^+(\tilde{f}_{**} + \mathring{E}_\Omega \mathcal{R}_* \tilde{f}_{**}, \mathcal{P}\tilde{f}_{**}) \quad \text{on } \partial\Omega, \tag{6.63}$$

where the couple $(\tilde{f}_{**}, \Phi_{**}) \in \widetilde{H}^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega)$ is given by (6.47)–(6.48). Up to the notations, system (6.62)–(6.63) is the same as (6.7)–(6.8) with the right-hand side given by (6.9), where $\psi_0 = 0$.

First, let $s = 1$. Then Theorems 6.1 and 6.3 imply that BDIE system (6.62)–(6.63) and hence (6.60)–(6.61) are solvable if and only if

$$\begin{aligned}
\langle \tilde{f}_{**}, u^0 \rangle_\Omega &= \langle \check{\Delta}_\Omega(a\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n a), u^0 \rangle_\Omega \\
&= \langle \nabla \cdot \mathring{E}_\Omega \nabla(a\mathcal{F}_1) + \gamma^*(\mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n a), u^0 \rangle_{\mathbb{R}^n} \\
&= -\langle \mathring{E}_\Omega \nabla(a\mathcal{F}_1), \nabla u^0 \rangle_{\mathbb{R}^n} + \langle \mathcal{F}_2 + (\gamma^+\mathcal{F}_1)\partial_n a, \gamma^+ u^0 \rangle_{\partial\Omega} \\
&= \langle (\gamma^+\mathcal{F}_1)\partial_n a + \mathcal{F}_2, \gamma^+ u^0 \rangle_{\partial\Omega} = 0,
\end{aligned}$$

where we took into account that $u^0 = 1$ in $\mathbb{R}^n$. Thus the functional g^{*1} defined by (6.59) generates the necessary and sufficient solvability condition of equation $\mathfrak{N}^1\mathcal{U} = (\mathcal{F}_1, \mathcal{F}_2)^\top$. Hence g^{*1} is a basis of the co-kernel of $\mathfrak{N}^1$ (and thus the kernel of the operator adjoint to $\mathfrak{N}^1$) for $s = 1$.

Let now $s \in (\frac{1}{2}, \frac{3}{2})$. By Theorem 6.4 operator (6.15) and thus its adjoint are Fredholm with zero index. We already proved that, at $s = 1$, the kernel of the adjoint operator is spanned over g^{*1}. Then Lemma 7.5 implies that the kernel is the same for any $s \in (\frac{1}{2}, \frac{3}{2})$. □

To find the co-kernel of operator (6.16), we need some auxiliary assertions. Lemma 6.10 and Theorem 6.11 were proved in [33, Lemma 6.4 and Theorem 6.5] for the infinitely smooth coefficient a and boundary $\partial\Omega$. We further only slightly modify these proofs for the non-smooth coefficients and Lipschitz boundary.

Lemma 6.10 *Let Ω be a bounded simply connected Lipschitz domain, $s > \frac{1}{2}$, $a \in C^s_+(\overline{\Omega})$, and $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$. If*

$$r_\Omega \mathbf{P}\tilde{f} = 0 \quad \text{in } \Omega, \tag{6.64}$$

then $\tilde{f} = 0$ in $\mathbb{R}^n$.

Proof Multiplying (6.64) by a, taking into account (3.16), and applying the Laplace operator, we obtain $r_\Omega \tilde{f} = 0$, which means $\tilde{f} \in H^{s-2}_{\partial\Omega}$. If $s \geq \frac{3}{2}$, then $\tilde{f} = 0$ by Theorem 2.10 from [31]. If $\frac{1}{2} < s < \frac{3}{2}$, then by the same theorem there exists $v \in H^{s-\frac{3}{2}}(\partial\Omega)$ such that $\tilde{f} = \gamma^* v$. This gives $\mathbf{P}\tilde{f} = \mathbf{P}\gamma^* v = -Vv$ in $\mathbb{R}^n$; see (3.53). Then (6.64) reduces to $Vv = 0$ in Ω, which implies $v = 0$ on $\partial\Omega$ by Lemma 4.8(i), and thus $\tilde{f} = 0$ in $\mathbb{R}^n$. □

Theorem 6.11 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, and $a \in C_+^s(\overline{\Omega})$. The operator*

$$r_\Omega \mathbf{P} : \widetilde{H}^{s-2}(\Omega) \to H^s(\Omega) \tag{6.65}$$

and its inverse

$$(r_\Omega \mathbf{P})^{-1} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega) \tag{6.66}$$

are continuous, and

$$(r_\Omega \mathbf{P})^{-1} g = \big[\Delta \mathring{E}_\Omega \big(I - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+\big) - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+\big](ag) \quad \text{in } \mathbb{R}^n,\ \forall\, g \in H^s(\Omega). \tag{6.67}$$

Proof The continuity of (6.65) is given by Theorem 3.2. By Lemma 6.10 operator (6.65) is injective. Let us prove its surjectivity. To this end, for arbitrary $g \in H^s(\Omega)$, let us consider the following equation with respect to $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$:

$$r_\Omega \mathbf{P}_\Delta \tilde{f} = g \quad \text{in } \Omega. \tag{6.68}$$

Let $g_1 \in H^s(\Omega)$ be the (unique) solution of the following Dirichlet problem: $\Delta g_1 = 0$ in Ω, $\gamma^+ g_1 = \gamma^+ g$, which can be particularly presented as $g_1 = V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+ g$; see, e.g., [11] or proof of Lemma 2.6 in [31]. Let $g_0 := g - r_\Omega g_1$. Then $g_0 \in H^s(\Omega)$ and $\gamma^+ g_0 = 0$, and thus g_0 can be uniquely extended to $\mathring{E}_\Omega g_0 \in \widetilde{H}^s(\Omega)$. Thus by (3.53) equation (6.68) takes form

$$r_\Omega \mathbf{P}_\Delta \big[\tilde{f} + \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g\big] = g_0 \quad \text{in } \Omega. \tag{6.69}$$

Any solution $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$ of the corresponding equation in $\mathbb{R}^n$,

$$\mathbf{P}_\Delta \big[\tilde{f} + \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g\big] = \mathring{E}_\Omega g_0 \quad \text{in } \mathbb{R}^n, \tag{6.70}$$

evidently solves (6.69). If $\tilde{f}$ solves (6.70), then applying the Laplace operator to (6.70), we obtain

$$\tilde{f} = \tilde{Q} g := \Delta \mathring{E}_\Omega g_0 - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g = \Delta \mathring{E}_\Omega \big(g - r_\Omega V_\Delta \mathcal{V}_\Delta^{-1} \gamma^+ g\big) - \gamma^* \mathcal{V}_\Delta^{-1} \gamma^+ g \quad \text{in } \mathbb{R}^n. \tag{6.71}$$

On the other hand, substituting $\tilde{f}$ given by (6.71) into (6.70) and taking into account that $\mathbf{P}_\Delta \Delta \tilde{h} = \tilde{h}$ for any $\tilde{h} \in \widetilde{H}^s(\Omega)$, $s \in \mathbb{R}$, we obtain that $\tilde{Q} g$ is indeed a solution of equation (6.70) and thus of (6.69). By Lemma 6.10 the solution of (6.69) is unique, which means that the operator $\tilde{Q}$ is inverse to operator (6.65), i.e., $\tilde{Q} = (r_\Omega \mathbf{P})^{-1}$. Since Δ is a continuous operator from $\widetilde{H}^s(\Omega)$ to $\widetilde{H}^{s-2}(\Omega)$, equation (6.71) implies that the operator $(r_\Omega \mathbf{P})^{-1} = \tilde{Q} : H^s(\Omega) \to \widetilde{H}^{s-2}(\Omega)$ is continuous. The relations $\mathbf{P} = \frac{1}{a} \mathbf{P}_\Delta$ and $a(x) \geq a_{\min} > 0$ then imply the invertibility of operator (6.65) and ansatz (6.67). □

Theorem 6.12 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, and $a \in C_+^\sigma(\overline{\Omega})$ with $\sigma = \max\{1, s\}$. The co-kernel of operator* (6.16) *is spanned over the functional*

$$g^{*2} := \begin{pmatrix} -a\gamma^{+*}(\frac{1}{2} + \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+ u^0 \\ -a(\frac{1}{2} - \mathcal{W}'_\Delta)\mathcal{V}_\Delta^{-1}\gamma^+ u^0 \end{pmatrix} \tag{6.72}$$

in $[H^s(\Omega)\times H^{s-\frac{1}{2}}(\partial\Omega)]^* = \widetilde{H}^{-s}(\Omega)\times H^{\frac{1}{2}-s}(\partial\Omega)$, *i.e.*,

$$g^{*2}(\mathcal{F}_1,\mathcal{F}_2) = \left\langle -a\gamma^{+*}\left(\frac{1}{2}+\mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0,\mathcal{F}_1\right\rangle_\Omega + \left\langle -a\left(\frac{1}{2}-\mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0,\mathcal{F}_2\right\rangle_{\partial\Omega},$$

where $u^0(x)=1$.

Proof Let us consider the equation $\mathfrak{N}^2\mathcal{U}=(\mathcal{F}_1,\mathcal{F}_2)^\top$, i.e., the BDIE system (N2),

$$u+\mathcal{R}u+W\varphi=\mathcal{F}_1 \quad \text{in } \Omega, \tag{6.73}$$

$$\frac{1}{2}\varphi+\gamma^+\mathcal{R}u+\mathcal{W}\varphi=\mathcal{F}_2 \quad \text{on } \partial\Omega, \tag{6.74}$$

with arbitrary $(\mathcal{F}_1,\mathcal{F}_2)\in H^s(\Omega)\times H^{s-\frac{1}{2}}(\partial\Omega)$ for $(u,\varphi)\in H^s(\Omega)\times H^{s-\frac{1}{2}}(\partial\Omega)$.

Introducing the new variable $\varphi'=\varphi-(\mathcal{F}_2-\gamma^+\mathcal{F}_1)$, BDIE system (6.73)–(6.74) takes form

$$u+\mathcal{R}u+W\varphi'=\mathcal{F}'_1 \quad \text{in } \Omega, \tag{6.75}$$

$$\frac{1}{2}\varphi'+\gamma^+\mathcal{R}u+\mathcal{W}\varphi'=\gamma^+\mathcal{F}'_1 \quad \text{on } \partial\Omega, \tag{6.76}$$

where

$$\mathcal{F}'_1=\mathcal{F}_1-W(\mathcal{F}_2-\gamma^+\mathcal{F}_1)\in H^s(\Omega).$$

On the other hand, by Theorem 6.11, we can always represent $\mathcal{F}'_1=\mathcal{P}\tilde{f}_*$, with

$$\tilde{f}_*=\left[\Delta\mathring{E}_\Omega\left(I-r_\Omega V_\Delta\mathcal{V}_\Delta^{-1}\gamma^+\right)-\gamma^{+*}\mathcal{V}_\Delta^{-1}\gamma^+\right]\left(a\mathcal{F}'_1\right)\in\widetilde{H}^{s-2}(\Omega).$$

For $\mathcal{F}'_1=\mathcal{P}\tilde{f}_*$, the right-hand side of BDIE system (6.73)–(6.74) is the same as in (6.12) with $\tilde{f}=\tilde{f}_*$ and $\psi_0=0$.

First, let $s=1$. Then Theorems 6.1 and 6.3 imply that BDIE system (6.75)–(6.76) is solvable if and only if

$$\begin{aligned}
\langle\tilde{f}_*,u^0\rangle_\Omega &= \left\langle\left[\Delta\mathring{E}_\Omega\left(I-r_\Omega V_\Delta\mathcal{V}_\Delta^{-1}\gamma^+\right)-\gamma^{+*}\mathcal{V}_\Delta^{-1}\gamma^+\right]\left(a\mathcal{F}'_1\right),u^0\right\rangle_{\mathbb{R}^n}\\
&= \left\langle\mathring{E}_\Omega\left(I-r_\Omega V_\Delta\mathcal{V}_\Delta^{-1}\gamma^+\right)\left(a\mathcal{F}'_1\right),\Delta u^0\right\rangle_{\mathbb{R}^n}-\left\langle\mathcal{V}_\Delta^{-1}\gamma^+\left(a\mathcal{F}'_1\right),\gamma^+u^0\right\rangle_{\partial\Omega}\\
&= -\left\langle\gamma^+\left(a\mathcal{F}'_1\right),\mathcal{V}_\Delta^{-1}\gamma^+u^0\right\rangle_{\partial\Omega}\\
&= -\left\langle\frac{1}{2}\left[\gamma^+(a\mathcal{F}_1)+(a\mathcal{F}_2)\right]-\mathcal{W}_\Delta\left[a\left(\mathcal{F}_2-\gamma^+\mathcal{F}_1\right)\right],\mathcal{V}_\Delta^{-1}\gamma^+u^0\right\rangle_{\partial\Omega}\\
&= -\left\langle\mathcal{F}_1,a\gamma^{+*}\left(\frac{1}{2}+\mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0\right\rangle_\Omega-\left\langle\mathcal{F}_2,a\left(\frac{1}{2}-\mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0\right\rangle_{\partial\Omega}\\
&= 0.
\end{aligned} \tag{6.77}$$

Thus the functional g^{*2} defined by (6.72) generates a necessary and sufficient solvability condition of equation $\mathfrak{N}^2\mathcal{U}=(\mathcal{F}_1,\mathcal{F}_2)^\top$. Hence g^{*2} is a basis of the co-kernel of operator (6.16) for $s=1$.

Let us now choose any $s \in (\frac{1}{2}, \frac{3}{2})$. By Theorem 6.4 operator (6.16) and thus its adjoint are Fredholm with zero index. We already proved that, at $s = 1$, the kernel of the adjoint operator is spanned over g^{*2}. For any fixed coefficient $a \in C_+^\sigma(\overline{\Omega})$, the operator

$$\mathfrak{N}^2 : H^{s'}(\Omega) \times H^{s'-\frac{1}{2}}(\partial\Omega) \to H^{s'}(\Omega) \times H^{s'-\frac{1}{2}}(\partial\Omega) \tag{6.78}$$

is continuous for any $s' \in (\frac{1}{2}, \sigma]$ and particularly for $s' = s$ and $s' = 1$. Then Lemma 7.5 implies that the co-kernel of operator (6.78) is the same for $s' = s$ and $s' = 1$ and is spanned over g^{*2}. □

Theorems 6.3, 6.5, and 6.7 (or 6.9) imply the following extension of Theorem 6.1 to the range $\frac{1}{2} < s < \frac{3}{2}$.

Corollary 6.13 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$, $\psi_0 \in H^{s-\frac{3}{2}}(\partial\Omega)$, and $a \in C_+^\sigma(\overline{\Omega})$ with $\sigma = \max\{1, s\}$.*

The homogeneous Neumann problem (6.1)–(6.2) *admits only one linearly independent solution $u^0 = 1$ in $H^s(\Omega)$. The non-homogeneous Neumann problem* (6.1)–(6.2) *is solvable in $H^s(\Omega)$ if and only if condition* (6.3) *is satisfied.*

Proof Assuming that a function u is a solution of the homogeneous Neumann problem, by Theorem 6.3 the couple $(u, \varphi) = (u, \gamma_+\varphi)$ solves the homogeneous BDIE system ($N1_\Delta$), and then Theorem 6.7 implies that u is spanned over $u^0 = 1$.

Assume that solvability condition (6.3) is satisfied. Then the right-hand side (6.6) of the BDIE system ($N1_\Delta$) satisfies its solvability condition $g^{*1\Delta}(\mathcal{F}_1, \mathcal{F}_2) = \langle \mathcal{F}_2, \gamma^+ u^0\rangle_{\partial\Omega} = 0$ given by Theorem 6.7. Indeed, due to the first Green identities (2.15) and (2.18) applied to the operator Δ and Remark 2.7, since $V_\Delta\psi_0$ is a harmonic function in Ω and $u^0 = 1$, we obtain

$$\begin{aligned}
\langle \mathcal{F}_2, \gamma^+ u^0\rangle_{\partial\Omega} &= \left\langle T_\Delta^+(\tilde{f}; \mathcal{P}_\Delta\tilde{f}) - \frac{1}{2}\psi_0 + \mathcal{W}'_\Delta\psi_0, \gamma^+ u^0\right\rangle_{\partial\Omega} \\
&= \langle T_\Delta^+(\tilde{f}; \mathcal{P}_\Delta\tilde{f}) - \psi_0 + T_\Delta^+ V_\Delta\psi_0, \gamma^+ u^0\rangle_{\partial\Omega} \\
&= \langle \tilde{f}, u^0\rangle_\Omega + \check{\mathcal{E}}_\Omega(\mathcal{P}_\Delta\tilde{f}, u^0) - \langle \psi_0, \gamma^+ u^0\rangle_{\partial\Omega} \\
&\quad + \langle \tilde{\Delta} V_\Delta\psi_0, u^0\rangle_\Omega + \check{\mathcal{E}}_\Omega(V_\Delta\psi_0, u^0) \\
&= \langle \tilde{f}, u^0\rangle_\Omega - \langle \psi_0, \gamma^+ u^0\rangle_{\partial\Omega}.
\end{aligned} \tag{6.79}$$

Hence the BDIE system ($N1_\Delta$) is solvable, implying solvability of the Neumann BVP due to Theorem 6.3(ii). This proves that condition (6.3) is sufficient.

Let us now assume that there exists a solution of the Neumann BVP. Hence Theorem 6.3(i) implies that the BDIE system ($N1_\Delta$) with the right-hand side (6.9) is solvable, implying that its solvability condition $\langle \mathcal{F}_2, \gamma^+ u^0\rangle_{\partial\Omega} = 0$ is satisfied. Then (6.79) implies condition (6.3), proving that it is necessary. □

6.3 Perturbed (stabilised) segregated BDIE systems for the Neumann problem

Theorem 6.5 implies that even when the solvability condition (6.3) is satisfied, the solutions of BDIE systems ($N1_\Delta$), (N1), and (N2) are not unique, and moreover, the BDIE left-hand side operators $\mathfrak{N}^{1\Delta}$, $\mathfrak{N}^1$, and $\mathfrak{N}^2$, have non-zero kernels and thus are not invertible. To find a solution (u,φ) from uniquely solvable BDIE systems with continuously invertible left-hand side operators, let us consider, following [28], some stabilised BDIE systems obtained from (N1$_\Delta$), (N1), and (N2) by finite-dimensional operator perturbations. Note that other choices of the perturbing operators are also possible.

We further use the notations $\mathcal{U}^N = (u,\varphi)^\top$, $\mathcal{U}^0 = (1,1)^\top$, and $|\partial\Omega| := \int_{\partial\Omega} dS$.

Let us introduce the perturbed counterparts of the BDIE systems (N1$_\Delta$), (N1), and (N2):

$$\hat{\mathfrak{N}}^{1\Delta}\mathcal{U}^N = \mathcal{F}^{N1\Delta}, \qquad \hat{\mathfrak{N}}^{1}\mathcal{U}^N = \mathcal{F}^{N1}, \qquad \hat{\mathfrak{N}}^{2}\mathcal{U}^N = \mathcal{F}^{N2}, \tag{6.80}$$

where $\hat{\mathfrak{N}}^{1\Delta} := \mathfrak{N}^{1\Delta} + \mathring{\mathfrak{N}}^{1\Delta}$, $\hat{\mathfrak{N}}^{1} := \mathfrak{N}^{1} + \mathring{\mathfrak{N}}^{1}$, $\hat{\mathfrak{N}}^{2} := \mathfrak{N}^{2} + \mathring{\mathfrak{N}}^{2}$, and

$$\mathring{\mathfrak{N}}^{1\Delta}\mathcal{U}^N(y) = \mathring{\mathfrak{N}}^{1}\mathcal{U}^N(y) := g^0\big(\mathcal{U}^N\big)\mathcal{G}^1(y) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi(x)\,dS\begin{pmatrix}0\\1\end{pmatrix}, \tag{6.81}$$

that is,

$$g^0\big(\mathcal{U}^N\big) := \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi(x)\,dS, \qquad \mathcal{G}^1(y) := \begin{pmatrix}0\\1\end{pmatrix}, \tag{6.82}$$

whereas

$$\mathring{\mathfrak{N}}^{2}\mathcal{U}^N := g^0\big(\mathcal{U}^N\big)\mathcal{G}^2 = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi(x)\,dS\begin{pmatrix}a^{-1}(y)\\ \gamma^+a^{-1}(y)\end{pmatrix},$$

that is, $g^0(\mathcal{U}^N)$ is as in (6.82), and

$$\mathcal{G}^2(y) := \begin{pmatrix}a^{-1}(y)\\ \gamma^+a^{-1}(y)\end{pmatrix}.$$

Theorem 6.14 *Let Ω be a bounded simply connected Lipschitz domain, $\frac{1}{2} < s < \frac{3}{2}$, and* $\sigma = \max\{1,s\}$.

(i) *The following operators are continuous and continuously invertible*:

$$\hat{\mathfrak{N}}^{1\Delta} : H^s(\Omega)\times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)\times H^{s-\frac{3}{2}}(\partial\Omega) \quad \text{if } a \in C^\sigma_+(\overline{\Omega}), \tag{6.83}$$

$$\hat{\mathfrak{N}}^{1} : H^s(\Omega)\times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)\times H^{s-\frac{3}{2}}(\partial\Omega) \quad \text{if } a \in C^{\frac{3}{2}}_+(\overline{\Omega}), \tag{6.84}$$

$$\hat{\mathfrak{N}}^{2} : H^s(\Omega)\times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)\times H^{s-\frac{1}{2}}(\partial\Omega) \quad \text{if } a \in C^\sigma_+(\overline{\Omega}). \tag{6.85}$$

(ii) *If the conditions $g^{*1\Delta}(\mathcal{F}^{N1\Delta}) = 0$, $g^{*1}(\mathcal{F}^{N1}) = 0$, or $g^{*2}(\mathcal{F}^{N2}) = 0$ are satisfied, then the unique solutions of the perturbed BDIE systems in* (6.80) *give the solutions $\mathcal{U}^N$ of the corresponding original BDIE systems* (*N1*$_\Delta$), (*N*1), *and* (*N*2) *such that*

$$g^0\big(\mathcal{U}^N\big) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\varphi\,dS = \frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma^+u\,dS = 0.$$

Proof For the functional $g^{*1\Delta}$ given by (6.38) in Theorem 6.7, $g^{*1\Delta}(\mathcal{G}^1) = |\partial\Omega|$. Similarly, for the functional g^{*1} given by (6.59) in Theorem 6.9, $g^{*1}(\mathcal{G}^1) = |\partial\Omega|$.

For the functional g^{*2} given by (6.72) in Theorem 6.12, since the operator $\mathcal{V}_\Delta^{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is positive definite and $u^0(x) = 1$, there exists a positive constant C such that

$$\begin{aligned} g^{*2}(\mathcal{G}^2) &= \left\langle -a\gamma^{+*}\left(\frac{1}{2} + \mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0, a^{-1}u^0\right\rangle_\Omega \\ &\quad + \left\langle -a\left(\frac{1}{2} - \mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+\left(a^{-1}u^0\right)\right\rangle_{\partial\Omega} \\ &= -\left\langle\left(\frac{1}{2} + \mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0 + \left(\frac{1}{2} - \mathcal{W}'_\Delta\right)\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\right\rangle_{\partial\Omega} \\ &= -\langle\mathcal{V}_\Delta^{-1}\gamma^+u^0, \gamma^+u^0\rangle_{\partial\Omega} \\ &\le -C\|\gamma^+u^0\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \le -C\|\gamma^+u^0\|^2_{L_2(\partial\Omega)} \\ &= -C|\partial\Omega|^2 < 0. \end{aligned} \tag{6.86}$$

On the other hand, $g^0(\mathcal{U}^0) = 1$. Hence Theorem 7.4 from [28] implies the claims of the theorem. □

7 Auxiliary assertions

We provide here some auxiliary results used in the main text.

Theorem 7.1 *Let $\frac{1}{2} < s < \frac{3}{2}$, $u \in H^s(\Omega)$, $a \in C_+^\sigma(\overline{\Omega})$ with $\sigma = \max\{1, s\}$, $Au = r_\Omega\tilde{f}$ in an interior or exterior Lipschitz domain Ω for some $\tilde{f} \in \widetilde{H}^{s-2}(\Omega)$. Let $\{f_k\} \in \widetilde{H}_\bullet^{-\frac{1}{2}}(\Omega)$ be a sequence such that $\|\tilde{f} - \mathring{E}_\Omega f_k\|_{\widetilde{H}^{s-2}(\Omega)} \to 0$ as $k \to \infty$.*

Then there exists a sequence $\{u_k\} \in H^{s,-\frac{1}{2}}(\Omega; A)$ such that $Au_k = f_k$ in Ω and $\|u - u_k\|_{H^s(\Omega)} \to 0$ as $k \to \infty$. Moreover, $\|T^+(u_k) - T^+(\tilde{f}; u)\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \to 0$ as $k \to \infty$.

Proof Let us consider the Dirichlet problem

$$Au_k = f_k \quad \text{in } \Omega, \tag{7.1}$$

$$\gamma^+u_k = \gamma^+u \quad \text{on } \partial\Omega, \tag{7.2}$$

By Corollary 5.5 the unique solution of problem (7.1)–(7.2) in $H^s(\Omega)$ is $u_k = (\mathcal{A}^D)^{-1}(f_k, \varphi_k)^\top$, where $(\mathcal{A}^D)^{-1} : H^{s-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$ is a continuous operator. Hence the functions u_k converge to u in $H^s(\Omega)$ as $k \to \infty$. Since $Au_k = f_k \in \widetilde{H}_\bullet^{-\frac{1}{2}}(\Omega)$, we obtain that $u_k \in H^{s,-\frac{1}{2}}(\Omega; A)$ and the canonical conormal derivative T^+u_k is well defined. Then subtracting (2.16) for u_k from (2.12), we obtain

$$T^+(\tilde{f}; u) - T^+u_k = (\gamma^{-1})^*[\tilde{f} - \mathring{E}_\Omega f_k + \check{A}_\Omega(u - u_k)].$$

Hence

$$\|T^+(\tilde{f}; u) - T^+u_k\|_{H^{s-\frac{3}{2}}(\partial\Omega)} \le C(\|\tilde{f} - \mathring{E}_\Omega f_k\|_{\widetilde{H}^{s-2}(\Omega)} + C_1\|u - u_k\|_{H^s(\Omega)}) \tag{7.3}$$

for some positive C and C_1. Since the right-hand side of (7.3) tends to zero as $k \to \infty$, so does the left-hand side. □

Note that since $\mathcal{D}(\Omega) \subset \widetilde{H}_{\bullet}^{-\frac{1}{2}}(\Omega)$ is dense in $\widetilde{H}^{s-2}(\Omega)$, the sequence $\{f_k\} \in \widetilde{H}_{\bullet}^{-\frac{1}{2}}(\Omega)$ from the hypotheses of Theorem 7.1 does always exist.

The following multiplication theorem is well known; see, e.g., [15, Theorems 1.4.1.1, 1.4.1.2], [54, Theorem 2(b)], [1, Theorems 1.9.1, 1.9.2, 1.9.5], [32, Theorem 3.2].

Theorem 7.2 *Let Ω_0 be an open set.*

(i) *If $g \in L_\infty(\Omega_0)$, then $gv \in L_2(\Omega_0)$ and $\|gv\|_{L_2(\Omega)} \le c\|g\|_{L_\infty(\Omega_0)}\|v\|_{L_2(\Omega_0)}$ for every $v \in L_2(\Omega_0)$.*

(ii) *If σ is a non-zero integer and $g \in C^{|\sigma|-1,1}(\overline{\Omega_0})$, then $gv \in H^\sigma(\Omega_0)$ for every $v \in H^\sigma(\Omega_0)$, and $\|gv\|_{H^\sigma(\Omega)} \le c\|g\|_{C^{|\sigma|-1,1}(\overline{\Omega_0})}\|v\|_{H^\sigma(\Omega_0)}$.*

(iii) *If σ is a non-integer, $|\sigma| = m + \theta$, where m is a non-negative integer and $0 < \theta < 1$, then for $g \in C^{m,\eta}(\overline{\Omega_0})$ with $\theta < \eta < 1$, we have $gv \in H^\sigma(\Omega_0)$ and $\|gv\|_{H^\sigma(\Omega)} \le c\|g\|_{C^{m,\eta}(\overline{\Omega_0})}\|v\|_{H^\sigma(\Omega_0)}$ for every $v \in H^\sigma(\Omega_0)$.*

In all cases, c is a positive constant independent of g, v, or Ω_0.

Theorem 7.3 *Let Ω be a bounded simply connected Lipschitz domain, and let $0 \le \sigma \le 1$. The operators*

$$\mathcal{V}_\Delta : H^{\sigma-1}(\partial\Omega) \to H^\sigma(\partial\Omega), \tag{7.4}$$

$$-\frac{1}{2}I + \mathcal{W}_\Delta : H^\sigma(\partial\Omega) \to H^\sigma(\partial\Omega), \tag{7.5}$$

$$-\frac{1}{2}I + \mathcal{W}'_\Delta : H^{-\sigma}(\partial\Omega) \to H^{-\sigma}(\partial\Omega) \tag{7.6}$$

are isomorphisms, and the operators

$$\frac{1}{2}I + \mathcal{W}_\Delta : H^\sigma(\partial\Omega) \to H^\sigma(\partial\Omega), \tag{7.7}$$

$$\frac{1}{2}I + \mathcal{W}'_\Delta : H^{-\sigma}(\partial\Omega) \to H^{-\sigma}(\partial\Omega), \tag{7.8}$$

$$\mathcal{L}_\Delta : H^\sigma(\partial\Omega) \to H^{\sigma-1}(\partial\Omega) \tag{7.9}$$

are Fredholm with zero index.

Proof The properties of the boundary integral operators (7.4)–(7.9) related to the harmonic layer potential are well known; see, e.g., [51], [39, Theorem 4.1], [14, Theorem 8.1] for the invertibility of operators (7.4)–(7.6) and the Fredholm properties of operators (7.7)–(7.8). The Fredholm property of operator (7.9) for $\sigma = \frac{1}{2}$ is also well known; see, e.g., [27, Theorem 7.8]. Then the corresponding result for $0 \le \sigma \le 1$ can be proved as in [27, Theorem 7.17] by using a sharper regularity result from [11, Theorem 3]. □

Theorem 7.4 further is implied by [28, Lemma 2] (see also [50, Sect. 21], [49, Sect. 21.4], where the particular case $h_i^*(\mathring{x}_j) = \mathring{x}_i^*(h_j) = \delta_{ij}$ has been considered). Another approach, although with hypotheses similar to those in Theorem 7.4, is presented in [17, Lemma 4.8.24].

Theorem 7.4 *Let B_1 and B_2 be two Banach spaces. Let $\underline{A} : B_1 \to B_2$ be a linear Fredholm operator with zero index, and let $\underline{A}^* : B_2^* \to B_1^*$ be the adjoint operator with* $\dim\ker\underline{A} = \dim\ker\underline{A}^* = n < \infty$, *where* $\ker\underline{A} = \operatorname{span}\{\mathring{x}_i\}_{i=1}^n \subset B_1$ *and* $\ker\underline{A}^* = \operatorname{span}\{\mathring{x}_i^*\}_{i=1}^n \subset B_2^*$. *Let*

$$\underline{A}_1 x := \sum_{i=1}^{k} h_i h_i^*(x),$$

where h_i^* and h_i $(i = 1,\dots,n)$ are elements from B_1^* and B_2, respectively, such that

$$\det\big[h_i^*(\mathring{x}_j)\big] \neq 0, \qquad \det\big[\mathring{x}_i^*(h_j)\big] \neq 0, \quad i,j = 1,\dots,n. \tag{7.10}$$

Then:

(i) *the operator* $\underline{A} - \underline{A}_1 : B_1 \to B_2$ *is an isomorphism*;

(ii) *if $y \in B_2$ satisfies the solvability conditions*

$$\mathring{x}_i^*(y) = 0, \quad i = 1,\dots,n, \tag{7.11}$$

of the equation

$$\underline{A}x = y, \tag{7.12}$$

then the unique solution x of equation

$$(\underline{A} - \underline{A}_1)x = y, \tag{7.13}$$

is a solution of equation (7.12) *such that*

$$h_i^*(x) = 0 \quad (i = 1,\dots,k). \tag{7.14}$$

(iii) *Vice versa, if x is a solution of equation* (7.13) *satisfying conditions* (7.14), *then conditions* (7.11) *are satisfied for the right-hand side y of equation* (7.13), *and x is a solution of equation* (7.12) *with the same right-hand side y.*

Note that more results about finite-dimensional operator perturbations are available in [28].

The following known result (see, e.g., [42, Lemma 11.9.21]) is useful for us.

Lemma 7.5 *Let X_1, X_2 and Y_1, Y_2, be Banach spaces such that the embeddings $X_1 \hookrightarrow X_2$ and $Y_1 \hookrightarrow Y_2$ are continuous, and the embedding $Y_1 \hookrightarrow Y_2$ has a dense range. Assume that $T : X_1 \to Y_1$ and $T : X_2 \to Y_2$ are Fredholm operators with the same index,* $\operatorname{ind}(T : X_1 \to Y_1) = \operatorname{ind}(T : X_2 \to Y_2)$. *Then* $\operatorname{Ker}\{T : X_1 \to Y_1\} = \operatorname{Ker}\{T : X_2 \to Y_2\}$.

8 Concluding remarks

The Dirichlet and Neumann problems on a bounded Lipschitz domain for a variable-coefficient second-order PDE with general right-hand side functions from $H^{s-2}(\Omega)$ and $\widetilde{H}^{s-2}(\Omega)$, $\frac{1}{2} < s < \frac{3}{2}$, respectively, were equivalently reduced to three direct segregated boundary-domain integral equation systems for each of the BVPs. This involved systematic use of the generalised co-normal derivatives. The operators associated with the left-hand sides of all the BDIE systems were analysed in the corresponding Sobolev spaces. It was shown that the operators of the BDIE systems for the Dirichlet problem are continuous and continuously invertible. For the Neumann problem, the BDIE system operators are continuous but only Fredholm with zero index; their kernels and co-kernels were analysed, and appropriate finite-dimensional perturbations were constructed to make the perturbed (stable) operators invertible and provide a solution of the original BDIE systems and the Neumann problem.

The same approach can be implemented to extend to the general PDE right-hand sides, non-smooth coefficients and Lipschitz domains: the BDIE systems for the mixed problems, unbounded domains, BDIEs of more general scalar PDEs and the systems of PDEs, and the united and localised BDIEs, for which the analysis is now available for the right-hand sides only from $L^2(\Omega)$, with smooth coefficients and smooth domain boundaries; see [2–10, 13, 30, 36, 37].

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Abbreviations
BDIE, Boundary-domain integral equation; BVP, Boundary value problem; PDE, Partial differential equation.

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