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Existence and multiplicity of solutions for fractional $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger-type equations with Robin boundary conditions

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Abstract

In this paper, we study fractional $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger-type equations for Robin boundary conditions. Under some suitable assumptions, we show that two solutions exist using the mountain pass lemma and Ekeland's variational principle. Then, the existence of infinitely many solutions is derived by applying the fountain theorem and the Krasnoselskii genus theory, respectively. Different from previous results, the topic of this paper is the Robin boundary conditions in $\mathbb{R}^N \setminus \overline{\Omega}$ for fractional order $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger-type equations, including concave-convex nonlinearities, which has not been studied before. In addition, two examples are given to illustrate our results.

Keywords: Schrödinger equations; $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian; Robin boundary conditions; Concave-convex nonlinearities; Krasnoselskii genus theory

1 Introduction and the main results

In this paper, we consider fractional $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger-type equations, including concave-convex nonlinearities with nonlocal Robin boundary conditions

$$\begin{cases} \sum_{i=1}^{2}[(-\Delta)^s_{p_i(x,\cdot)}\varphi + V_i(x)|\varphi|^{\bar{p}_i(x)-2}\varphi] \\ \quad = \lambda_1 A_1(x)|\varphi|^{r_1(x)-2}\varphi + \lambda_2 A_2(x)|\varphi|^{r_2(x)-2}\varphi, \quad x \in \Omega, \\ \sum_{i=1}^{2}[\mathcal{N}_{s,p_i(x,\cdot)}\varphi + \beta(x)|\varphi|^{\bar{p}_i(x)-2}\varphi] = \sum_{i=1}^{2} g_i(x), \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{1}$$

where $V_i(x)$ ($x \in \Omega$, $i = 1,2$) is a potential function, $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with the Lipschitz boundary $\partial\Omega$, $s \in (0,1)$, $p_i(x,\cdot) : \mathbb{R}^{2N} \to (1,+\infty)$, $\overline{p}_i(x) = p_i(x,x)$, $r_1(x)$, $r_2(x)$ are continuous functions, λ_1, λ_2 are positive constants, $A_1(x)$, $A_2(x)$ are positive weighted functions, $g_i(x) \geq 0 \in L^1(\mathbb{R}^N \setminus \Omega)$, $\beta(x) \geq 0 \in L^\infty(\mathbb{R}^N \setminus \Omega)$,

$$\mathcal{N}_{s,p_i(x,\cdot)}\varphi(x) = \int_\Omega \frac{|\varphi(x)-\varphi(y)|^{p_i(x,y)-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_i(x,y)}}\,dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega},$$

and

$$(-\Delta)^s_{p_i(x,\cdot)}\varphi(x) := P.V.\int_{\mathbb{R}^N}\frac{|\varphi(x)-\varphi(y)|^{p_i(x,y)-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_i(x,y)}}\,dy, \quad x\in\Omega,$$

where $P.V.$ stands for the Cauchy principal value.

Equations (1) arise from general reaction-diffusion equation

$$\varphi_t = \nabla\cdot\left[A(\varphi)\nabla\varphi\right] + r(x,\varphi), \tag{2}$$

where $A(\varphi) = |\nabla\varphi|^{p-2} + |\nabla\varphi|^{q-2}$. Problem (2) has applications in biophysics, plasma physics, and chemical reactions. For more details on equation (2), readers are referred to [1, 2]. Combining with a $\mathbb{Z}_2$-symmetric version of the mountain pass lemma for even functionals and some adequate variational methods, Mihăilescu [3] proved that the equations

$$\begin{cases} -\operatorname{div}((|\nabla\varphi|^{p_1(x)} + |\nabla\varphi|^{p_2(x)})\nabla\varphi) = f(x,\varphi), & x\in\Omega,\\ \varphi = 0, & x\in\partial\Omega \end{cases} \tag{3}$$

have infinitely many weak solutions. In addition, Chung and Toan [4] considered a class of fractional Laplacian problems

$$\begin{cases} (\Delta)^s_{p_1(x,\cdot)}\varphi(x) + (\Delta)^s_{p_2(x,\cdot)}\varphi(x) + |\varphi|^{q(x)-2}\varphi \\ \quad = \lambda A_1(x)|\varphi(x)|^{r_1(x)-2}\varphi(x) - \mu A_2(x)|\varphi(x)|^{r_2(x)-2}\varphi(x), & x\in\Omega,\\ \varphi(x) = 0, & x\in\partial\Omega \end{cases} \tag{4}$$

using variational techniques and Ekeland's variational principle. The authors used the variational techniques to discuss the results of the existence of solutions in fractional cases [5–7]. In addition, Heidarkhani et al. [8–10] studied the existence results of variable exponent equations using variational methods and established the critical point theory. Zuo et al. [11] investigated the existence and multiplicity of solutions for the $p(x,\cdot)\&q(x,\cdot)$ fractional Choquard problems with variable order. On a similar issue, a related study was conducted by Biswas et al. For more details, see [12].

The classical Schrödinger equation is of the following form:

$$ih\frac{\partial}{\partial t}\varphi = -\frac{h^2}{2m}\nabla^2\varphi + V\varphi,$$

where V, φ denote the potential function and wave function, respectively, and i, h are constants ([13]). Recently, Xiang et al. in [14] and Bu et al. in [15] discussed the fractional Laplace operator Schrödinger equations with variable order and Schrödinger–Kirchhof-type equations, respectively.

The critical local problem involving concave-convex nonlinearities was first studied by Ambrosetti et al. in [16]. Subsequently, variational methods were used [17] to discuss the following equations:

$$\begin{cases} -\operatorname{div}(\omega(x)|\nabla\varphi|^{p(x)-2}\nabla\varphi) = \lambda a(x)|\varphi|^{q(x)-2}\varphi + \mu b(x)|\varphi|^{h(x)-2}\varphi, & x\in\Omega,\\ \varphi = 0, & x\in\partial\Omega \end{cases} \tag{5}$$

with the variable order concave-convex term. For other similar types of equations, see [18, 19] and the references therein.

The Robin and Neumann boundary problems are interesting topics [20]. Mugnai et al. [21] investigated fractional p-Laplacian problems with nonlocal Neumann boundary conditions. Moreover, Deng [22] considered the following equations:

$$\begin{cases} -\Delta_{p(x)}\varphi = \lambda f(x,\varphi), & x \in \Omega, \\ |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta} + \beta(x)|\varphi|^{p(x)-2}\varphi = 0, & x \in \partial\Omega. \end{cases} \tag{6}$$

For double-phase problems depending on Robin and Steklov eigenvalues for the p-Laplacian, Manouni et al. [23] proved the existence of solutions by variational tools, truncation techniques, and comparison methods. In many papers, the Robin and Neumann boundary problems of fractional equations were studied in different ways; e.g., the Morse theory was used in [24, 25], the mountain pass lemma in [26, 27], Ekeland's variational principle in [28, 29], and the topological degree in [30].

To our knowledge, there is no previous work on the problem (1). This paper is devoted to this topic. We obtain new results by applying the mountain pass lemma, Ekeland's variational principle, the fountain theorem, and the Krasnoselskii genus theory. Our problem differs from problems (3), (4), and (5) in that we discuss Robin boundary conditions, and it also differs from problem the (6) in that we consider $p_1(x,\cdot)\&p_2(x,\cdot)$-Laplacian Schrödinger-type equations with concave-convex nonlinearities.

Before stating the main results, we introduce the basic assumptions.

(P) $p_i(x,y)$ is a symmetric and continuous function, that is,

$$p_i(x,y) = p_i(y,x), \quad \text{for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N$$

with

$$1 < p_i^- := \min_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p_i(x,y) \le p_i(x,y) \le p_i^+ := \max_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} p_i(x,y) < +\infty,$$

and

$$1 < p_1^- \le p_1^+ \le p_2^- \le p_2^+ < +\infty,$$

such that $sp_i^+ < N$. Let $0 < s < 1 < p(x,\cdot)$, the fractional critical exponent $p_s^*(x)$ be defined as $p_s^*(x) = \frac{Np(x,x)}{N-sp(x,x)}$ and $p(x,\cdot) < p_s^*(x)$ for all $x \in \overline{\Omega}$.

(G) $\int g_i(x)\varphi\,dx = -\int g_i(x)\varphi\,dx$.

(V) $V_i(x)$ is a continuous function, satisfying $\inf_{x\in\Omega} V_i(x) > V_{i0} > 0$, for all $d_i > 0$, means$(\{x \in \Omega : V_i(x) < d_i\}) < +\infty$.

(H) $A_1(x)$ and $A_2(x)$ are weighted functions in $C(\overline{\Omega})$ and satisfy $A_1(x) \in L^{s_1(x)}(\Omega)$ such that $1 < s_1(x) \in C(\overline{\Omega})$ and $1 < s_1'(x)r_1(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, $A_2(x) \in L^{s_2(x)}(\Omega)$ such that $1 < s_2(x) \in C(\overline{\Omega})$ and $1 < s_2'(x)r_2(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Here, $s_1'(x)$ and $s_2'(x)$ are conjugate exponents of the functions $s_1(x)$ and $s_2(x)$, respectively.

The main results of this paper are as follows:

Theorem 1.1 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. Equations* (1) *have two nontrivial weak solutions.*

Theorem 1.2 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. Then, equations* (1) *have infinitely many nontrivial weak solutions in* X.

Theorem 1.3 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. Then, equations* (1) *possess infinitely many solutions.*

In Sect. 2, we state some basic results of the Lebesgue space $L^{q(x)}(\Omega)$. In Sect. 3, we introduce the workspaces associated with equations (1). In Sect. 4, we verify the (PS) conditions and prove Theorem 1.1 by the mountain pass lemma and Ekeland's variational principle. In Sect. 5, we prove Theorem 1.2 by applying the fountain theorem. Finally, using the Krasnoselskii genus theory, we give the proof of Theorem 1.3.

2 Preliminaries

In this section, we recall some basic results of the Lebesgue space $L^{q(x)}(\Omega)$ with a variable exponent. Assume that domain Ω is bounded in $\mathbb{R}^N$ with the Lipschitz boundary $\partial\Omega$. Let

$$q^- = \min_{x\in\overline{\Omega}} q(x), \qquad q^+ = \max_{x\in\overline{\Omega}} q(x),$$

where $C_+(\overline{\Omega}) = \{q \in C(\overline{\Omega}) : q(x) > 1, \text{for all } x \in \overline{\Omega}\}$.

The variable exponent Lebesgue space $L^{q(x)}(\Omega)$, which is defined by

$$L^{q(x)}(\Omega) = \left\{\varphi|\varphi : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \left|\varphi(x)\right|^{q(x)} dx < \infty\right\},$$

equipped with the Luxemburg norm

$$\|\varphi\|_{q(x)} = \inf\left\{\iota > 0 : \int_\Omega \left|\frac{\varphi(x)^{q(x)}}{\iota}\right| dx \le 1\right\},$$

where $(L^{q(x)}(\Omega), \|\cdot\|_{q(x)})$ is a separable, uniformly convex, and reflexive Banach space [31].

Let $L^{q'(x)}(\Omega)$ be the conjugate space of $L^{q(x)}(\Omega)$ and $1/q(x) + 1/q'(x) = 1$ ($p(x)$ and $q'(x)$ are conjugate indices to each other). For $\varphi \in L^{q(x)}(\Omega)$ and $\nu \in L^{q'(x)}(\Omega)$, the Hölder inequality

$$\left|\int_\Omega \varphi(x)\nu(x)\,dx\right| < \left(\frac{1}{q^-} + \frac{1}{q'^-}\right)\|\varphi\|_{q(x)}\|\nu\|_{q'(x)} \le 2\|\varphi\|_{q(x)}\|\nu\|_{q'(x)} \tag{7}$$

holds. If $q_i(x) \in C_+(\overline{\Omega})$ $(i = 1, 2, \ldots, \widehat{n})$ and

$$\frac{1}{q_1(x)} + \frac{1}{q_2(x)} + \cdots + \frac{1}{q_{\widehat{n}}(x)} = 1,$$

for all $\varphi_i(x) \in L^{q_i(x)}(\Omega)$, there exists

$$\left|\int_\Omega \varphi_1(x)\varphi_2(x)\cdots\varphi_{\widehat{n}}(x)\,dx\right| \le \left(\frac{1}{q_1^-} + \frac{1}{q_2^-} + \cdots \frac{1}{q_{\widehat{n}}^-}\right)\|\varphi_1\|_{q_1(x)}\|\varphi_2\|_{q_2(x)}\cdots\|\varphi_{\widehat{n}}\|_{q_{\widehat{n}}(x)}.$$

Lemma 2.1 ([32]) *Let* $\rho_{q(x)}$ *be the modular of the* $L^{q(x)}(\Omega)$ *space, and* $\rho_{q(x)} : L^{q(x)}(\Omega) \to \mathbb{R}$ *defined by* $\rho_{q(x)}(\varphi) = \int_\Omega |\varphi(x)|^{q(x)}\,dx$. *Then, the following properties hold*:

(i) $\|\varphi\|_{q(x)} < 1(=1,>1) \Longleftrightarrow \rho_{q(x)}(\varphi) < 1(=1,>1)$;
(ii) $\|\varphi\|_{q(x)} > 1 \Longrightarrow \|\varphi\|_{q(x)}^{q^-} \le \rho_{q(x)}(\varphi) \le \|\varphi\|_{q(x)}^{q^+}$;
(iii) $\|\varphi\|_{q(x)} < 1 \Longrightarrow \|\varphi\|_{q(x)}^{q^+} \le \rho_{q(x)}(\varphi) \le \|\varphi\|_{q(x)}^{q^-}$.

Lemma 2.2 ([32]) *If $\varphi,\varphi_n \in L^{q(x)}(\Omega)$ with $n\in\mathbb{N}$, then*
(i) $\lim_{n\to+\infty}\|\varphi_n-\varphi\|_{q(x)}=0$;
(ii) $\lim_{n\to+\infty}\rho_{q(x)}(\varphi_n-\varphi)=0$;
(iii) $\varphi_n(x)\to\varphi(x)$ *a. e. in* Ω *and* $\lim_{n\to+\infty}\rho_{q(x)}(\varphi_n)=\rho_{q(x)}(\varphi)$.

Lemma 2.3 ([33]) *Let $p(x)$, $q(x)$ be measurable functions such that $p(x)\in L^\infty(\mathbb{R}^N)$ and $1<p(x)q(x)<\infty$, for any $x\in\mathbb{R}^N$. Then, there is*

$$\min\left\{\|\varphi\|_{p(x)q(x)}^{p^+},\|\varphi\|_{p(x)q(x)}^{p^-}\right\} \le \left\| |\varphi|^{p(x)}\right\|_{q(x)} \le \max\left\{\|\varphi\|_{p(x)q(x)}^{p^+},\|\varphi\|_{p(x)q(x)}^{p^-}\right\}$$

with $\varphi\in L^{q(x)}(\mathbb{R}^N)$, $\varphi\neq 0$.

3 The basic properties of functionals and operators

In this section, we state some properties of functionals and operators, and give the definition of weak solutions of equations (1) with Robin boundary conditions. We first introduce the workspaces $(W,\|\cdot\|_W)$ and $(X,\|\cdot\|_X)$ associated with equations (1).

The fractional variable Sobolev space $W := W^{s,q(x),p(x,\cdot)}(\Omega)$ is given by

$$W^{s,q(x),p(x,\cdot)}(\Omega) := \left\{\varphi:\Omega\to\mathbb{R}|\varphi\in L^{q(x)}(\Omega),\right.$$
$$\left.\int_{\Omega\times\Omega}\frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy<\infty,\text{for some }\mu>0\right\}.$$

Set

$$[\varphi]_{s,p(x,\cdot)} = \inf\left\{\mu>0:\int_{\Omega\times\Omega}\frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy<1\right\}$$

as the variable exponent Gagliardo seminorm. W is a Banach space with the norm

$$\|\varphi\|_W = \|\varphi\|_{W^{s,q(x),p(x,\cdot)}} = \|\varphi\|_{q(x)} + [\varphi]_{s,p(x,\cdot)}.$$

We take into account three continuous functions $p(x,y):\overline{\Omega}\times\overline{\Omega}\to(1,\infty)$ and $r_1(x)$, $r_2(x)\in C_+(\overline{\Omega})$. From condition (P), we know that

$$\begin{aligned}
&p(x,y)=p(y,x), \quad \text{for all } x,y\in\overline{\Omega};\\
&1<p^- := \min p(x,y)\le p(x,y)\le p^+ := \max p(x,y)<\infty;\\
&1<r_1^- := \min r_1(x)\le r_1(x)\le r_1^+ := \max r_1(x)<\infty;\\
&1<r_2^- := \min r_2(x)\le r_2(x)\le r_2^+ := \max r_2(x)<\infty.
\end{aligned}\tag{8}$$

Lemma 3.1 ([34]) *Suppose that $\Omega\subset\mathbb{R}^N$ is a bounded open domain and (8) holds. Then, W is a separable and reflexive space.*

Lemma 3.2 ([35]) *Let smooth bounded domain $\Omega \subset \mathbb{R}^N$, $sp(x,y) < N$ for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ with $s \in (0,1)$, and $q(x) \geq p(x,x)$ for $x \in \overline{\Omega}$. Suppose that continuous function $\widehat{h}(x) : \overline{\Omega} \to (1,\infty)$ satisfies*

$$p_s^*(x) \geq \widehat{h}(x) \geq \widehat{h}^- = \inf_{x\in\Omega} \widehat{h}(x) > 1, \quad \text{for all } x \in \overline{\Omega}.$$

There exists a positive constant $C_0 = C_0(N,s,p,\widehat{h},\Omega)$ such that for every $\varphi \in W$, it holds that

$$\|\varphi\|_{\widehat{h}(x)} \leq C_0 \|\varphi\|_W.$$

Then, the embedding $W \hookrightarrow L^{\widehat{h}(x)}$ for all $\widehat{h} \in (1, p_s^*)$ is compact.

Lemma 3.3 ([35, 36]) *If $1 < sp^-$ and*

$$p_\partial^*(x) := \frac{(N-1)p(x,x)}{N - sp(x,x)} > \widehat{h}(x), \quad \text{in } \partial\Omega \cap \left\{x \in \overline{\Omega} : N - sp(x,x) > 0\right\}.$$

There exists a positive constant $C_1 = C_1(N,s,p,\widehat{h},\partial\Omega)$ such that

$$\|\varphi\|_{L^{\widehat{h}(x)}(\partial\Omega)} \leq C_1 \|\varphi\|_{s,\overline{p}(x),p(x,y)}, \quad \text{for all } \varphi \in W^{s,\overline{p}(x),p(x,y)}(\Omega).$$

Then, the embedding $W^{s,\overline{p}(x),p(x,y)}(\Omega) \hookrightarrow L^{\widehat{h}(x)}(\partial\Omega)$ is compact.

Define nonlinear map $\mathcal{L} : W \to W^*$

$$\left\langle \mathcal{L}(\varphi), \psi \right\rangle = \int_\Omega \frac{|\varphi(x) - \varphi(y)|^{p(x,y)-2}(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp(x,y)}}\, dy, \tag{9}$$

for all $\varphi, \psi \in W$, $\mathcal{L}$ has the following properties.

Lemma 3.4 ([28])

(i) *$\mathcal{L}$ is a bounded and strictly monotone operator;*
(ii) *$\mathcal{L}$ is a mapping of (S_+), i.e., if $\varphi_n \rightharpoonup \varphi$ in W and $\lim_{n\to\infty} \sup\langle \mathcal{L}(\varphi_n) - \mathcal{L}(\varphi), \varphi_n - \varphi\rangle \leq 0$, then $\varphi_n \to \varphi$ in W;*
(iii) *$\mathcal{L} : W \to W^*$ is a homeomorphism.*

Define function $S : W \to \mathbb{R}$

$$S(\varphi) = \int_{\Omega\times\Omega} \frac{|\varphi(x) - \varphi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}\, dx\, dy, \quad \text{for all } \varphi \in W,$$

which is related to (9). The derivative of S is

$$\left\langle S'(\varphi), \psi \right\rangle = \int_\Omega \frac{|\varphi(x) - \varphi(y)|^{p(x,y)-2}(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x-y|^{N+sp(x,y)}}\, dy = \left\langle \mathcal{L}(\varphi), \psi \right\rangle,$$

for all $\varphi, \psi \in W$; for more details, see [34].

Let

$$X_i = \left\{\varphi \in W : \int_\Omega \frac{V_i(x)|\varphi|^{\overline{p}_i(x)}}{\mu^{\overline{p}_i(x)}}\,dx < +\infty, \text{for some } \mu > 0\right\},$$

equipped with the norm

$$\|\varphi\|_{X_i} := [\varphi]_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2} + [\varphi]_{V_i(\Omega)} + \left\|\beta^{\frac{1}{\overline{p}_i(x)}}\varphi\right\|_{L^{\overline{p}_i(x)}(C\Omega)} + \left\||g|^{\frac{1}{\overline{p}_i(x)}}\varphi\right\|_{L^{\overline{p}_i(x)}(C\Omega)},$$

where

$$[\varphi]_{V_i(\Omega)} = \inf\left\{\mu > 0 : \int_\Omega V_i(x)\frac{|\varphi(x)|^{\overline{p}_i(x)}}{\mu^{\overline{p}_i(x)}}\,dx < 1\right\},$$

and

$$[\varphi]_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2} = \inf\left\{\mu \geq 0 : \frac{1}{2}\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} \frac{|\varphi(x)-\varphi(y)|^{p_i(x,y)}}{\mu^{p_i(x,y)}|x-y|^{N+sp_i(x,y)}}\,dx\,dy \leq 1\right\}$$

with $\Omega^c = \mathbb{R}^N\setminus\Omega$.

Lemma 3.5 ([28]) *Assume that assumptions* (P), (G), *and* (V) *hold. Then,* $(X_i, \|\cdot\|_{X_i})$ *is a reflexive Banach space.*

The norm $\|\cdot\|_{X_i}$ on X_i is equivalent to

$$\begin{aligned}
\|\varphi\|_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2} &= \inf\left\{\mu \geq 0 | \rho_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}\left(\frac{\varphi}{\mu} \leq 1\right)\right\} \\
&= \inf\left\{\mu \geq 0 | \int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} \frac{|\varphi(x)-\varphi(y)|^{p_i(x,y)}}{\mu^{p_i(x,y)}p_i(x,y)|x-y|^{N+sp_i(x,y)}}\,dx\,dy\right. \\
&\quad + \int_\Omega \frac{V_i(x)}{\mu^{\overline{p}_i(x)}\overline{p}_i(x)}|\varphi|^{\overline{p}_i(x)}\,dx + \int_{C\Omega} \frac{\beta(x)}{\mu^{\overline{p}_i(x)}\overline{p}_i(x)}|\varphi|^{\overline{p}_i(x)}\,dx \\
&\quad \left. + \int_{C\Omega} \frac{g(x)}{\mu^{\overline{p}_i(x)}\overline{p}_i(x)}|\varphi|^{\overline{p}_i x)}\,dx \leq 1\right\},
\end{aligned}$$

where the modular $\rho_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2} : X_i \to \mathbb{R}$ is defined by

$$\begin{aligned}
\rho_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2} = &\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} \frac{|\varphi(x)-\varphi(y)|^{p_i(x,y)}}{p_i(x,y)|x-y|^{N+sp_i(x,y)}}\,dx\,dy + \int_\Omega \frac{V_i(x)}{\overline{p}_i(x)}|\varphi|^{\overline{p}_i(x)}\,dx \\
&+ \int_{C\Omega} \frac{\beta(x)}{\overline{p}_i(x)}|\varphi|^{\overline{p}_i(x)}\,dx + \int_{C\Omega} \frac{g(x)}{\overline{p}_i(x)}|\varphi|^{\overline{p}_i(x)}\,dx.
\end{aligned} \tag{10}$$

Lemma 3.6 *Assume that assumptions* (P), (G), *and* (V) *hold. The following properties hold*:

(i) $\|\varphi\|_{X_i} < 1(= 1, > 1) \Longleftrightarrow \rho_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}(\varphi) < 1(= 1, > 1)$;

(ii) $\|\varphi\|_{X_i} \geq 1 \Longrightarrow \|\varphi\|_{X_i}^{p_i^-} \leq \rho_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}(\varphi) \leq \|\varphi\|_{X_i}^{p_i^+}$;

(iii) $\|\varphi\|_{X_i} \leq 1 \Longrightarrow \|\varphi\|_{X_i}^{p_i^+} \leq \rho_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}(\varphi) \leq \|\varphi\|_{X_i}^{p_i^-}$;

(iv) $\rho_{s,p_i(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}(\varphi - \nu) \to 0 \Leftrightarrow \|\varphi - \nu\|_{X_i} \to 0$.

Let $X = X_1 \cap X_2$ with norm $\|\varphi\|_X = \|\varphi\|_{X_1} + \|\varphi\|_{X_2}$, which is a separable and reflexive Banach space. The dual space of X is X^*. The modular $\rho_{s,p(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2} = \rho_{s,p_1(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2} + \rho_{s,p_2(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}$. We have the following result.

Lemma 3.7 ([28]) *Assume that assumptions* (P), (G), *and* (V) *hold. Then, from* (10), *the following properties hold*:

(i) *The function $\rho_{s,p(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2)}$ is of class $C^1(X,\mathbb{R})$*;

(ii) *The strictly monotone operator $\rho'_{s,p(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2} : X \to X^*$ is coercive, then*

$$\frac{\langle \rho'_{s,p(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}, \varphi\rangle_X}{\|\varphi\|_X} \to +\infty, \qquad \|\varphi\|_X \to +\infty;$$

(iii) *$\rho'_{s,p(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}$ is a mapping of type (S_+), that is, if $\varphi_n \rightharpoonup \varphi$ in X and $\limsup_{n\to+\infty} \langle \rho'_{s,p(x,\cdot),\mathbb{R}^{2N}\setminus(\Omega^c)^2}, \varphi\rangle_X \le 0$, then $\varphi_n \to \varphi$ in X.*

Lemma 3.8 ([35, 36]) *Assume that assumptions* (P), (G), (V), *and* (H) *hold. Then, for any $\widehat{r} \in C_+(\overline{\Omega})$ with $1 < \widehat{r}(x) < p_s^*(x)$ for all $x \in \Omega$, there is a positive constant $\varpi^* = \varpi^*(s,p_i,N,\widehat{r},\Omega) > 0$ such that*

$$\|\varphi\|_{L^{\widehat{r}(x)}(\Omega)} \le \varpi^* \|\varphi\|_X, \quad \textit{for all } \varphi \in X.$$

Moreover, this embedding is compact.

Lemma 3.9 ([35]) *Assume that assumptions* (P), (G), (V), *and* (H) *hold. Then, for any $\widehat{r} \in C_+(\mathbb{R}^N\setminus\Omega)$ with $1 < \widehat{r}(x) < p_\partial^*(x)$ for all $x \in \mathbb{R}^N\setminus\Omega$, there is a positive constant $\widehat{\varpi}^* = \widehat{\varpi}^*(s,p_i,N,\widehat{r},\partial\Omega) > 0$ such that*

$$\|\varphi\|_{L^{\widehat{r}(x)}(\mathbb{R}^N\setminus\Omega)} \le \widehat{\varpi}^* \|\varphi\|_X, \quad \textit{for all } \varphi \in X.$$

Moreover, this embedding is compact.

More precisely, we now present the divergence theorem and the analogous formula for the partition integral formula in nonlocal case [37].

Lemma 3.10 ([29]) *Let the hypotheses* (P) *hold, and let φ be any bounded C^2-function in R^N. Then,*

$$\begin{aligned}&\int_\Omega (-\Delta)^s_{p_1(x,\cdot)}\varphi(x)\,dx + \int_\Omega (-\Delta)^s_{p_2(x,\cdot)}\varphi(x)\,dx \\ &\quad = -\left(\int_{\mathbb{R}^N\setminus\Omega} \mathcal{N}_{s,p_1(x,\cdot)}\varphi(x)\,dx + \int_{\mathbb{R}^N\setminus\Omega} \mathcal{N}_{s,p_2(x,\cdot)}\varphi(x)\,dx\right).\end{aligned}$$

Lemma 3.11 *Let the hypotheses* (P) *hold. Suppose that φ and ν are bounded C^2-functions in $\mathbb{R}^N$. Then,*

$$\frac{1}{2}\left(\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \left|\varphi(x)-\varphi(y)\right|^{p_1(x,y)-2} \frac{(\varphi(x)-\varphi(y))(\nu(x)-\nu(y))}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy\right.$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\big|\varphi(x)-\varphi(y)\big|^{p_2(x,y)-2}\frac{(\varphi(x)-\varphi(y))(\nu(x)-\nu(y))}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy\Big)\\
&=\int_\Omega \nu(-\Delta)^s_{p_1(x,\cdot)}\varphi(x)\,dx+\int_\Omega \nu(-\Delta)^s_{p_2(x,\cdot)}\varphi(x)\,dx\\
&\quad+\int_{C\Omega}\nu\mathcal{N}_{s,p_1(x,\cdot)}\varphi(x)\,dx+\int_{C\Omega}\nu\mathcal{N}_{s,p_2(x,\cdot)}\varphi(x)\,dx,
\end{aligned}
$$

for every $\nu \in X$.

Proof According to symmetry, we obtain

$$
\begin{aligned}
&\frac{1}{2}\Big(\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\big|\varphi(x)-\varphi(y)\big|^{p_1(x,y)-2}\frac{(\varphi(x)-\varphi(y))(\nu(x)-\nu(y))}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy\\
&\quad+\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\big|\varphi(x)-\varphi(y)\big|^{p_2(x,y)-2}\frac{(\varphi(x)-\varphi(y))(\nu(x)-\nu(y))}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy\Big)\\
&=\int_\Omega\int_{\mathbb{R}^N}\nu(x)\big|\varphi(x)-\varphi(y)\big|^{p_1(x,y)-2}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+sp_1(x,y)}}\,dy\,dx\\
&\quad+\int_\Omega\int_{\mathbb{R}^N}\nu(x)\big|\varphi(x)-\varphi(y)\big|^{p_2(x,y)-2}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+sp_2(x,y)}}\,dy\,dx\\
&\quad+\int_{C\Omega}\int_\Omega\nu(x)\big|\varphi(x)-\varphi(y)\big|^{p_1(x,y)-2}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+sp_1(x,y)}}\,dy\,dx\\
&\quad+\int_{C\Omega}\int_\Omega\nu(x)\big|\varphi(x)-\varphi(y)\big|^{p_2(x,y)-2}\frac{\varphi(x)-\varphi(y)}{|x-y|^{N+sp_2(x,y)}}\,dy\,dx.
\end{aligned}
$$

□

Lemma 3.12 *Assuming that assumption* (P) *holds and letting φ be a weak solutions of equations* (1), *we have*

$$
\begin{aligned}
&\mathcal{N}_{s,p_1(x,\cdot)}\varphi+\mathcal{N}_{s,p_2(x,\cdot)}\varphi+\beta(x)|\varphi|^{\bar{p}_1(x)-2}\varphi+\beta(x)|\varphi|^{\bar{p}_2(x)-2}\varphi\\
&=g_1(x)+g_2(x),\quad a.e.\ in\ \mathbb{R}^N\setminus\overline{\Omega}.
\end{aligned}
$$

Lemma 3.13 *Assuming that assumptions* (P), (G), (V), *and* (H) *hold, let $I_\lambda : X \to \mathbb{R}$ be a energy function defined by*

$$
\begin{aligned}
I_\lambda(\varphi)&=\frac{1}{2}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\frac{|\varphi(x)-\varphi(y)|^{p_1(x,y)}}{p_1(x,y)|x-y|^{N+sp_1(x,y)}}\,dx\,dy\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\frac{|\varphi(x)-\varphi(y)|^{p_2(x,y)}}{p_2(x,y)|x-y|^{N+sp_2(x,y)}}\,dx\,dy\\
&\quad+\int_\Omega V_1(x)\frac{|\varphi|^{\bar{p}_1(x)}}{\bar{p}_1(x)}\,dx+\int_\Omega V_2(x)\frac{|\varphi|^{\bar{p}_2(x)}}{\bar{p}_2(x)}\,dx+\int_{\mathbb{R}^N\setminus\Omega}\beta(x)\frac{|\varphi|^{\bar{p}_1(x)}}{\bar{p}_1(x)}\,dx\\
&\quad+\int_{\mathbb{R}^N\setminus\Omega}\beta(x)\frac{|\varphi|^{\bar{p}_2(x)}}{\bar{p}_2(x)}\,dx-\lambda_1\int_\Omega A_1(x)\frac{|\varphi|^{r_1(x)}}{r_1(x)}\,dx-\lambda_2\int_\Omega A_2(x)\frac{|\varphi|^{r_2(x)}}{r_2(x)}\,dx\\
&\quad-\int_{\mathbb{R}^N\setminus\Omega}g_1(x)\varphi\,dx-\int_{\mathbb{R}^N\setminus\Omega}g_2(x)\varphi\,dx,
\end{aligned}
$$

for every $\varphi \in X$. Then, any critical point of I_λ is a weak solution of equations (1).

4 Proof of Theorem 1.1

To prove Theorem 1.1, we need a well-known mountain pass lemma.

Theorem 4.1 *Let X be a real Banach space and $I_\lambda \in C^1(X,\mathbb{R})$ with $I_\lambda(0) = 0$. Assume that the following conditions hold*:

(i) *I_λ satisfies (PS) conditions*;

(ii) *there exist $\rho, \sigma > 0$ such that $I_\lambda(\varphi) \geq \sigma$, for all $\varphi \in X$, with $\|\varphi\|_X = \rho$*;

(iii) *there exists $\upsilon \in X$, satisfying $\|\upsilon\|_X > \rho$ such that $I_\lambda(\upsilon) < 0$.*

Then, I_λ has a critical value $c > \sigma$, that is,

$$c = \inf_{\gamma\in\Upsilon} \max_{0\leq t\leq 1} I_\lambda\big(\gamma(t)\big),$$

where $\Upsilon = \{\gamma \in C^1([0,1];X) : \gamma(0) = 1, \gamma(1) = \upsilon\}$.

Definition 1 Let X be a Banach space, $I_\lambda \in C^1(E,\mathbb{R})$. We say that I_λ satisfies the (PS) conditions if every sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset X$ satisfying

$$I_\lambda(\varphi_n) \to c, \qquad I_\lambda(\varphi_n) \to 0, \quad n \to \infty$$

has a convergent subsequence in X.

Next, we prove that the I_λ defined in Lemma 3.13 satisfies the (PS) conditions.

Lemma 4.1 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. Then, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in X.*

Proof According to (H), we get

$$s_1'(x)r_1(x) < p_s^*(x), \qquad s_2'(x)r_2(x) < p_s^*(x), \quad \text{for all } x \in \Omega,$$

so from Lemmas 3.8 and 3.9, there exist constants M_1 and M_2 such that

$$\|\varphi\|_{L^{s_1'(x)r_1(x)}(\Omega)} \leq M_1\|\varphi\|_X, \qquad \|\varphi\|_{L^{s_2'(x)r_2(x)}(\Omega)} \leq M_2\|\varphi\|_X, \quad \text{for all } \varphi \in X. \tag{11}$$

Let $\rho > \max\{1, \frac{1}{M_1}, \frac{1}{M_2}\}$ and

$$\|\varphi\|_{L^{s_1'(x)r_1(x)}(\Omega)} > 1, \qquad \|\varphi\|_{L^{s_2'(x)r_2(x)}(\Omega)} > 1.$$

Thus, by the Hölder inequality and Lemma 2.3, for all $\varphi \in X$ with $\|\varphi\|_X = \rho$, we obtain

$$\int_\Omega A_1(x)\big|\varphi(x)\big|^{r_1(x)}\,dx \leq 2\|A_1\|_{s_1(x)}\|\varphi\|^{r_1^+}_{s_1'(x)r_1(x)} \leq 2M_1^{r_1^+}\|A_1\|_{s_1(x)}\|\varphi\|_X^{r_1^+} \tag{12}$$

and

$$\int_\Omega A_2(x)\big|\varphi(x)\big|^{r_2(x)}\,dx \leq 2\|A_2\|_{s_2(x)}\|\varphi\|^{r_2^+}_{s_2'(x)r_2(x)} \leq 2M_2^{r_2^+}\|A_2\|_{s_2(x)}\|\varphi\|_X^{r_2^+}. \tag{13}$$

We use the counterfactual method. Suppose $\|\varphi_n\|_X \to \infty$, $n \to \infty$. Combining conditions (P), (G), (V), (H), and Lemma 3.8 and letting $0 < \vartheta < \min\{r_1^- r_2^- \frac{Z_A + Z_B - 1}{r_2^- Z_A K_A + r_1^- Z_B K_A}, r_1^-, r_2^-\}$, where $K_A = \max\{\|A_1\|_X, \|A_2\|_X\}$, $Z_A = 4\lambda_1 M_1^{r_1^+} C_{A_1}$, $Z_B = 4\lambda_2 M_2^{r_2^+} C_{A_2}$, we have

$$
\begin{aligned}
& c\frac{1}{\|\varphi_n\|_X^{p_1^-}} + o(1)\frac{1}{\|\varphi_n\|_X^{p_1^-}} \\
& \geq \frac{1}{\|\varphi_n\|_X^{p_1^-}}\left(I_\lambda(\varphi_n) - \left\langle \frac{1}{\vartheta} I_\lambda'(\varphi_n), \varphi_n \right\rangle\right) \\
& = \frac{1}{\|\varphi_n\|_X^{p_1^-}} \sum_{i=1}^{2} \left[\int_{\mathbb{R}^{2N}\setminus(\mathcal{C}\Omega)^2} \frac{|\varphi(x) - \varphi(y)|^{p_i(x,y)}}{2p_i(x,y)|x-y|^{N+sp_i(x,y)}}\,dx\,dy + \int_\Omega V_i(x)\frac{|\varphi_n|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx \right. \\
& \quad + \int_{\mathbb{R}^N\setminus\Omega} \beta(x)\frac{|\varphi_n|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx - \lambda_1 \int_\Omega A_1(x)\frac{|\varphi_n|^{r_1(x)}}{r_1(x)}\,dx - \lambda_2 \int_\Omega A_2(x)\frac{|\varphi_n|^{r_2(x)}}{r_2(x)}\,dx \\
& \quad - \int_{\mathbb{R}^N\setminus\Omega} g_i(x)\varphi_n\,dx - \frac{1}{\vartheta}\left(\int_{\mathbb{R}^{2N}\setminus(\mathcal{C}\Omega)^2} \frac{|\varphi_n(x) - \varphi_n(y)|^{p_i(x,y)}}{2|x-y|^{N+sp_i(x,y)}}\,dx\,dy \right. \\
& \quad + \int_\Omega V_i(x)|\varphi_n|^{\bar{p}_i(x)}\,dx + \int_{\mathbb{R}^N\setminus\Omega} \beta(x)|\varphi_n|^{\bar{p}_i(x)}\,dx - \lambda_1 \int_\Omega A_1(x)|\varphi_n|^{r_1(x)}\,dx \\
& \quad \left.\left. - \lambda_2 \int_\Omega A_2(x)|\varphi_n|^{r_2(x)}\,dx - \int_{\mathbb{R}^N\setminus\Omega} g_i(x)\varphi_n\,dx \right)\right] \\
& \geq \frac{1}{\|\varphi_n\|_X^{p_1^-}} \left[\left(\frac{1}{2p_2^+} - \frac{1}{2\vartheta}\right) \sum_{i=1}^{2} \left(\int_{\mathbb{R}^{2N}\setminus(\mathcal{C}\Omega)^2} \frac{|\varphi_n(x) - \varphi_n(y)|^{p_i(x,y)}}{|x-y|^{N+sp_i(x,y)}}\,dx\,dy \right.\right. \\
& \quad \left. + \int_\Omega V_i(x)|\varphi_n|^{\bar{p}_i(x)}\,dx + \int_{\mathbb{R}^N\setminus\Omega} \beta(x)|\varphi_n|^{\bar{p}_i(x)}\,dx - \int_{\mathbb{R}^N\setminus\Omega} g_i(x)\varphi_n\,dx \right) \\
& \quad \left. - 2\left(\vartheta - r_1^+\right)\frac{\lambda_1 M_1^{r_1^+}}{\vartheta r_1^-}\|A_1\|_{s_1(x)}\|\varphi_n\|_X^{r_1^+} - 2\left(\vartheta - r_2^+\right)\frac{\lambda_2 M_2^{r_2^+}}{\vartheta r_2^-}\|A_2\|_{s_2(x)}\|\varphi_n\|_X^{r_2^+} \right] \\
& \geq \left(\frac{1}{2p_2^+} - \frac{1}{2\vartheta}\right) - 2\left(\vartheta - r_1^-\right)\frac{\lambda_1 M_1^{r_1^+} C_{A_1}}{\vartheta r_1^-}\|A_1\|_X - 2\left(\vartheta - r_2^-\right)\frac{\lambda_2 M_2^{r_2^+} C_{A_2}}{\vartheta r_2^-}\|A_2\|_X.
\end{aligned}
$$

In addition, we obtain $c\frac{1}{\|\varphi_n\|_X^{p_1^-}} + o(1)\frac{1}{\|\varphi_n\|_X^{p_1^-}} \to 0$ because $\|\varphi_n\|_X \to \infty$, $n \to \infty$. Due to

$$
0 < \vartheta < \min\left\{ r_1^- r_2^- \frac{Z_A + Z_B - 1}{r_2^- Z_A K_A + r_1^- Z_B K_A}, r_1^-, r_2^- \right\},
$$

there is a contradiction. Thus, $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded. □

Inspired by [15], we have the following lemma.

Lemma 4.2 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. Then,* I_λ *satisfies the* (PS) *conditions.*

Proof According to Lemma 4.1, $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded, that is, there is a subsequence $\{\varphi_n\}_{n\in\mathbb{N}}$ and φ_0 in X such that

$$
\varphi_n \rightharpoonup \varphi_0 \quad \text{in } X;
$$

$$\begin{aligned}
&\varphi_n \to \varphi_0 \quad \text{a.e. in } \Omega;\\
&\varphi_n \to \varphi_0 \quad \text{in } L^{\widehat{r}(x)}(\Omega), \widehat{r}(x) < p_s^*(x);\\
&\varphi_n \to \varphi_0 \quad \text{in } L^{\widehat{r}(x)}(\mathbb{R}^N \setminus \Omega), \widehat{r}(x) < p_\partial^*(x).
\end{aligned}$$

Due to $\varphi_n \to \varphi_0$ in $L^{\widehat{r}(x)}(\mathbb{R}^N \setminus \Omega)$, then $|\varphi_n|^{\bar{p}_i(x)-2}\varphi_n \to |\varphi_0|^{\bar{p}_i(x)-2}\varphi_0$. We get

$$\lim_{n\to\infty} \int_{\mathbb{R}\setminus\Omega} \beta(x)\big(|\varphi_n|^{\bar{p}_i(x)-2}\varphi_n - |\varphi_0|^{\bar{p}_i(x)-2}\varphi_0\big) = 0.$$

Since $\lambda_1 A_1(x)|\varphi|^{r_1(x)-2}\varphi$ and $\lambda_2 A_2(x)|\varphi|^{r_2(x)-2}\varphi$ in X are sequentially weakly lower semicontinuous, for $v \in X$ and measurable for all $\Omega \subset \mathbb{R}^N$, we obtain

$$\begin{aligned}
&\left|\int_\Omega A_1(x)\big(|\varphi|_n^{r_1(x)-2}\varphi_n - |\varphi|_0^{r_1(x)-2}\varphi_0\big)\right| v|\,dx|\\
&\quad\le \int_\Omega A_1(x)\big(|\varphi|_n^{r_1(x)-1} - |\varphi|_0^{r_1(x)-1}\big) v\,dx\\
&\quad= \int_\Omega A_1(x)^{\frac{r_1(x)-1}{r_1(x)}}\big(|\varphi|_n^{r_1(x)-1} - |\varphi|_0^{r_1(x)-1}\big) A_1(x)^{\frac{1}{r_1(x)}}|v|\,dx\\
&\quad\le \big\|A_1(x)^{\frac{r_1(x)-1}{r_1(x)}}\big(|\varphi|_n^{r_1(x)-1} - |\varphi|_0^{r_1(x)-1}\big\|_{L^{\frac{r_1(x)}{r_1(x)-1}}}\big\|A_1(x)^{\frac{1}{r_1(x)}}\big)|v|\big\|_{L^{r_1(x)}}.
\end{aligned}$$

Hence, $\{A_1(x)(|\varphi|_n^{r_1(x)-2}\varphi_n - |\varphi|_0^{r_1(x)-2}\varphi_0)|v|\}_{n\in\mathbb{N}}$ is uniformly integrable in $\mathbb{R}^N$. Then, using the Vitali convergence theorem, we get

$$\lim_{n\to\infty} \int_\Omega A_1(x)\big(|\varphi_n|^{r_1(x)-2}\varphi_n - |\varphi_0|^{r_1(x)-2}\varphi_0\big) = 0.$$

Similarly, there is

$$\lim_{n\to\infty} \int_\Omega A_2(x)\big(|\varphi_n|^{r_2(x)-2}\varphi_n - |\varphi_0|^{r_2(x)-2}\varphi_0\big) = 0.$$

We need to prove that $\{\varphi_n\}_{n\in\mathbb{N}}$ is strongly convergent,

$$\begin{aligned}
o(1) &= \big\langle I'_\lambda(\varphi_n) - I'_\lambda(\varphi_0), \varphi_n - \varphi_0\big\rangle\\
&= \sum_{i=1}^2 \Bigg[\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|\varphi_n(x)-\varphi_n(y)|^{p_i(x,y)-2}(\varphi_n(x)-\varphi_n(y))(\varphi_n(x)-\varphi_n(y)-\varphi_0(x)+\varphi_0(y))}{2|x-y|^{N+sp_i(x,y)}}\,dx\,dy\\
&\quad - \int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|\varphi_0(x)-\varphi_0(y)|^{p_i(x,y)-2}(\varphi_0(x)-\varphi_0(y))(\varphi_n(x)-\varphi_n(y)-\varphi_0(x)+\varphi_0(y))}{2|x-y|^{N+sp_i(x,y)}}\,dx\,dy\\
&\quad + \int_\Omega V_i(x)\big(|\varphi_n|^{\bar{p}_i(x)-2}\varphi_n - |\varphi_0|^{\bar{p}_i(x)-2}\varphi_0\big)(\varphi_n - \varphi_0)\,dx\\
&\quad + \int_{\mathbb{R}^N\setminus\Omega} \beta(x)\big(|\varphi_n|^{\bar{p}_i(x)-2}\varphi_n - |\varphi_0|^{\bar{p}_i(x)-2}\varphi_0\big)(\varphi_n - \varphi_0)\,dx\\
&\quad - \lambda_1 \int_\Omega A_1(x)\big(|\varphi_n|^{\bar{p}_i(x)-2}\varphi_n - |\varphi_0|^{\bar{p}_i(x)-2}\varphi_0\big)(\varphi_n - \varphi_0)\,dx\\
&\quad - \lambda_2 \int_\Omega A_2(x)\big(|\varphi_n|^{\bar{p}_i(x)-2}\varphi_n - |\varphi_0|^{\bar{p}_i(x)-2}\varphi_0\big)(\varphi_n - \varphi_0)\,dx\Bigg]
\end{aligned}$$

$$= \sum_{i=1}^{2}\Bigg[\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|\varphi_n(x)-\varphi_n(y)|^{p_i(x,y)-2}(\varphi_n(x)-\varphi_n(y))(\varphi_n(x)-\varphi_n(y)-\varphi_0(x)+\varphi_0(y))}{2|x-y|^{N+sp_i(x,y)}}\,dx\,dy$$
$$-\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|\varphi_0(x)-\varphi_0(y)|^{p_i(x,y)-2}(\varphi_0(x)-\varphi_0(y))(\varphi_n(x)-\varphi_n(y)-\varphi_0(x)+\varphi_0(y))}{2|x-y|^{N+sp_i(x,y)}}\,dx\,dy\Bigg].$$

A discussion similar to Lemma 3.7 gives that $\varphi_n \to \varphi_0$ in X. Combining the Definition 1 and the Lemma 4.1, we complete the proof. □

Lemma 4.3 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. There exist* $\rho > 0$ *and* $\sigma > 0$ *such that, for all* $\varphi \in X$ *with* $\|\varphi\|_X = \rho$,

$$I_\lambda(\varphi) \geq \sigma > 0$$

holds.

Proof Combining (12) with (13), for any $\varphi \in X$ with $\|\varphi\|_X = \rho > 1$, we have

$$\begin{aligned} I_\lambda(\varphi) \geq{}& \frac{1}{p_2^+}\sum_{i=1}^{2}\bigg(\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2}\frac{|\varphi(x)-\varphi(y)|^{p_i(x,y)}}{2|x-y|^{N+sp_i(x,y)}}\,dx\,dy + \int_\Omega V_i(x)|\varphi|^{\bar{p}_i(x)}\,dx \\ &+ \int_{\mathbb{R}^N\setminus\Omega}\beta(x)|\varphi|^{\bar{p}_i(x)}\,dx - \int_{\mathbb{R}^N\setminus\Omega} g_i(x)\varphi\,dx\bigg) \\ &- \frac{\lambda_1}{r_1^-}\int_\Omega A_1(x)|\varphi|^{r_1(x)}\,dx - \frac{\lambda_2}{r_2^-}\int_\Omega A_2(x)|\varphi|^{r_2(x)}\,dx \\ \geq{}& \frac{1}{p_2^+}\|\varphi\|_X^{p_2^+} - \lambda_1\frac{2M_1^{r_1^-}\|A_1\|_{s_1(x)}}{r_1^-}\|\varphi\|_X^{r_1^-} - \lambda_2\frac{2M_2^{r_2^-}\|A_2\|_{s_2(x)}}{r_2^-}\|\varphi\|_X^{r_2^-} \\ ={}& \|\varphi\|_X^{r_1^-}\bigg(\frac{1}{p_2^+}\|\varphi\|_X^{p_2^+-r_1^-} - \lambda_2\frac{2M_2^{r_2^-}\|A_2\|_{s_2(x)}}{r_2^-}\|\varphi\|_X^{r_2^--r_1^-} - \lambda_1\frac{2M_1^{r_1^-}\|A_1\|_{s_1(x)}}{r_1^-}\bigg). \end{aligned}$$

Let

$$f(t) = \frac{1}{p_2^+}t^{p_2^+-r_1^-} - \lambda_2\frac{2M_2^{r_2^-}\|A_2\|_{s_2(x)}}{r_2^-}t^{r_2^--r_1^-}, \quad t \geq 0,$$

where $r_1^- < r_2^- < p_2^+$. Then, there exists $\chi > 0$ such that $f(t) = f(\chi) > 0$. Choosing $\|A_1\|_{s_1(x)}^{r_1^-} < \sigma^* = \frac{r_1^-}{4\lambda_1 M_1^{r_1^-}}f(\chi)$, we get

$$I_\lambda(\varphi) \geq \sigma = \chi^{r_1^-}\frac{f(\chi)}{2} > 0,$$

for $\|\varphi\|_X = \chi = \rho$. □

Lemma 4.4 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. Then, there exists* υ, *which satisfies* $\|\upsilon\|_X > \rho$. *Then, there exists* $\upsilon \in X$ *such that*

$$I_\lambda(\upsilon) < 0.$$

Proof Choosing $\widehat{\upsilon} \in X$ such that $\|\widehat{\upsilon}\|_X = 1$, and for $t \in (0,1)$ small enough, we obtain

$$\begin{aligned}
I_\lambda(t\widehat{\upsilon}) &\le t^{p_2^+} \sum_{i=1}^{2} \biggl(\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|\widehat{\upsilon}(x)-\widehat{\upsilon}(y)|^{p_i(x,y)}}{2p_i(x,y)|x-y|^{N+sp_i(x,y)}}\,dx\,dy + \int_\Omega V_i(x)\frac{|\widehat{\upsilon}|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx \\
&\quad + \int_{\mathbb{R}^N\setminus\Omega} \frac{\beta(x)|\widehat{\upsilon}|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx \biggr) - \lambda_1 t^{r_1^-} \int_\Omega \frac{A_1(x)|\widehat{\upsilon}|^{r_1(x)}}{r_1(x)}\,dx \\
&\quad - \lambda_2 t^{r_2^-} \int_\Omega \frac{A_2(x)|\widehat{\upsilon}|^{r_2(x)}}{r_2(x)}\,dx - \int_{\mathbb{R}^N\setminus\Omega} t\widehat{\upsilon} g_1(x)\,dx - \int_{\mathbb{R}^N\setminus\Omega} t\widehat{\upsilon} g_2(x)\,dx < 0
\end{aligned}$$

with the fact that $1 < r_1^- < p_2^+$. Thus, $I_\lambda(t\widehat{\upsilon}) < 0$ with $\|t\widehat{\upsilon}\|_X > \rho$. The proof is proved by letting $\upsilon = t\widehat{\upsilon}$. □

Proof of Theorem 1.1 Combining Lemmas 4.1 and 4.2, it can be inferred that I_λ satisfies (PS) conditions. According to Lemmas 4.3 and 4.4, we know that I_λ satisfies the mountain pass lemma. Therefore, we have a subsequence $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\varphi_0^{(1)} \in X$ such that $\varphi_n \to \varphi_0^{(1)}$ in X by Lemma 4.1 and $0 < \sigma < c < \infty$. Therefore, $I_\lambda(\varphi_n) = c > \sigma$, that is, $\varphi_0^{(1)}$ is a solution of problem (1) with positive energy.

Next, we will apply Ekeland's variational principle to prove that (1) has a solution with negative energy.

By Lemma 4.3, we derive that

$$\widehat{c} = \inf_{\partial B_\rho(0)} I_\lambda > 0,$$

where ρ is the positive constant introduced in Lemma 4.3.

From condition (H), there exist $\epsilon_1, \epsilon_2 > 0$ and an open set $\Omega_0 \subset\subset \Omega$ such that

$$\bigl|r_1(x) - r_1{}^-\bigr| < \epsilon_1, \qquad \bigl|r_2(x) - r_2{}^-\bigr| < \epsilon_2, \quad \text{for all } x \in \Omega_0,$$

and we get

$$r_1{}^- + \epsilon_1 < p_1^-, \qquad r_2{}^- + \epsilon_2 < p_1^-, \quad \text{for all } x \in \overline{\Omega}_0.$$

Hence,

$$r_1(x) \le r_1{}^- + \epsilon_1 < p_1^-, \qquad r_2(x) \le r_2{}^- + \epsilon_2 < p_1^-, \quad \text{for all } x \in \overline{\Omega}_0. \tag{14}$$

By Lemma 2.1 and $g_i(x) > 0$, we conclude

$$\begin{aligned}
\int_{\mathbb{R}\setminus\Omega} g_i\varphi\,dx &\le \int_{\mathbb{R}\setminus\Omega} |g_i|^{\frac{1}{\bar{p}_i'(x)}} |g_i|^{\frac{1}{\bar{p}_i(x)}} |\varphi|\,dx \\
&\le 2\|g_i\|_{L^1(\mathbb{R}\setminus\Omega)} \Bigl\| |g_i|^{\frac{1}{\bar{p}_i(x)}} |\varphi| \Bigr\|_{L^{\bar{p}_i(x)}(\mathbb{R}\setminus\Omega)} \le K_1\|\varphi\|_X.
\end{aligned}$$

For sufficiently small $\tau \in (0,1)$, let $\eta \in C_0^\infty(\Omega)$ such that $\overline{\Omega}_0 \subset \operatorname{supp}(\eta)$, $\eta = 1$, for all $x \in \overline{\Omega}_0$ and $0 \le \eta \le 1$ in Ω. Then, by applying (14), it follows that

$$I_\lambda(\tau\eta) \le \tau^{p_1^-} \sum_{i=1}^{2} \biggl(\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{|\eta(x)-\eta(y)|^{p_i(x,y)}}{2p_i(x,y)|x-y|^{N+sp_i(x,y)}}\,dx\,dy + \int_\Omega \frac{V_i(x)|\eta|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N\setminus\Omega}\frac{\beta(x)|\eta|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx\Big)-\lambda_1\tau^{r_1{}^-+\epsilon_1}\int_{\Omega_0}\frac{A_1(x)|\eta|^{r_1(x)}}{r_1(x)}\,dx\\
&-\lambda_2\tau^{r_2{}^-+\epsilon_2}\int_{\Omega_0}\frac{A_2(x)|\eta|^{r_2(x)}}{r_2(x)}\,dx-\int_{\mathbb{R}^N\setminus\Omega}\tau\eta g_1(x)\,dx-\int_{\mathbb{R}^N\setminus\Omega}\tau\eta g_2(x)\,dx\\
\leq{}&\frac{\tau^{p_1^-}}{p_1^-}3K_1\max\big\{\|\eta\|_X^{p_2^-},\|\eta\|_X^{p_2^+}\big\}-\frac{\lambda_2\tau^{r_2{}^-+\epsilon_2}}{r_2^+}\int_{\Omega_0}A_2(x)|\eta|_X^{r_2(x)}\,dx.
\end{aligned}
$$

Since $r_2{}^- + \epsilon_2 < p_1^-$, we have $I_\lambda(t\eta) < 0$.

In addition, combining the Hölder inequality and inequality (11), for any $\varphi \in B_\rho(0)$, we have

$$
I_\lambda(\varphi)\geq\frac{1}{p_2^+}\|\varphi\|_X^{p_1^-}-\frac{2\lambda_1M_1^{r_1^-}}{r_1^-}\|A_1\|_{s_1(x)}\|\varphi\|_X^{r_1^-}-\frac{2\lambda_2M_2^{r_2^-}}{r_2^-}\|A_2\|_{s_2(x)}\|\varphi\|_X^{r_2^-}-2K_1\|\varphi\|_X.
$$

This fact gives

$$
-\infty<\widehat{c}:=\underline{\inf}_{B_\rho(0)}I_\lambda<0. \tag{15}
$$

Set

$$
\frac{1}{n}\in\big(0,\underline{\inf}_{\partial B_\rho(0)}I_\lambda(\varphi)-\underline{\inf}_{B_\rho(0)}I_\lambda(\varphi)\big),\quad n\in\mathbb{N}.
$$

By (15), $I_\lambda:\overline{B_\rho(0)}\to\mathbb{R}$ is lower bounded on $\overline{B_\rho(0)}$ and $I_\lambda\in C^1(\overline{B_\rho(0)},\mathbb{R})$. Using Ekeland's variational principle, there exists $\{\varphi_n\}_{n\in\mathbb{N}}\in\overline{B_\rho(0)}$ such that

$$
\begin{cases}
\widehat{c}\leq I_\lambda(\varphi_n)\leq\widehat{c}+\frac{1}{n},\\
0<I_\lambda(\varphi_n)-I_\lambda(\varphi)+\frac{1}{n}\|\varphi_n-\varphi\|_X,\quad \varphi_n\neq\varphi.
\end{cases} \tag{16}
$$

Since

$$
I_\lambda(\varphi_n)\leq\widehat{c}+\frac{1}{n}\leq\inf_{\overline{B_\rho(0)}}I_\lambda+\frac{1}{n}\leq\inf_{B_\rho(0)}I_\lambda+\frac{1}{n}<\inf_{\partial B_\rho(0)}I_\lambda,
$$

we have $\varphi_n\in B_\rho(0)$. Define function $\zeta:\overline{B_\rho(0)}\to\mathbb{R}$ by

$$
\zeta(\varphi)=I_\lambda(\varphi)+\frac{1}{n}\cdot\|\varphi_n-\varphi\|_X,
$$

which implies $\zeta(\varphi_n)<\zeta(\varphi)$ from (16). Then, φ_n is a minimum point of ζ, and we have

$$
\frac{\zeta(\varphi_n+t\cdot\nu)-\zeta(\varphi_n)}{t}\geq 0,
$$

for small $t>0$ and any $\nu\in B_1(0)=\{\nu\in X:\|\nu\|_X=1\}$. Hence,

$$
\frac{I_\lambda(\varphi_n+t\cdot\nu)-I_\lambda(\varphi_n)}{t}+\frac{1}{n}\|\nu\|_X\geq 0.
$$

Let $t \to 0$, then $\langle I'_\lambda, \nu\rangle + \frac{1}{n}\|\nu\|_X \geq 0$. Replace ν with $-\nu$. Then, we obtain $\langle -I'_\lambda, \nu\rangle + \frac{1}{n}\|\nu\|_X \geq 0$. Thus, $\|I'_\lambda(\varphi_n)\|_{X^*} \leq \frac{1}{n}$. We infer that there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset B_\rho(0)$ such that

$$I_\lambda(\varphi_n) \to \widehat{c} < 0, \qquad \left\|I'_\lambda(\varphi_n)\right\|_{X^*} \to 0, \quad n \to \infty.$$

By Lemma 4.2, there is $\varphi_n \to \varphi_0^{(2)}$ in X. Then, we have $I'_\lambda(\varphi_0^{(2)}) = 0$ and $I_\lambda(\varphi_0^{(2)}) = \widehat{c} < 0$, that is, $\varphi_0^{(2)}$ is another solution of equations (1) with negative energy, which ends the proof. □

Here, we give an example of application of Theorem 1.1.

Example 4.1 Let $\Omega = \{(x,y) \in \mathbb{R} : x^2 + y^2 \leq 1\}$. Consider the problem

$$\begin{cases} (-\Delta)^{\frac{1}{2}}_{x^2+y^2+3}\varphi + (-\Delta)^{\frac{1}{2}}_{x^2+y^2+5}\varphi + 2|\varphi|^{2x^2+1}\varphi + 4|\varphi|^{2x^2+3}\varphi \\ \quad = |x||\varphi|^{\frac{1}{6}}\varphi + x^2|\varphi|^{\frac{1}{2}}\varphi, \quad x \in \Omega, \\ \mathcal{N}_{\frac{1}{2},x^2+y^2+3}\varphi + \mathcal{N}_{\frac{1}{2},x^2+y^2+5}\varphi + \ln|x||\varphi|^{2x^2+1}\varphi \\ \quad + \ln|x||\varphi|^{2x^2+3}\varphi = 0, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{17}$$

By simple calculations, we obtain $\text{meas}(\partial\Omega) = 2\pi$, $p_1^- = 3$, $p_1^+ = 4$, $p_2^- = 5$, $p_2^+ = 6$. Conditions (P), (G), (H), and (V) are satisfied. We observe that all assumptions of Theorem 1.1 are fulfilled. Hence, Theorem 1.1 implies that problem (17) admits two nontrivial weak solutions.

5 Proof of Theorem 1.2

To prove Theorem 1.2, we first recall the following lemmas.

Lemma 5.1 ([15]) *Let X be a reflexive and separable Banach space. Then, there are $\{e_n\} \subset E$ and $\{e_n^*\} \subset E^*$ such that*

$$E = \overline{\text{span}\{e_n : n = 1,2,3\ldots\}}, \qquad E^* = \overline{\text{span}\left\{e_n^* : n = 1,2,3\ldots\right\}}$$

and

$$\left\langle e_i^*, e_j\right\rangle = \begin{cases} 1, & \textit{if } i = j, \\ 0, & \textit{if } i \neq j. \end{cases}$$

Denote

$$E_n = \text{span}\{e_n\}, \qquad X_k = \bigoplus_{n=1}^{k} E_n, \quad \textit{and} \quad Y_k = \overline{\bigoplus_{n=k}^{\infty} E_n}.$$

Lemma 5.2 ([15]) *Assume that $q(x) \in C_+(\overline{\Omega})$, $q(x) < p^*(x)$, for any $x \in \overline{\Omega}$ and denote*

$$\widetilde{\xi}_k = \sup_{\varphi\in Y_k,\|\varphi\|_X=1} \|\varphi\|_{L^{q(x)}(\Omega)},$$

then $\lim_{k\to\infty}\widetilde{\xi}_k = 0$.

Now, we recall the fountain theorem.

Theorem 5.1 ([15]) *Let X be a real Banach space and $I_\lambda \in C^1(X,\mathbb{R})$ be a even functional satisfying the (PS) conditions. There exists $r_k > 0$ such that $\widehat{\rho}_k > r_k > 0$ for every $k \in \mathbb{N}$. Then, the following conditions hold*:

(i) $\alpha_k = \max\{I_\lambda(\varphi) : \varphi \in X_k, \|\varphi\| = \widehat{\rho}_k\} \le 0$;

(ii) $\beta_k = \inf\{I_\lambda(\varphi) : \varphi \in Y_k, \|\varphi\| = r_k\} \to +\infty$ *as* $k \to \infty$.

Then, I_λ *possesses a series of critical points* φ_k *such that* $I_\lambda(\varphi_k) \to +\infty$.

Lemma 5.3 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. There exists* $\widehat{\rho}_k > 0$ *such that*

$$\max_{\varphi\in X_k,\|\varphi\|=\widehat{\rho}_k} I_\lambda(\varphi) < 0.$$

Proof Let $t \in (0,1)$. For $\|\widehat{\varphi}\|_X = \widetilde{\rho}_k \ge 1$ and $\widehat{\rho}_k > \widetilde{\rho}_k$, there exists $\widehat{\varphi}$ such that

$$\begin{aligned} I_\lambda(t\widehat{\varphi}) &\le t^{p_2^+}\sum_{i=1}^{2}\biggl(\int_{\mathbb{R}^{2N}\setminus(\mathcal{C}\Omega)^2}\frac{|\widehat{\varphi}(x)-\widehat{\varphi}(y)|^{p_i(x,y)}}{2p_i(x,y)|x-y|^{N+sp_i(x,y)}}\,dx\,dy + \int_\Omega V_i(x)\frac{|\widehat{\varphi}|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx \\ &\quad + \int_{\mathbb{R}^N\setminus\Omega}\frac{\beta(x)|\widehat{\varphi}|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx\biggr) - \lambda_1 t^{r_1^-}\int_\Omega\frac{A_1(x)|\widehat{\varphi}|^{r_1(x)}}{r_1(x)}\,dx \\ &\quad - \lambda_2 t^{r_2^-}\int_\Omega\frac{A_2(x)|\widehat{\varphi}|^{r_2(x)}}{r_2(x)}\,dx - \int_{\mathbb{R}^N\setminus\Omega} t\widehat{\varphi}g_1(x)\,dx - \int_{\mathbb{R}^N\setminus\Omega} t\widehat{\varphi}g_2(x)\,dx \\ &\le \frac{t^{p_2^+}}{p_1^-}\|\widehat{\varphi}\|_X^{p_2^+} - \frac{\lambda_1 t^{r_1^-}}{r_1^+}\int_\Omega A_1(x)|\widehat{\varphi}|^{r_1(x)}\,dx < 0 \end{aligned}$$

with $p_1^- > r_1^+ > 1$. Taking $\varphi = t\widehat{\varphi}$, for sufficiently small t, it follows that

$$\alpha_k = \max_{\varphi\in X_k,\|\varphi\|=\widehat{\rho}_k} I_\lambda(\varphi) \le 0.$$ □

Lemma 5.4 *Assume that assumptions* (P), (G), (V), *and* (H) *hold. There exists* $r_k > 0$ *such that*

$$\inf_{\varphi\in Y_k,\|\varphi\|=r_k} I_\lambda(\varphi) > +\infty.$$

Proof According to Lemma 5.2, for $\|\varphi\|_X = r_k > 1$, we obtain

$$\begin{aligned} I_\lambda(\varphi) &\ge \frac{1}{p_2^+}\|\varphi\|_X^{p_1^-} - \frac{\lambda_1}{r_1^-}\int_\Omega A_1(x)|\varphi|^{r_1(x)}\,dx - \frac{\lambda_2}{r_2^-}\int_\Omega A_2(x)|\varphi|^{r_2(x)}\,dx \\ &\ge \frac{1}{p_2^+}\|\varphi\|_X^{p_1^-} - \frac{2\lambda_1 M_1^{r_1^+}}{r_1^-}\|A_1\|_{s_1(x)}\|\varphi\|_X^{r_1^+} - \frac{2\lambda_2 M_2^{r_2^+}}{r_2^-}\|A_2\|_{s_2(x)}\|\varphi\|_X^{r_2^+} \\ &\ge \frac{1}{p_2^+}\|\varphi\|_X^{p_1^-} - \frac{2\lambda_1 M_1^{r_1^+}\widetilde{\xi}_k^{r_1^+}}{r_1^-}\|A_1\|_{s_1(x)}\|\varphi\|_X^{r_1^+} - \frac{2\lambda_2 M_2^{r_2^+}\widetilde{\xi}_k^{r_2^+}}{r_2^-}\|A_2\|_{s_2(x)}\|\varphi\|_X^{r_2^+}. \end{aligned}$$

Let

$$\max\biggl\{\frac{2\lambda_1 M_1^{r_1^+}\widetilde{\xi}_k^{r_1^+}}{r_1^-}\|A_1\|_{s_1(x)}\|\varphi\|_X^{r_1^+}, \frac{2\lambda_2 M_2^{r_2^+}\widetilde{\xi}_k^{r_2^+}}{r_2^-}\|A_2\|_{s_2(x)}\|\varphi\|_X^{r_2^+}\biggr\}$$

$$= 2\frac{\lambda_\varphi M_\varphi^{r_\varphi}\widetilde{\xi}_k^{\varphi}}{r_\varphi^-}A_\varphi^*\|\varphi\|_X^{r_\varphi},$$

and there exists a constant $\widetilde{C}$ such that $r_\varphi = \max\{r_1^+ + \widetilde{C}, r_2^+ + \widetilde{C}\}$, where $1 < p_1^- < r_\varphi^- < r_\varphi$. Therefore,

$$\begin{aligned}
&\frac{1}{p_2^+}\|\varphi\|_X^{p_1^-} - \frac{2\lambda_1 M_1^{r_1^+}\widetilde{\xi}_k^{r_1^+}}{r_1^-}\|A_1\|_{s_1(x)}\|\varphi\|_X^{r_1^+} - \frac{2\lambda_2 M_2^{r_2^+}\widetilde{\xi}_k^{r_2^+}}{r_2^-}\|A_2\|_{s_2(x)}\|\varphi\|_X^{r_2^+}\\
&\geq \|\varphi\|_X^{p_1^-}\left(\frac{1}{p_2^+} - 4\frac{\lambda_\varphi M_\varphi^{r_\varphi}\widetilde{\xi}_k^{\varphi}}{r_\varphi^-}A_\varphi^*\|\varphi\|_X^{r_\varphi - p_1^-}\right).
\end{aligned}$$

Choose

$$r_k = \left(\frac{5\lambda_\varphi M_\varphi^{r_\varphi}p_2^+\widetilde{\xi}_k^{\varphi}}{r_\varphi^-}A_\varphi^*\right)^{\frac{1}{p_1^- - r_\varphi}}.$$

Since $p_1^- < r_\varphi$, we have $r_k \to +\infty$ as $k \to +\infty$. By the choice of r_k with $\|\varphi\|_X = r_k$ such that $\widehat{\rho}_k > \widetilde{\rho}_k > r_k > 0$, we obtain

$$\beta_k = \inf_{\varphi\in Y_k, \|\varphi\| = r_k} I_\lambda(\varphi) \to +\infty, \quad k \to +\infty. \qquad \square$$

Proof of Theorem 1.2 Let hypotheses (P), (G), (V), and (H) be satisfied. By Lemma 4.2, I_λ satisfies (*PS*) conditions. Under the definition of I_λ in Lemma 3.13, it follows that $I_\lambda(0) = 0$ and I_λ is an even function. Therefore, from Lemmas 5.3 and 5.4, it can be deduced that I_λ satisfies Theorem 5.1. Then, I_λ possesses a series of critical points φ_k as $k \to +\infty$. In conclusion, equations (1) possess infinitely many nontrivial weak solutions. $\square$

Here, we give an example of application of Theorem 1.2.

Example 5.1 Let $\Omega = \{(x,y) \in \mathbb{R} : x^2 + y^2 \leq 1\}$. Consider the problem

$$\begin{cases}
(-\Delta)^{\frac{1}{2}}_{x^2+y^2+3}\varphi + (-\Delta)^{\frac{1}{2}}_{x^2+y^2+5}\varphi + (x^2+1)|\varphi|^{2x^2+1}\varphi + (e^{|x|}+2)|\varphi|^{2x^2+3}\varphi\\
\quad = |x||\varphi|^{\sqrt{|x|}-\frac{3}{4}}\varphi + x^2|\varphi|^{\sqrt{|x|}-\frac{1}{4}}\varphi, \quad x \in \Omega,\\
\mathcal{N}_{\frac{1}{2},x^2+y^2+3}\varphi + \mathcal{N}_{\frac{1}{2},x^2+y^2+5}\varphi + \ln|x||\varphi|^{2x^2+1}\varphi\\
\quad + \ln|x||\varphi|^{2x^2+3}\varphi = 0, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}.
\end{cases} \tag{18}$$

By simple calculations, we obtain $\mathrm{meas}(\partial\Omega) = 2\pi$, $p_1^- = 3$, $p_1^+ = 4$, $p_2^- = 5$, $p_2^+ = 6$, $r_1^- = 5/4$, $r_1^+ = 9/4$, $r_2^- = 7/4$, and $r_2^+ = 11/4$. That is, conditions (P), (G), (H), and (V) are satisfied. We observe that all assumptions of Theorem 1.2 are fulfilled. Hence, Theorem 1.2 implies that problem (18) admits infinitely many nontrivial weak solutions.

6 Proof of Theorem 1.3

We give some results with the aid of the Krasnoselskii genus. Let E be a real Banach space and set

$$\mathfrak{R} = \{\mathcal{A} \subset E \setminus \{0\} : \mathcal{A} \text{ is compact and } \mathcal{A} = -\mathcal{A}\}.$$

Let $\mathcal{A} \subset \mathfrak{R}$ and $E = \mathbb{R}^k$. We define genus

$$\gamma(\mathcal{A}) = \min\{k \geq 1 : \text{there exists an odd continuous mapping } \psi : \mathcal{A} \to \mathbb{R}^k \setminus \{0\}\}.$$

If the mapping ψ does not exist for any $k > 0$, and set $\gamma(\mathcal{A}) = \infty$. If $\mathcal{A}$ is a subset consisting of a finite number of pairs of points, then, $\gamma(\mathcal{A}) = 1$. Furthermore, from definition, $\gamma(\emptyset) = 0$.

Lemma 6.1 ([18]) *Let $E = R^N$ and $\partial\Omega$ be the boundary of an open, symmetric, and bounded subset $\partial\Omega \subset R^N$ with $0 \in \Omega$. Then, $\gamma(\partial\Omega) = N$.*

Corollary 6.1 ([18]) $\gamma(S^{N-1}) = N$.

Theorem 6.1 ([18]) *Let $I_\lambda \in C^1(X)$ be a functional satisfying the (PS) conditions and assume that*

(i) *I_λ is bounded from below and even;*
(ii) *there is a compact set $K \in \mathbb{R}$ such that $\gamma(K) = k$ and $\sup_{x\in K} I_\lambda(x) < I_\lambda(0)$.*

Then, I_λ has at least k pairs of distinct critical points whose corresponding critical values are all less than $I_\lambda(0)$.

Proof of Theorem 1.3 Combining (12), (13), and the Hölder inequality (7) for $\|\varphi\|_X > 1$, we obtain

$$\begin{aligned} I_\lambda(\varphi) &\geq \frac{1}{p_2^+}\|\varphi\|_X^{p_1^-} - \frac{\lambda_1}{r_1^-}\|A_1\|_{s_1(x)}\|\varphi\|^{r_1^+}_{L^{s_1'(x)r_1(x)}(\Omega)} - \frac{\lambda_2}{r_2^-}\|A_2\|_{s_2(x)}\|\varphi\|^{r_2^+}_{L^{s_2'(x)r_2(x)}(\Omega)} \\ &\geq \frac{1}{p_2^+}\|\varphi\|_X^{p_1^-} - \frac{\lambda_1}{r_1^-}2\|A_1\|_{s_1(x)}M_1^{r_1^+}\|\varphi\|_X^{r_1^+} - \frac{\lambda_2}{r_2^-}2\|A_2\|_{s_2(x)}M_2^{r_2^+}\|\varphi\|_X^{r_2^+}. \end{aligned}$$

Since $\max\{1, r_1^+, r_2^+\} < p_1^-$, for $\|\varphi\|_X$ large enough, I_λ is bounded from below. $I_\lambda \in X$ is an even function by the definition and $I_\lambda(0) = 0$. Moreover, I_λ is coercive in X and satisfies the (PS) conditions by Lemma 4.2. Let

$$\mathfrak{R}_k = \{\mathcal{M} \subset \mathfrak{R} : \gamma(\mathcal{M}) \geq k\}, \qquad c_k = \inf_{\mathcal{M}\in\mathfrak{R}_k}\sup_{\varphi\in\mathcal{M}} I(\varphi), \quad k = 1, 2, \ldots.$$

We obtain

$$-\infty < c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots.$$

Now we prove that for any $k \in \mathbb{N}$, there is $c_k < 0$. For each k, we take k disjoint open sets $\widetilde{K}_i$ such that $\bigcup_{i=1}^k \widetilde{K}_i \subset \Omega$. For $i = 1, \ldots, k$, let $\varphi_i \in (X \bigcap C_0^\infty(\widetilde{K}_i) \setminus \{0\}$ with $\|\varphi_i\|_X = 1$, and

$$\mathcal{M}_k = \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_k\}.$$

Since each norm on $\mathcal{M}_k$ is equivalent, there is $\rho_k^* \in (0, 1)$ such that $\varphi \in \mathcal{M}_k$ with $\|\varphi\|_X \leq \rho_k^*$, which means that $\|\varphi\|_\infty < C_{\rho_k^*} < 1$. Set

$$S^{(k)}_{\rho_k^*} = \{\varphi \in \mathcal{M}_k : \|\varphi\|_X = \rho_k^*\}.$$

Combining the compactness of $S^{(k)}_{\rho^*_k}$ and $t \in (0,1)$ for all $\varphi \in S^{(k)}_{\rho^*_k}$,

$$\begin{aligned}
I_\lambda(t\varphi) \le{} & t^{p_1^-} \sum_{i=1}^{2} \biggl(\int_{\mathbb{R}^{2N}\setminus(\mathcal{C}\Omega)^2} \frac{|\varphi(x)-\varphi(y)|^{p_i(x,y)}}{2p_i(x,y)|x-y|^{N+sp_i(x,y)}}\,dx\,dy + \int_\Omega V_i(x)\frac{|\varphi|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx \\
& + \int_{\mathbb{R}^N\setminus\Omega} \frac{\beta(x)|\varphi|^{\bar{p}_i(x)}}{\bar{p}_i(x)}\,dx \biggr) - \int_{\mathbb{R}^N\setminus\Omega} t\varphi g_1(x)\,dx - \int_{\mathbb{R}^N\setminus\Omega} t\varphi g_2(x)\,dx \\
& - \lambda_1 t^{r_1^-} \int_\Omega \frac{A_1(x)|\varphi|^{r_1(x)}}{r_1(x)}\,dx - \lambda_2 t^{r_2^-} \int_\Omega \frac{A_2(x)|\varphi|^{r_2(x)}}{r_2(x)}\,dx \\
\le{} & 3K_1 t^{p_1^-} \|\varphi\|_X^{p_2^+} - \frac{\lambda_1 t^{r_1^-}}{p_1^-} \int_\Omega A_1(x)|\varphi|^{r_1(x)}\,dx.
\end{aligned}$$

Sine $1 < r_1^- < p_1^-$, there exist $t_k \in (0,1)$ and ε_k such that

$$I_\lambda(t_k\varphi) < -\varepsilon_k < 0.$$

Thus, $I_\lambda(\varphi) < 0$ for all $\varphi \in S^{(k)}_{t_k\rho^*_k}$. Furthermore, $\gamma(S^{(k)}_{t_k\rho^*_k}) = k$ such that $c_k < -\varepsilon_k < 0$ for all k, and the assertion is proved. Each c_k is a critical value by the Krasnoselskii genus theory. Combining Theorem 6.1, I_λ has at least k pairs of different critical points. In addition, since k is arbitrary, we obtain an infinite number of critical points of equations (1). □

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
Zhenfeng Zhang wrote the main manuscript text and Tianqing An, Weichun Bu, and Shuai Li verified this article. All authors reviewed the manuscript.

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