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Blow-up of solutions for a nonlinear Petrovsky type equation with initial data at arbitrary high energy level

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Abstract

In this paper, we study the initial boundary value problem for a Petrovsky type equation with a memory term, nonlinear weak damping, and a superlinear source:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(\tau) \, d\tau + |u_t|^{m-2} u_t = |u|^{p-2} u, \quad \text{in } \Omega \times (0, T).$$

When the source is stronger than dissipations, we obtain the existence of certain weak solutions which blow up in finite time with initial energy $E(0) = R$ for any given $R \geq 0$.

MSC: 35L70; 65M60

Keywords: Petrovsky type equation; Memory term; Nonlinear damping; Blow-up

1 Introduction

In this paper, we are concerned with the following initial boundary value problem for a nonlinear Petrovsky type equation with a memory term and nonlinear damping:

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t g(t-\tau) \Delta^2 u(\tau) \, d\tau + |u_t|^{m-2} u_t = |u|^{p-2} u, & \text{in } \Omega \times (0, T), \\ u(x, t) = \partial_\nu u(x, t) = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $m \geq 2$, $p > 2$, Ω is a bounded domain of \mathbb{R}^n ($n \geq 1$) with a smooth boundary $\partial\Omega$, $\partial_\nu u$ is the outer normal derivative of u , the relaxation function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nonincreasing with $\int_0^\infty g(s) \, ds < 1$, the initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. The Petrovsky type of equation

$$u_{tt} + \Delta^2 u = g(x, t, u, u_t)$$

originated from the study of beams and plates, and it can also be used in many branches of physics such as nuclear physics, optics, geophysics, and ocean acoustics. In the past

decades, a great deal of mathematical effort has been devoted to the study of well-posedness, regularity, asymptotic behavior for such kind of fourth order wave equations, as reported in [1–9] and the references therein.

In [10], Messaoudi studied the following nonlinearly damped semilinear Petrovsky equation:

$$\begin{cases} u_{tt} + \Delta^2 u + a|u_t|^{m-2}u_t = b|u|^{p-2}u, & \text{in } \Omega \times (0, T), \\ u(x, t) = \partial_\nu u(x, t) = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $a, b > 0$. It is showed that the solution blows up in finite time if $p > m$ and the initial energy is negative, and it globally exists if $p < m$. In [11], Messaoudi considered a nonlinear wave equation under the influence of a linear memory term and a nonlinear damping

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t - \tau)\Delta u(\tau) d\tau + |u_t|^{m-1}u_t = |u|^{p-1}u, & \text{in } \Omega \times (0, T), \\ u(x, t) = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega. \end{cases} \tag{1.3}$$

By using the method in [12], the author proved that the solution is global if $p \leq m$ and blows up in finite time with negative initial energy if $p > m$. In [13], Messaoudi improved the blow-up result in [11] to solutions with positive bounded initial energy. In [14], Chen and Zhou studied problem (1.2) and obtained the finite time blow-up results provided that the initial energy is positive bounded or vanishing. If the solution u of problem (1.2) blows up in finite time, Zhou [5] presented an estimate for the lower bound of blow-up time. Following the strategy similar to that in [13], Li and Gao [15] studied problem (1.1). Under the following assumptions:

- (a1) $p > m \geq 2$,
- (a2) $\int_0^\infty g(s) ds < 1 - \frac{1}{(p-1)^2}$,
- (a3) $E(0) < E_1$ and $\|\Delta u_0\|_{L^2} > C$, where E_1 and C are positive constants,

Li and Gao proved that the solution of problem (1.1) blows up in finite time.

In recent years, some researchers obtained the finite time blow-up results for different kinds of evolution equations with initial data at arbitrary high energy level. In [16], Todorova and Vitillaro studied the Cauchy problem for equation (1.3) without the memory term. The authors proved that, for any given $\alpha > 0, \lambda \geq 0$, there exist infinitely many data u_0, u_1 in the energy space such that the initial energy $E(0) = \lambda$, the gradient norm $\|\nabla u_0\|_{L^2} = \alpha$, and the solution of the Cauchy problem blows up in finite time. By establishing a special second order differential inequality, Wang [4] studied the following initial boundary value problem of the fourth order wave equation with the nonlinear strain and source terms:

$$\begin{cases} u_{tt} + u_t + \Delta^2 u - \alpha \Delta u - \sum_{i=1}^n \frac{\partial}{\partial x_i}(\theta_i(x)) = f(u), & \text{in } \Omega \times (0, T), \\ u(x, t) = \partial_\nu u(x, t) = 0, & \text{on } \partial\Omega \times [0, +\infty), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \end{cases}$$

and established the finite time blow-up results of solutions with arbitrary initial energy, $-\infty < E(0) < +\infty$. In [17], Ye considered the following initial boundary value problem of

higher order nonlinear viscoelastic wave equation:

$$\begin{cases} u_{tt} + (-\Delta)^m u - \int_0^t g(t-s)(-\Delta)^m u(s) \, d\tau = |u|^{p-2}u, & \text{in } \Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \\ \frac{\partial_i u}{\partial \nu_i} |_{\partial \Omega} = 0, \quad i = 0, 1, 2, \dots, m-1, \end{cases}$$

where $m \geq 1, p > 2$. By using a special differential inequality, the author obtained the existence of certain solutions with nonpositive, positive bounded, or arbitrary high initial energy that blow up in finite time and also derived the life span estimates of these solutions. In [18], Song and Xue studied an equation similar to (1.3) except for the nonlinear weak damping term which is replaced by a linear strong damping term $-\Delta u_t$. By applying the technique similar to that in [18], Song [19] considered problem (1.3) and obtained the finite time blow-up result for certain solutions whose initial data have arbitrary high initial energy. By combining Song’s method and the concavity method, we studied some parabolic type equations and hyperbolic equations with damping terms. We also obtained the finite time blow-up result for certain solutions whose initial data have arbitrary high initial energy, see [20–23]. For details of the study of other kinds of evolution equations, we refer the reader to [24–35].

To the best of our knowledge, there are no results about the finite time blow-up for solutions of problem (1.1) with arbitrary high initial energy. In this paper, following the strategy similar to that in [19] with minor modifications and under suitable conditions on m, p and g :

- (a1) $p > m \geq 2$,
- (a2) $\int_0^\infty g(s) \, ds < 1 - \frac{1}{(p-1)^2}$,
- (a3') $0 < E(0) < C \int_\Omega u_0 u_1 \, dx$, where C is a positive constant,

we prove that there exist solutions of problem (1.1) that blow up in finite time. Moreover, we prove that, for any given $R > 0$, there exists $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ such that $E(0) = R$ and (u_0, u_1) satisfies assumption (a3'). We finally obtain a set $\mathbb{B}_1 \cup \mathbb{B}_2$ in the space $H_0^2(\Omega) \times L^2(\Omega)$ such that the solution u of problem (1.1) blows up in finite time provided that the initial data $(u_0, u_1) \in \mathbb{B}_1 \cup \mathbb{B}_2$. Hence, we have extended the result of [15].

The rest of the paper is organized as follows. In Sect. 2, we present some notations and definitions and give the local existence theorem and some important lemmas. In Sect. 3, we prove our main results.

2 Preliminaries

Throughout this paper, we denote $\mathbb{R}^+ = [0, +\infty)$ and the standard $L^s(\Omega)$ -norm by $\|u\|_s = \|u\|_{L^s(\Omega)}$. In view of the Sobolev–Poincaré inequality, the norm $\|w\| := |\Delta w|_2$ is equivalent to the usual norm

$$\|w\|_{H^2} = (|\Delta w|_2^2 + |\nabla w|_2^2 + |w|_2^2)^{\frac{1}{2}}$$

on the Sobolev space

$$H_0^2(\Omega) = \{w \in H^2(\Omega) : w = \partial_\nu w = 0 \text{ on } \partial \Omega\}.$$

We assume that p, m and the relaxation function g satisfy the following assumptions.

- (H1) $2 < p < \infty$, if $n \leq 4$; $2 < p \leq \frac{2(n-2)}{n-4}$, if $n \geq 5$.
- (H2) $2 \leq m < \infty$, if $n \leq 4$; $2 \leq m \leq \frac{2n}{n-4}$, if $n \geq 5$.
- (H3) $g : [0, +\infty) \rightarrow [0, +\infty)$ is a C^1 non-increasing function satisfying

$$l := 1 - \int_0^\infty g(s) \, ds > 0.$$

Lemma 2.1 *Suppose that (H1) holds. Then there exists a sequence of functions $\{v_k\} \in H_0^2(\Omega)$ such that*

$$\frac{1}{2} \int_\Omega |\Delta v_k|^2 \, dx + \frac{1}{2} \int_\Omega |v_k|^2 \, dx - \frac{1}{p} \int_\Omega |v_k|^p \, dx \rightarrow +\infty \quad \text{as } k \rightarrow \infty.$$

Proof It is easy to verify that the C^2 functional

$$J(v) := \frac{1}{2} \int_\Omega |\Delta v(x)|^2 \, dx + \frac{1}{2} \int_\Omega |v(x)|^2 \, dx - \frac{1}{p} \int_\Omega |v(x)|^p \, dx, \quad v \in H_0^2(\Omega)$$

is the Euler–Lagrange functional for the following boundary value problem:

$$\begin{cases} \Delta^2 v + v = |v|^{p-2}v, & \text{in } \Omega, \\ v = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Similar to the proof of Theorem 3.7 in [36], by using the fountain theorem, we can prove that problem (2.1) has a sequence of weak solutions $\{v_k\} \subset H_0^2(\Omega)$ such that $J(v_k) \rightarrow +\infty$ as $k \rightarrow \infty$. □

Similar to Definition 1.2 in [37, 38], we define the weak solution of (1.1) as follows.

Definition 2.2 A function $u = u(x, t)$ is said to be a weak solution of (1.1) defined on the time interval $[0, T]$ for some $T > 0$ provided that $u \in C([0, T]; H_0^2(\Omega))$ is such that $u_t \in C([0, T]; L^2(\Omega)) \cap L^m(\Omega \times (0, T))$ and

- (i) $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$;
- (ii) The following variational identity holds for all $t \in [0, T]$ and all test functions $\phi \in \mathcal{F}$:

$$\begin{aligned} & \int_\Omega u_t(t)\phi(t) \, dx - \int_\Omega u_t(0)\phi(0) \, dx - \int_0^t \int_\Omega u_t(\tau)\phi_t(\tau) \, dx \, d\tau \\ & + \int_0^t \int_\Omega \Delta u(\tau)\Delta\phi(\tau) \, dx \, d\tau - \int_0^t \int_0^s g(s-\tau) \int_\Omega \Delta u(\tau)\Delta\phi(s) \, dx \, d\tau \, ds \\ & + \int_0^t \int_\Omega |u_t(\tau)|^{m-2} u_t(\tau)\phi(\tau) \, dx \, d\tau \\ & = \int_0^t \int_\Omega |u(\tau)|^{p-2} u(\tau)\phi(\tau) \, dx \, d\tau, \end{aligned} \tag{2.2}$$

where

$$\mathcal{F} = \{ \phi : \phi \in C([0, T]; H_0^2(\Omega)) \text{ with } \phi_t \in C([0, T]; L^2(\Omega)) \}.$$

Suppose that u is a weak solution of (1.1), then we define the energy functional $E(t)$ by

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} (1 - G(t)) \|\Delta u\|_2^2 + \frac{1}{2} (g \circ \Delta u)(t) - \frac{1}{p} \|u\|_p^p, \tag{2.3}$$

where $G(t) := \int_0^t g(s) \, ds$ and

$$(g \circ \Delta u)(t) := \int_0^t g(t - \tau) \|\Delta u(t) - \Delta u(\tau)\|_2^2 \, d\tau.$$

We state the local existence theorem as follows.

Theorem 2.3 *Assume that $m, p,$ and g satisfy (H1)–(H3). Then, for any given $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, problem (1.1) has a unique weak solution u such that*

$$u \in C([0, T]; H_0^2(\Omega)), \quad u_t \in C([0, T]; L^2(\Omega)) \cap L^m(\Omega \times (0, T))$$

for some $T > 0$. Moreover, we have the following energy identity:

$$\begin{aligned} E(t) - \frac{1}{2} \int_0^t (g' \circ \Delta u)(\tau) \, d\tau + \frac{1}{2} \int_0^t G(\tau) \|\Delta u(\tau)\|_2^2 \, d\tau \\ + \int_0^t \|u_t(\tau)\|_m^m \, d\tau = E(0) \quad \text{for all } t \in [0, T^*), \end{aligned} \tag{2.4}$$

where T^* is the maximal existence time for the weak solution u .

Proof Since p satisfies (H1), the embedding $H_0^2(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$ is continuous and it holds that $|u|^{p-2}u \in L^2(\Omega)$ provided $u \in H_0^2(\Omega)$. We claim that the Nemytskii operator $f : u \mapsto |u|^{p-2}u$ is locally Lipschitz continuous from $H_0^2(\Omega)$ into $L^2(\Omega)$. In fact, for any $u, v \in H_0^2(\Omega)$, we have

$$\begin{aligned} & \|f(u) - f(v)\|_2 \\ &= \left[\int_{\Omega} (|u|^{p-2}u - |v|^{p-2}v)^2 \, dx \right]^{\frac{1}{2}} \\ &\leq (p-1) \left[\int_{\Omega} |\xi|^{2(p-2)} |u - v|^2 \, dx \right]^{\frac{1}{2}} \\ &\leq (p-1) \left[\int_{\Omega} |u|^{2(p-2)} |u - v|^2 \, dx + \int_{\Omega} |v|^{2(p-2)} |u - v|^2 \, dx \right]^{\frac{1}{2}} \\ &\leq (p-1) \left[\left(\int_{\Omega} |u|^{2(p-2) \cdot \frac{p-1}{p-2}} \, dx \right)^{\frac{p-2}{p-1}} \left(\int_{\Omega} |u - v|^{2(p-1)} \, dx \right)^{\frac{1}{p-1}} \right. \\ &\quad \left. + \left(\int_{\Omega} |v|^{2(p-2) \cdot \frac{p-1}{p-2}} \, dx \right)^{\frac{p-2}{p-1}} \left(\int_{\Omega} |u - v|^{2(p-1)} \, dx \right)^{\frac{1}{p-1}} \right]^{\frac{1}{2}} \\ &= (p-1) \left[\|u\|_{2(p-1)}^{2(p-2)} \|u - v\|_{2(p-1)}^2 + \|v\|_{2(p-1)}^{2(p-2)} \|u - v\|_{2(p-1)}^2 \right]^{\frac{1}{2}} \\ &\leq C (\|u\|^{2(p-2)} + \|v\|^{2(p-2)})^{\frac{1}{2}} \|u - v\|, \end{aligned}$$

where $\xi(x)$ is between $u(x)$ and $v(x)$. So f is locally Lipschitz continuous from $H_0^2(\Omega)$ into $L^2(\Omega)$.

Then, following the framework in [37, Sect. 2], by employing the theory of monotone operators and nonlinear semigroups, with energy methods, we can establish the existence of a unique local weak solution for problem (1.1). \square

Remark 2.4 Using the approximation technique and the contraction mapping principle, the local existence result can also be established similar to those of [10, 39].

Lemma 2.5 *Assume that (H1)–(H3) hold and u is the weak solution of problem (1.1). Then $E(t)$ is nonincreasing and*

$$\frac{d}{dt}E(t) = \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}G(t)|\Delta u|_2^2 - |u_t|_m^m \leq 0, \quad \text{a.e. } t \in [0, T^*]. \tag{2.5}$$

Proof In view of the regularity of u and u_t , we have $\int_0^t |u_t(\tau)|_m^m \, d\tau < \infty$,

$$\int_0^t G(\tau)|\Delta u(\tau)|_2^2 \, d\tau \leq \max_{\tau \in [0,t]} |\Delta u(\tau)|_2^2 \int_0^t G(\tau) \, d\tau < \infty,$$

and

$$\begin{aligned} \int_0^t (g' \circ \Delta u)(\tau) \, d\tau &= \int_0^t g'(t - \tau)|\Delta u(t) - \Delta u(\tau)|_2^2 \, d\tau \\ &\leq 2 \max_{\tau \in [0,t]} |\Delta u(\tau)|_2^2 \int_0^t g'(\tau) \, d\tau < \infty. \end{aligned}$$

Therefore, by (2.4), $E(t) - E(0)$ is absolutely continuous on any closed subsegment of $[0, T^*)$. Then, by differentiating (2.4) with respect to t , we obtain (2.5) for a.e. $t \in [0, T^*)$. \square

By constructing proper Lyapunov’s functions, Li and Gao [15] obtained the following blow-up result.

Theorem 2.6 ([15]) *Assume that (H1)–(H3) hold,*

$$p > m \quad \text{and} \quad l > \frac{1}{(p - 1)^2}.$$

Then the weak solution of problem (1.1) blows up in finite time provided that the initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfies $E(0) < E_1$ and $l^{\frac{1}{2}}|\Delta u_0|_2 > \lambda_1$, where

$$E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1^2, \quad \lambda_1 = B_1^{-\frac{p}{p-2}}, \quad B_1 = Bl^{-\frac{1}{2}}, \quad B^{-1} = \inf_{w \in H_0^2(\Omega), w \neq 0} \frac{|\Delta w|_2}{|w|_p}.$$

For the case of $E(0) < 0$, the following blow-up result is a natural conclusion of Theorem 2.6.

Proposition 2.7 *Assume that (H1)–(H3) hold,*

$$p > m \quad \text{and} \quad l > \frac{1}{(p - 1)^2}.$$

Then the weak solution of problem (1.1) blows up in finite time provided that the initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfies $E(0) < 0$.

Proof For any $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ such that

$$E(0) = \frac{1}{2}|u_1|_2^2 + \frac{1}{2}|\Delta u_0|_2^2 - \frac{1}{p}|u_0|_p^p < 0,$$

it holds that

$$0 < \frac{1}{2}|\Delta u_0|_2^2 < \frac{1}{p}|u_0|_p^p \leq \frac{1}{p}(B|\Delta u_0|_2)^p.$$

Then

$$|\Delta u_0|_2 > \left(\frac{p}{2}\right)^{\frac{1}{p-2}} B^{-\frac{p}{p-2}} > B^{-\frac{p}{p-2}} = (l^{\frac{1}{2}} B_1)^{-\frac{p}{p-2}} = l^{-\frac{p}{2(p-2)}} \lambda_1.$$

Since $0 < l < 1$, we finally obtain

$$l^{\frac{1}{2}}|\Delta u_0|_2 > l^{-\frac{1}{p-2}} \lambda_1 > \lambda_1.$$

So, by Theorem 2.6, the solution u with initial data (u_0, u_1) blows up in finite time. □

3 Main results

Now, we state our main results as follows.

Theorem 3.1 *Assume that (H1)–(H3) hold,*

$$p > m \quad \text{and} \quad l > \frac{1}{(p-1)^2}.$$

Then, for any $R \geq 0$, there exists a weak solution u of problem (1.1) blowing up in finite time provided that the initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfies $E(0) = R$.

Proof In the following, we present the proof of Theorem 3.1 in three steps.

Step 1. Suppose that u is a weak solution of problem (1.1) and let $\rho(t) = \int_{\Omega} u_t u \, dx$. We claim that there exist positive constants A and B such that

$$\frac{d}{dt}(\rho(t) - BE(t)) \geq A(\rho(t) - BE(t)) \tag{3.1}$$

for all $t \in [0, T^*)$, where T^* is the maximal existence time of u .

Because of the lack of the regularity of u_{tt} , the following *formal* calculation

$$\frac{d}{dt}\rho(t) = \int_{\Omega} u_t^2 \, dx + \int_{\Omega} u_{tt} u \, dx$$

is not legitimate. Following the procedure in the proof of Theorem 1.7 in [38], we firstly verify that we can differentiate $\rho(t)$ with respect to t . From (H2) and Theorem 2.3, it follows that

$$u \in C([0, T]; H_0^2(\Omega)) \subset L^m(\Omega \times (0, T))$$

for any $T \in (0, T^*)$, thus $u \in \mathcal{F}$. Letting $\phi = u$ in (2.2), we obtain

$$\begin{aligned} \rho(t) &= \int_{\Omega} u_0 u_1 \, dx + \int_0^t |u_t(\tau)|_2^2 \, d\tau - \int_0^t |\Delta u(\tau)|_2^2 \, d\tau + \int_0^t |u(\tau)|_p^p \, d\tau \\ &\quad - \int_0^t \int_{\Omega} |u_t(\tau)|^{m-2} u_t(\tau) u(\tau) \, dx \, d\tau + \int_0^t \int_0^s g(s-\tau) \int_{\Omega} \Delta u(\tau) \Delta u(s) \, dx \, d\tau \, ds \end{aligned}$$

for any $t \in [0, T]$. In view of (H1)–(H3) and the regularity of u , for any $t \in [0, T]$, we have $\int_0^t |\Delta u(\tau)|_2^2 \, d\tau < \infty$,

$$\begin{aligned} \int_0^t |u_t(\tau)|_2^2 \, d\tau &\leq \int_0^T |u_t(\tau)|_2^2 \, d\tau \leq T \max_{\tau \in [0, T]} |u_t(\tau)|_2^2 < \infty, \\ \int_0^t |u(\tau)|_p^p \, d\tau &\leq C \int_0^T |\Delta u(\tau)|_2^p \, d\tau \leq CT \max_{\tau \in [0, T]} |\Delta u(\tau)|_2^p < \infty, \\ \int_0^t \int_{\Omega} |u_t(\tau)|^{m-2} u_t(\tau) u(\tau) \, dx \, d\tau &\leq \int_0^T |u_t(\tau)|_m^{m-1} |u(\tau)|_m \, d\tau \leq \frac{m-1}{m} \int_0^T |u_t(\tau)|_m^m \, d\tau + \frac{1}{m} \int_0^T |u(\tau)|_m^m \, d\tau < \infty, \\ \int_0^t \int_0^s g(s-\tau) \int_{\Omega} \Delta u(\tau) \Delta u(s) \, dx \, d\tau \, ds &\leq \frac{1}{2} \int_0^T \int_0^s g(s-\tau) (|\Delta u(\tau)|_2^2 + |\Delta u(s)|_2^2) \, d\tau \, ds \\ &\leq \max_{\tau \in [0, T]} |\Delta u(\tau)|_2^2 \int_0^T \int_0^s g(s-\tau) \, d\tau \, ds < \infty. \end{aligned}$$

Therefore, $\rho(t)$ is absolutely continuous on any closed subsegment of $[0, T^*)$. Thus, we have

$$\begin{aligned} \frac{d}{dt} \rho(t) &= |u_t|_2^2 - |\Delta u|_2^2 + |u|_p^p + \int_0^t g(t-\tau) \int_{\Omega} \Delta u(t) \Delta u(\tau) \, dx \, d\tau \\ &\quad - \int_{\Omega} |u_t|^{m-2} u_t u \, dx \end{aligned} \tag{3.2}$$

for a.e. $t \in [0, T^*)$. In the following, we estimate the last two terms on the right-hand side of (3.2).

Since

$$\begin{aligned} &\int_0^t g(t-\tau) \int_{\Omega} \Delta u(\tau) \Delta u(t) \, dx \, d\tau \\ &= \int_0^t g(t-\tau) \int_{\Omega} (\Delta u(\tau) - \Delta u(t)) \Delta u(t) \, dx \, d\tau + \int_0^t g(t-\tau) \int_{\Omega} |\Delta u(t)|^2 \, dx \, d\tau \\ &= \int_0^t g(t-\tau) \int_{\Omega} (\Delta u(\tau) - \Delta u(t)) \Delta u(t) \, dx \, d\tau + G(t) |\Delta u(t)|_2^2 \end{aligned}$$

and

$$\left| \int_0^t g(t-\tau) \int_{\Omega} (\Delta u(\tau) - \Delta u(t)) \Delta u(t) \, dx \, d\tau \right|$$

$$\begin{aligned} &\leq \int_0^t g(t-\tau) \left(\frac{1}{2} \alpha |\Delta u(t) - \Delta u(\tau)|_2^2 + \frac{1}{2\alpha} |\Delta u(t)|_2^2 \right) d\tau \\ &= \frac{1}{2} \alpha (g \circ \Delta u)(t) + \frac{1}{2\alpha} G(t) |\Delta u|_2^2 \end{aligned}$$

for any $\alpha > 0$, we have

$$\int_0^t g(t-\tau) \int_{\Omega} \Delta u(\tau) \Delta u(t) dx d\tau \geq -\frac{1}{2} \alpha (g \circ \Delta u)(t) + \left(1 - \frac{1}{2\alpha}\right) G(t) |\Delta u|_2^2 \tag{3.3}$$

for any $\alpha > 0$.

(i) For the case of $2 < m < p$, by using Hölder’s inequality and Young’s inequality, we obtain

$$\begin{aligned} &\left| \int_{\Omega} |u_t|^{m-2} u_t u dx \right| \\ &\leq \int_{\Omega} |u_t|^{m-1} |u| dx \leq |u_t|_m^{m-1} |u|_m \\ &\leq \frac{m-1}{m} (\epsilon |u_t|_m^{m-1})^{\frac{m}{m-1}} + \frac{1}{m} \left(\frac{1}{\epsilon} |u|_m\right)^m \\ &= \frac{m-1}{m} \epsilon^{\frac{m}{m-1}} |u_t|_m^m + \frac{1}{m} \frac{1}{\epsilon^m} \int_{\Omega} (\delta |u|^{\frac{p(m-2)}{p-2}}) \left(\frac{1}{\delta} |u|^{\frac{2(p-m)}{p-2}}\right) dx \\ &\leq \frac{m-1}{m} \epsilon^{\frac{m}{m-1}} |u_t|_m^m \\ &\quad + \frac{1}{m} \frac{1}{\epsilon^m} \int_{\Omega} \left[\frac{m-2}{p-2} (\delta |u|^{\frac{p(m-2)}{p-2}})^{\frac{p-2}{m-2}} + \frac{p-m}{p-2} \left(\frac{1}{\delta} |u|^{\frac{2(p-m)}{p-2}}\right)^{\frac{p-2}{p-m}} \right] dx \\ &= \frac{m-1}{m} \epsilon^{\frac{m}{m-1}} |u_t|_m^m + \frac{1}{m} \frac{m-2}{p-2} \frac{1}{\epsilon^m} \delta^{\frac{p-2}{m-2}} |u|_p^p + \frac{1}{m} \frac{p-m}{p-2} \frac{1}{\epsilon^m} \left(\frac{1}{\delta}\right)^{\frac{p-2}{p-m}} |u|_2^2 \end{aligned} \tag{3.4}$$

for any $\epsilon, \delta > 0$. For any $\beta > 0$, there exists unique $\epsilon > 0$ such that $\beta = \frac{1}{\epsilon^m}$. For any $\lambda > 0$, the equation

$$(m-2) \delta^{\frac{p-2}{m-2}} \lambda = (p-m) \left(\frac{1}{\delta}\right)^{\frac{p-2}{p-m}}$$

admits a unique positive root

$$\delta = \delta_{\lambda} := \left(\frac{p-m}{m-2} \frac{1}{\lambda}\right)^{\frac{(m-2)(p-m)}{(p-2)^2}}.$$

So, from (3.4), it follows that

$$\begin{aligned} \left| \int_{\Omega} |u_t|^{m-2} u_t u dx \right| &\leq \frac{m-1}{m} \left(\frac{1}{\beta}\right)^{\frac{1}{m-1}} |u_t|_m^m \\ &\quad + \frac{1}{m} \left(\frac{m-2}{p-2}\right)^{\frac{m-2}{p-2}} \left(\frac{1}{\lambda}\right)^{\frac{p-m}{p-2}} \beta (|u|_p^p + \lambda |u|_2^2) \end{aligned} \tag{3.5}$$

for any $\beta, \lambda > 0$.

By (H3) and (2.5), we have

$$\frac{d}{dt}E(t) \leq -|u_t|_m^m. \tag{3.6}$$

Choosing arbitrary $\theta > 0$, in view of (2.3) and (3.2)–(3.6), we have

$$\begin{aligned} & \frac{d}{dt}(\rho(t) - \theta E(t)) \\ & \geq \theta |u_t|_m^m + |u_t|_2^2 - |\Delta u|_2^2 + |u|_p^p + \left[-\frac{1}{2}\alpha(g \circ \Delta u)(t) + \left(1 - \frac{1}{2\alpha}\right)G(t)|\Delta u|_2^2 \right] \\ & \quad + \left[-\frac{m-1}{m} \left(\frac{1}{\beta}\right)^{\frac{1}{m-1}} |u_t|_m^m - \frac{1}{m} \left(\frac{m-2}{p-2}\right)^{\frac{m-2}{p-2}} \left(\frac{1}{\lambda}\right)^{\frac{p-m}{p-2}} \beta (|u|_p^p + \lambda |u|_2^2) \right] \\ & = \left[\left(1 - \frac{1}{2\alpha}\right)G(t) - 1 \right] |\Delta u|_2^2 - \frac{1}{m} \left(\frac{m-2}{p-2}\right)^{\frac{m-2}{p-2}} \left(\frac{1}{\lambda}\right)^{\frac{p-m}{p-2}} \beta \lambda |u|_2^2 \\ & \quad + \left[1 - \frac{1}{m} \left(\frac{m-2}{p-2}\right)^{\frac{m-2}{p-2}} \left(\frac{1}{\lambda}\right)^{\frac{p-m}{p-2}} \beta \right] |u|_p^p \\ & \quad + |u_t|_2^2 - \frac{1}{2}\alpha(g \circ \Delta u)(t) + \left[\theta - \frac{m-1}{m} \left(\frac{1}{\beta}\right)^{\frac{1}{m-1}} \right] |u_t|_m^m \\ & = \left[\left(1 - \frac{1}{2\alpha}\right)G(t) - 1 \right] |\Delta u|_2^2 - \frac{1}{m} \left(\frac{m-2}{p-2}\right)^{\frac{m-2}{p-2}} \left(\frac{1}{\lambda}\right)^{\frac{p-m}{p-2}} \beta \lambda |u|_2^2 \\ & \quad + \left[1 - \frac{1}{m} \left(\frac{m-2}{p-2}\right)^{\frac{m-2}{p-2}} \left(\frac{1}{\lambda}\right)^{\frac{p-m}{p-2}} \beta \right] |u|_p^p \\ & \quad + |u_t|_2^2 - \alpha \left[E(t) - \frac{1}{2}|u_t|_2^2 - \frac{1}{2}(1 - G(t))|\Delta u|_2^2 + \frac{1}{p}|u|_p^p \right] \\ & \quad + \left[\theta - \frac{m-1}{m} \left(\frac{1}{\beta}\right)^{\frac{1}{m-1}} \right] |u_t|_m^m \\ & = \frac{1}{2\alpha} [(\alpha - 1)^2(1 - G(t)) - 1] |\Delta u|_2^2 - \frac{1}{m} \left(\frac{m-2}{p-2}\right)^{\frac{m-2}{p-2}} \left(\frac{1}{\lambda}\right)^{\frac{p-m}{p-2}} \beta \lambda |u|_2^2 \\ & \quad + \left[1 - \frac{1}{m} \left(\frac{m-2}{p-2}\right)^{\frac{m-2}{p-2}} \left(\frac{1}{\lambda}\right)^{\frac{p-m}{p-2}} \beta - \frac{\alpha}{p} \right] |u|_p^p \\ & \quad + \left(1 + \frac{1}{2}\alpha\right) |u_t|_2^2 - \alpha E(t) + \left[\theta - \frac{m-1}{m} \left(\frac{1}{\beta}\right)^{\frac{1}{m-1}} \right] |u_t|_m^m \end{aligned} \tag{3.7}$$

for any $\alpha, \beta, \lambda > 0$.

For any $\lambda > 0$ and $0 < \alpha < p$, we have

$$\beta = \beta(\alpha, \lambda) := \left(1 - \frac{\alpha}{p}\right) \left(\frac{p-2}{m-2}\right)^{\frac{m-2}{p-2}} \lambda^{\frac{p-m}{p-2}} m > 0.$$

Since $l > \frac{1}{(p-1)^2}$, it holds that $1 + \frac{1}{\sqrt{l}} < p$. Letting $\beta = \beta(\alpha, \lambda)$ and $\theta = \frac{m-1}{m} \left(\frac{1}{\beta(\alpha, \lambda)}\right)^{\frac{1}{m-1}}$ in (3.7) and denoting

$$\Lambda_1 := \inf_{\substack{w \in H_0^2(\Omega) \\ w \neq 0}} \frac{|\Delta w|_2^2}{|w|_2^2} > 0,$$

we have

$$\begin{aligned}
 & \frac{d}{dt} \left(\rho(t) - \frac{m-1}{m} \left(\frac{1}{\beta(\alpha, \lambda)} \right)^{\frac{1}{m-1}} E(t) \right) \\
 & \geq \frac{1}{2\alpha} [(\alpha - 1)^2(1 - G(t)) - 1] |\Delta u|_2^2 - \left(1 - \frac{\alpha}{p} \right) \lambda |u|_2^2 + \left(1 + \frac{1}{2}\alpha \right) |u_t|_2^2 - \alpha E(t) \\
 & \geq \frac{1}{2\alpha} [(\alpha - 1)^2 l - 1] |\Delta u|_2^2 - \left(1 - \frac{\alpha}{p} \right) \lambda |u|_2^2 + \left(1 + \frac{1}{2}\alpha \right) |u_t|_2^2 - \alpha E(t) \\
 & \geq \left[\frac{1}{2\alpha} ((\alpha - 1)^2 l - 1) \Lambda_1 - \left(1 - \frac{\alpha}{p} \right) \lambda \right] |u|_2^2 + \left(1 + \frac{1}{2}\alpha \right) |u_t|_2^2 - \alpha E(t) \tag{3.8}
 \end{aligned}$$

for any $\alpha \in (1 + \frac{1}{\sqrt{l}}, p)$ and $\lambda > 0$. Since the mapping $\alpha \mapsto \frac{1}{2\alpha} [(\alpha - 1)^2 l - 1] \Lambda_1$ is strictly increasing on $(1 + \frac{1}{\sqrt{l}}, p)$ with

$$\frac{1}{2\alpha} [(\alpha - 1)^2 l - 1] \Lambda_1 \rightarrow 0^+ \quad \text{as } \alpha \rightarrow \left(1 + \frac{1}{\sqrt{l}} \right)^+,$$

and the mapping $\alpha \rightarrow (1 - \frac{\alpha}{p})\lambda$ is strictly decreasing on $(1 + \frac{1}{\sqrt{l}}, p)$ with

$$\left(1 - \frac{\alpha}{p} \right) \lambda \rightarrow 0^+ \quad \text{as } \alpha \rightarrow p^-,$$

it is easy to verify that, for any fixing $\lambda > 0$, the equation

$$\frac{1}{2\alpha} [(\alpha - 1)^2 l - 1] \Lambda_1 = \left(1 - \frac{\alpha}{p} \right) \lambda$$

admits a unique root $\alpha(\lambda) \in (1 + \frac{1}{\sqrt{l}}, p)$. Moreover, we have

$$C(\alpha, \lambda) := \frac{1}{2\alpha} [(\alpha - 1)^2 l - 1] \Lambda_1 - \left(1 - \frac{\alpha}{p} \right) \lambda > 0 \quad \text{for all } \alpha \in (\alpha(\lambda), p).$$

So, from (3.8), it follows that

$$\begin{aligned}
 & \frac{d}{dt} \left(\rho(t) - \frac{m-1}{m} \left(\frac{1}{\beta(\alpha, \lambda)} \right)^{\frac{1}{m-1}} E(t) \right) \\
 & \geq C(\alpha, \lambda) |u|_2^2 + \left(1 + \frac{1}{2}\alpha \right) |u_t|_2^2 - \alpha E(t) \\
 & \geq 2\sqrt{\left(1 + \frac{1}{2}\alpha \right) C(\alpha, \lambda)} \int_{\Omega} u_t u \, dx - \alpha E(t) \\
 & \geq A(\alpha, \lambda) (\rho(t) - B(\alpha, \lambda) E(t)) \tag{3.9}
 \end{aligned}$$

for any $\lambda > 0$ and $\alpha \in (\alpha(\lambda), p)$, where

$$A(\alpha, \lambda) := 2\sqrt{\left(1 + \frac{1}{2}\alpha \right) C(\alpha, \lambda)} > 0, \quad B(\alpha, \lambda) := \frac{\alpha}{A(\alpha, \lambda)} > 0.$$

For any fixed $\lambda > 0$, it is easy to verify that the mapping

$$\alpha \mapsto \frac{m-1}{m} \left(\frac{1}{\beta(\alpha, \lambda)} \right)^{\frac{1}{m-1}}$$

is strictly increasing on $(\alpha(\lambda), p)$ with

$$\frac{m-1}{m} \left(\frac{1}{\beta(\alpha, \lambda)} \right)^{\frac{1}{m-1}} \rightarrow +\infty \text{ as } \alpha \rightarrow p^-.$$

Since $p > 2, 0 < l < 1$ and

$$\begin{aligned} B(\alpha, \lambda) &= \frac{\alpha}{A(\alpha, \lambda)} \\ &= \frac{\alpha}{2\sqrt{(1 + \frac{1}{2}\alpha)C(\alpha, \lambda)}} \\ &= \frac{\alpha}{2\sqrt{(1 + \frac{1}{2}\alpha)[\frac{1}{2\alpha}((\alpha - 1)^2l - 1)\Lambda_1 - (1 - \frac{\alpha}{p})\lambda]}} \\ &= \frac{\alpha}{2\sqrt{\frac{1}{2}(\frac{l\Lambda_1}{2} + \frac{\lambda}{p})\alpha^2 - (\frac{\lambda}{2} - \frac{\lambda}{p})\alpha - (\lambda + \frac{1}{4}(1 + 3l)\Lambda_1) - \frac{\Lambda_1(1-l)}{2} \frac{1}{\alpha}}} \\ &= \frac{1}{2\sqrt{\frac{1}{2}(\frac{l\Lambda_1}{2} + \frac{\lambda}{p}) - (\frac{\lambda}{2} - \frac{\lambda}{p})\frac{1}{\alpha} - (\lambda + \frac{1}{4}(1 + 3l)\Lambda_1)\frac{1}{\alpha^2} - \frac{\Lambda_1(1-l)}{2} \frac{1}{\alpha^2}}}, \end{aligned} \tag{3.10}$$

it is easy to verify that the mapping $\alpha \mapsto B(\alpha, \lambda)$ is strictly decreasing on $(\alpha(\lambda), p)$ with $B(\alpha, \lambda) \rightarrow +\infty$ as $\alpha \rightarrow (\alpha(\lambda))^+$. So, for fixed $\lambda > 0$, the equation

$$B(\alpha, \lambda) = \frac{m-1}{m} \left(\frac{1}{\beta(\alpha, \lambda)} \right)^{\frac{1}{m-1}}$$

for α on $(\alpha(\lambda), p)$ admits a unique root $\alpha_0(\lambda)$. From (3.9), it follows that

$$\frac{d}{dt}(\rho(t) - B(\alpha_0(\lambda), \lambda)E(t)) \geq A(\alpha_0(\lambda), \lambda)(\rho(t) - B(\alpha_0(\lambda), \lambda)E(t))$$

for any $\lambda > 0$ and $t \in [0, T^*)$. Choosing a fixed $\lambda > 0$ and letting $A = A(\alpha_0(\lambda), \lambda)$ and $B = B(\alpha_0(\lambda), \lambda)$, we get (3.1).

(ii) For the case of $m = 2$, by (3.2), (3.3), and (3.6), we have

$$\begin{aligned} &\frac{d}{dt}(\rho(t) - \theta E(t)) \\ &\geq |u_t|_2^2 - |\Delta u|_2^2 + |u|_p^p + \left[\left(1 - \frac{1}{2\alpha}\right)G(t)|\Delta u|_2^2 - \frac{1}{2}\alpha(g \circ \Delta u)(t) \right] - \rho(t) + \theta |u_t|_2^2 \\ &= (1 + \theta)|u_t|_2^2 + \left[\left(1 - \frac{1}{2\alpha}\right)G(t) - 1 \right] |\Delta u|_2^2 + |u|_p^p - \rho(t) \\ &\quad - \alpha \left[E(t) - \frac{1}{2}|u_t|_2^2 - \frac{1}{2}(1 - G(t))|\Delta u|_2^2 + \frac{1}{p}|u|_p^p \right] \end{aligned}$$

$$\begin{aligned}
 &= \left(1 + \theta + \frac{1}{2}\alpha\right) |u_t|_2^2 + \frac{1}{2\alpha} [(\alpha - 1)^2(1 - G(t)) - 1] |\Delta u|_2^2 \\
 &\quad + \left(1 - \frac{\alpha}{p}\right) |u|_p^p - \rho(t) - \alpha E(t)
 \end{aligned} \tag{3.11}$$

for any $\alpha, \theta > 0$. Since $l > \frac{1}{(p-1)^2}$, by letting $\alpha = p$ in (3.11), we have

$$\frac{1}{2p} [(p - 1)^2(1 - G(t)) - 1] \geq \frac{1}{2p} [(p - 1)^2l - 1] > 0.$$

Then, from (3.11), it follows that

$$\begin{aligned}
 &\frac{d}{dt}(\rho(t) - \theta E(t)) \\
 &= \left(1 + \theta + \frac{p}{2}\right) |u_t|_2^2 + \frac{1}{2p} [(p - 1)^2(1 - G(t)) - 1] |\Delta u|_2^2 - \rho(t) - pE(t) \\
 &\geq \left(1 + \theta + \frac{p}{2}\right) |u_t|_2^2 + \frac{1}{2p} ((p - 1)^2l - 1) \Lambda_1 |u|_2^2 - \rho(t) - pE(t) \\
 &\geq \left(2\sqrt{\frac{\Lambda_1}{2p}} ((p - 1)^2l - 1) \left(1 + \frac{p}{2} + \theta\right) - 1\right) \rho(t) - pE(t) \\
 &= A(\theta)(\rho(t) - B(\theta)E(t))
 \end{aligned} \tag{3.12}$$

for any $\theta > \theta_1$, where

$$\begin{aligned}
 \theta_1 &:= \max\left\{0, \frac{p}{2\Lambda_1[(p - 1)^2 - 1]} - \frac{p}{2} - 1\right\}, \\
 A(\theta) &:= 2\sqrt{\frac{\Lambda_1}{2p}} ((p - 1)^2l - 1) \left(1 + \frac{p}{2} + \theta\right) - 1, \quad B(\theta) := \frac{p}{A(\theta)}.
 \end{aligned}$$

It is easy to verify that the equation

$$\theta = B(\theta) = \frac{p}{2\sqrt{\frac{\Lambda_1}{2p}} ((p - 1)^2l - 1) \left(1 + \frac{p}{2} + \theta\right) - 1} \tag{3.13}$$

for θ on $(\theta_1, +\infty)$ admits a unique positive root θ_0 . By (3.12), we obtain

$$\frac{d}{dt}(\rho(t) - \theta_0 E(t)) \geq A(\theta_0)(\rho(t) - \theta_0 E(t)) \tag{3.14}$$

for all $t \in [0, T^*)$. Letting $A = A(\theta_0)$ and $B = \theta_0$, we get (3.1).

Step 2. We claim that the solution u of problem (1.1) blows up in finite time provided that the initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfies

$$0 \leq E(0) < \frac{1}{B} \rho(0). \tag{3.15}$$

Suppose, on the contrary, that u is global.

On the one hand, since $m \geq 2$ and Ω is bounded, there exists a positive constant S depending on m and $|\Omega|$ such that $|w|_2 \leq S|w|_m$ for all $w \in L^m(\Omega)$. Then, by Theorem 2.3, (2.4) and using Hölder’s inequality, we obtain

$$\begin{aligned} |u(t)|_2 &= \left| u(0) + \int_0^t u_t(\tau) \, d\tau \right|_2 \\ &\leq |u_0|_2 + \int_0^t |u_t(\tau)|_2 \, d\tau \\ &\leq |u_0|_2 + S \int_0^t |u_t(\tau)|_m \, d\tau \\ &\leq |u_0|_2 + S \left(\int_0^t 1^{\frac{m}{m-1}} \, d\tau \right)^{\frac{m-1}{m}} \left(\int_0^t |u_t(\tau)|_m^m \, d\tau \right)^{\frac{1}{m}} \\ &\leq |u_0|_2 + St^{\frac{m-1}{m}} (E(0) - E(t))^{\frac{1}{m}} \quad \text{for all } t \in [0, +\infty). \end{aligned} \tag{3.16}$$

Since $u = u(t)$ is a global solution of problem (1.1), we have $E(t) \geq 0$ for all $t \in [0, \infty)$. Otherwise, there exists $t_0 \in (0, \infty)$ such that $E(t_0) < 0$. By Proposition 2.7, u blows up in finite time, which is a contradiction. Thus, due to (2.4), we have $0 \leq E(t) \leq E(0)$. Finally, in view of (3.16) we obtain

$$|u(t)|_2 \leq |u_0|_2 + St^{\frac{m-1}{m}} (E(0))^{\frac{1}{m}} \quad \text{for all } t \in [0, +\infty). \tag{3.17}$$

On the other hand, (3.1) implies that

$$\frac{d}{dt} H(t) \geq AH(t) \quad \text{for all } t \in [0, +\infty),$$

where $H(t) := \rho(t) - BE(t)$. By using Gronwall’s inequality, we obtain $H(t) \geq e^{At}H(0)$, i.e.,

$$\int_{\Omega} u_t u \, dx = \rho(t) \geq e^{At}H(0) + BE(t) \quad \text{for all } t \in [0, +\infty).$$

In view of (2.4) and Theorem 2.6, and since $u(t)$ is global, we have $0 \leq E(t) \leq E(0)$, which yields that

$$\rho(t) \geq e^{At}H(0) \quad \text{for all } t \in [0, +\infty).$$

Inequality (3.15) implies that $H(0) = \rho(0) - BE(0) > 0$, so we obtain

$$\begin{aligned} |u(t)|_2^2 &= |u(0)|_2^2 + 2 \int_0^t \rho(\tau) \, d\tau \\ &\geq |u_0|_2^2 + 2 \int_0^t e^{A\tau} H(0) \, d\tau \\ &= |u_0|_2^2 + \frac{2}{A} (e^{At} - 1) H(0) \end{aligned}$$

for all $t \in [0, +\infty)$, which contradicts (3.17) for t sufficiently large. So $T^* < +\infty$.

Step 3. For any $R \geq 0$, we can find $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ such that

$$E(0) = R < \frac{1}{B} \rho(0).$$

In fact, let Ω_1 and Ω_2 be two disjoint open subdomains of Ω such that $\text{dist}(\Omega_1, \partial\Omega) > 0$ and $\text{dist}(\Omega_2, \partial\Omega) > 0$, and choose an arbitrary nonzero function $w \in C_0^\infty(\Omega)$ such that $\text{supp } w \subset \Omega_1$. Then there exists $r_1 > 0$ such that

$$\int_{\Omega_1} |rw|^2 \, dx = r^2 \int_{\Omega_1} |w|^2 \, dx > BR \quad \text{for all } r \geq r_1. \tag{3.18}$$

On the other hand, by Lemma 2.1, there exists a sequence of functions $\{v_k\} \subset H_0^2(\Omega_2)$ such that

$$\frac{1}{2} \int_{\Omega_2} |\Delta v_k|^2 \, dx + \frac{1}{2} \int_{\Omega_2} |v_k|^2 \, dx - \frac{1}{p} \int_{\Omega_2} |v_k|^p \, dx \rightarrow +\infty \quad \text{as } k \rightarrow \infty.$$

Since $p > 2$, there exists $r_2 > r_1$ such that the function

$$\begin{aligned} f(r) &:= \frac{1}{2} \int_{\Omega} |rw|^2 \, dx + \frac{1}{2} \int_{\Omega} |\Delta(rw)|^2 \, dx - \frac{1}{p} \int_{\Omega} |rw|^p \, dx \\ &= \left(\frac{1}{2} \int_{\Omega} |w|^2 \, dx + \frac{1}{2} \int_{\Omega} |\Delta w|^2 \, dx \right) r^2 - \left(\frac{1}{p} \int_{\Omega} |w|^p \, dx \right) r^p \end{aligned}$$

is continuous and strictly increasing on $[r_2, +\infty)$, and

$$f(r) \rightarrow -\infty \quad \text{as } r \rightarrow +\infty.$$

So there exist $k_0 \in \mathbb{N}$ and $r_0 \geq r_2$ both sufficiently large such that

$$\begin{aligned} f(r_2) &\leq \frac{1}{2} \int_{\Omega_2} |\Delta v_{k_0}|^2 \, dx + \frac{1}{2} \int_{\Omega_2} |v_{k_0}|^2 \, dx - \frac{1}{p} \int_{\Omega_2} |v_{k_0}|^p \, dx = f(r_0) \\ &= R - \left(\frac{1}{2} \int_{\Omega} |r_0 w|^2 \, dx + \frac{1}{2} \int_{\Omega} |\Delta(r_0 w)|^2 \, dx - \frac{1}{p} \int_{\Omega} |r_0 w|^p \, dx \right). \end{aligned} \tag{3.19}$$

Let $u_1 = u_0 = r_0 w + \tilde{v}$, where

$$\tilde{v} = \begin{cases} 0, & x \in \Omega \setminus \Omega_2, \\ v_{k_0}, & x \in \Omega_2. \end{cases}$$

Then $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$. By Theorem 2.3, problem (1.1) has a unique weak solution u with initial data (u_0, u_1) such that

$$u \in C([0, T]; H_0^2(\Omega)), \quad u_t \in C([0, T]; L^2(\Omega)) \cap L^m(\Omega \times (0, T))$$

for some $T > 0$. In view of (2.3), (3.18), and (3.19), we have

$$\rho(0) = \int_{\Omega} u_0 u_1 \, dx = \int_{\Omega} |r_0 w + \tilde{v}|^2 \, dx \geq \int_{\Omega_1} |r_0 w + \tilde{v}|^2 \, dx = \int_{\Omega_1} |r_0 w|^2 \, dx > BR$$

and

$$\begin{aligned}
 E(0) &= \frac{1}{2} |u_t(0)|_2^2 + \frac{1}{2} (1 - G(0)) |\Delta u(0)|_2^2 + \frac{1}{2} (g \circ \Delta u)(0) - \frac{1}{p} |u(0)|_p^p \\
 &= \frac{1}{2} |u_1|_2^2 + \frac{1}{2} |\Delta u_0|_2^2 - \frac{1}{p} |u_0|_p^p \\
 &= \frac{1}{2} \left(\int_{\Omega_1} |r_0 w|^2 dx + \int_{\Omega_2} |\tilde{v}|^2 dx \right) + \frac{1}{2} \left(\int_{\Omega_1} |\Delta(r_0 w)|^2 dx + \int_{\Omega_2} |\Delta \tilde{v}|^2 dx \right) \\
 &\quad - \frac{1}{p} \left(\int_{\Omega_1} |r_0 w|^p dx + \int_{\Omega_2} |\tilde{v}|^p dx \right) \\
 &= R < \frac{1}{B} \rho(0).
 \end{aligned}$$

Hence, according to the conclusion in Step 2, the weak solution u blows up in finite time. This completes the proof. \square

In view of (3.10), $B(\alpha, \lambda)$ is strictly decreasing for α on $(\alpha(\lambda), p)$, then we have

$$B_0 := \inf_{\lambda > 0} B(\alpha_0(\lambda), \lambda) \geq B(p, \lambda) = \frac{p}{\sqrt{(1 + \frac{2}{p})((p-1)^2 l - 1) \Lambda_1}} > 0. \tag{3.20}$$

For the case of $m = 2$, we denote $B_0 = \theta_0$, where θ_0 is the unique root of equation (3.13). Then, in view of the proof of Theorem 3.1, we obtain the following result.

Corollary 3.2 *Assume that (H1)–(H3) hold,*

$$p > m \quad \text{and} \quad l > \frac{1}{(p-1)^2}.$$

Then the weak solution of problem (1.1) blows up in finite time provided that the initial data $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfies

$$0 \leq E(0) < \frac{1}{B_0} \int_{\Omega} u_0 u_1 dx. \tag{3.21}$$

Proof We only need to consider the case of $2 < m < p$. Noticing that the mapping $\lambda \mapsto B(\alpha_0(\lambda), \lambda)$ is continuous on $(0, +\infty)$, by (3.20) and (3.21), there exists $\lambda_0 > 0$ such that

$$0 \leq E(0) < \frac{1}{B(\alpha_0(\lambda_0), \lambda_0)} \int_{\Omega} u_0 u_1 dx \leq \frac{1}{B_0} \int_{\Omega} u_0 u_1 dx.$$

Following the proof of Theorem 3.1, the weak solution u blows up in finite time. \square

Remark 3.3 If we follow the strategy in [19] directly without the modifications, we will obtain the coefficient $B(\alpha_0(\lambda), \lambda)$ only for the specified parameter $\lambda = \frac{p-m}{p-2}$.

Remark 3.4 Denote

$$\mathbb{B}_1 := \{ (u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega) : E(0) < E_1, l^{\frac{1}{2}} |\Delta u_0| > \lambda_1 \}$$

and

$$\mathbb{B}_2 := \left\{ (u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega) : 0 \leq E(0) < \frac{1}{B_0} \int_{\Omega} u_0 u_1 \, dx \right\}.$$

In view of Theorem 2.6 and Corollary 3.2, the solution u for problem (1.1) with initial data $(u_0, u_1) \in \mathbb{B}_1 \cup \mathbb{B}_2$ blows up in finite time. On the one hand, for any $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ satisfying $E(0) < 0$, Proposition 2.7 implies that $(u_0, u_1) \in \mathbb{B}_1 \setminus \mathbb{B}_2$. On the other hand, Theorem 3.1 implies that $\mathbb{B}_2 \setminus \mathbb{B}_1 \neq \emptyset$.

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Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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